Weyl closure of a differential operator

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Outline

Definition

- K an algebraically closed field of characteristic 0
- $D = K[x]\langle\partial\rangle$ Weyl algebra in one variable

Definition

For $L \in D$, define $Cl(L) := K(x)\langle\partial\rangle L \cap D$, the Weyl closure of the operator L

Definition

For $I \trianglelefteq D$, define $Cl(I) := Cl(L)$, where L is the generator of $K(x)\langle\partial\rangle L$

Motivation

- Let $V(L)$ be the solution space of L in a neighbourhood of a nonsingular point λ. Then

$$Cl(L) = ann_D(V(L))$$

- $Cl(L)/_{DL} \leq D/_{DL}$ is the submodule of elements with finite support on K.

Goals

- Compute the local closure $Cl_\lambda(L)$ of a differantial operator L
- Compute the initial ideal $in_\lambda(Cl_\lambda(L))$ of $Cl_\lambda(L)$ with respect to the order filtration.
- Compute the global closure $Cl(L)$ of a differential operator L
- Compute the initial ideal $in(Cl(L))$ of $Cl(L)$ with respect to the order filtration.

Definition

Definition

The local closure $Cl_\lambda(L)$ of $L \in D$ at the point $x = \lambda$ is the ideal

$$Cl_\lambda(L) = K[x, (x-\lambda)^{-1}]\langle\partial\rangle L \cap D$$

Algorithm

- Input: $L = p_n(x)\partial^n + \cdots + p_0(x) \in D$
- Rewrite L as

$$L = \sum_{i=r}^{s} \zeta_i q_i((x-\lambda)\partial) \quad \zeta_i = \begin{cases} \partial^{-i} & i \leq 0 \\ (x-\lambda)^i & i > 0 \end{cases}$$

  with $q_i \in K[\theta]$ and $q_r \neq 0$
- Determine m as the maximum integer root of q_r if it is > 0, otherwise (or if q_r has no integer roots) set $m = 0$. m is called the critical exponent.
- If $m + r < 0$, set $B = \{\}$. Otherwise, compute a basis B of the kernel of the Matrix $(R_\lambda(L)_{i,j})_{0\leq i\leq m, 0\leq j\leq m+r}$ with

$$R_\lambda(L)_{i,j} = \begin{cases} q_{j-i}(i) & i \geq j \\ j(j-1)\ldots(i+1)q_{j-i}(i) & i < j \end{cases}$$

- For each $v \in B$, set $p_v = \sum_{i=0}^{m+r} v_i \partial^i$
- Output: Set of generators of $Cl_\lambda(L)$: $\{L, (x-\lambda)^{-1}p_v L | v \in B\}$

Example

Consider the operator

$$L = x^2(x-1)(x-3)\partial^2 - (6x^3 - 20x^2 + 12x)\partial + (12x^2 - 32x + 12)$$

at the point $x = 0$:

- Rewrite L as

$$(3\theta^2 - 15\theta + 12) + x(-4\theta^2 + 24\theta - 32) + x^2(\theta^2 - 7\theta + 12)$$

  with $\theta = x\partial$, thus $r = 0$
- $q_r(\theta) = q_0(\theta) = 3\theta^2 - 15\theta + 12 = 3(t-1)(t-4)$

Example

- Determine a set of generators for the kernel of the $(m+1) \times (m+r+1) = 5 \times 5$ Matrix

$$\begin{pmatrix} q_0(0) & q_1(0) & 2q_2(0) & 6q_3(0) & 24q_4(0) \\ q_{-1}(1) & q_0(1) & 2q_1(1) & 6q_2(1) & 24q_3(1) \\ q_{-2}(2) & q_{-1}(2) & q_0(2) & 3q_1(2) & 12q_2(2) \\ q_{-3}(3) & q_{-2}(3) & q_{-1}(3) & q_0(3) & 4q_1(3) \\ q_{-4}(4) & q_{-3}(4) & q_{-2}(4) & q_{-1}(4) & q_0(4) \end{pmatrix} = \begin{pmatrix} 12 & -32 & 24 & 0 & 0 \\ 0 & 0 & -24 & 36 & 0 \\ 0 & 0 & -6 & 0 & 24 \\ 0 & 0 & 0 & -6 & 16 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

in this case $B = \{B_1 := (8,3,0,0,0)^T, B_2 := (0,9,12,8,3)^T\}$ and set $p_{B_1} = 3\partial + 8$, $p_{B_2} = 3\partial^4 + 8\partial^3 + 12\partial^2 + 9\partial$.

Example

- We can now write down a set of generators for $C_0(L)$:

$$C_0(L) = \langle L, \frac{1}{x} p_{B_1} L, \frac{1}{x} p_{B_2} L \rangle$$

with

$$\begin{aligned} L &= x^2(x-1)(x-3)\partial^2 - (6x^3 - 20x^2 + 12x)\partial \\ &\quad + (12x^2 - 32x + 12) \end{aligned}$$

$$\begin{aligned} \tfrac{1}{x} p_{B_1} L &= (3x^3 - 12x^2 + 9x)\partial^3 + (8x^3 - 38x^2 + 48x - 18)\partial^2 \\ &\quad - (48x^2 - 142x + 72)\partial + (96x - 184) \end{aligned}$$

$$\begin{aligned} \tfrac{1}{x} p_{B_2} L &= (3x^3 - 12x^2 + 9x)\partial^6 + (8x^3 - 2x^2 - 60x + 36)\partial^5 \\ &\quad + (12x^3 - 56x)\partial^4 + (9x^3 - 12x^2 - 69x + 56)\partial^3 \\ &\quad - (18x^2 + 72x - 138)\partial^2 - (54x - 216)\partial + 216 \end{aligned}$$

Definition

Definition

For $T = p_n(x)\partial^n + \cdots + p_0(x) \in D$ define the initial as

$$in_\lambda(T) := (x-\lambda)^{ord_\lambda(p_n)}\partial^n \in K[x,\partial]$$

where $ord_\lambda(f)$ is the order of vanishing of f at the point $x = \lambda$

Definition

For $I \trianglelefteq D$, define the initial ideal as

$$in_\lambda(I) := \langle in_\lambda(T) | T \in I \rangle_K \trianglelefteq K[x,\partial]$$

Note that the initial ideal is an ideal of the commutative ring $K[x,\partial]$!

Calculating the initial ideal

Theorem (without proof)

Let $V \leq ker\left(D/_{(x-\lambda)D} \xrightarrow{\circ L} D/_{(x-\lambda)D}\right)$ *be a linear subspace, let* $\{f_0(\partial), \ldots, f_s(\partial)\}$ *be a basis of* V *with the property that* $deg(f_i) < deg(f_{i+1})$ *for all* i *and let* $I(V) \trianglelefteq D$ *be the left ideal generated by* $\left\{L, (x-\lambda)^{-1}vL | v \in V\right\}$. *Then*

$$in_\lambda(I(V)) = D\{in_\lambda(L), (x-\lambda)^{-(i+1)}\partial^{deg(f_i)-i} in_\lambda(L) | 0 \leq i \leq s\}$$

Recall that $Cl_\lambda(L) = I(V)$ if

$$V = ker\left(D/_{(x-\lambda)D} \xrightarrow{\circ L} D/_{(x-\lambda)D}\right)$$

Example

Recall the previous example: We already computed the basis $\{p_{B_1} = 3\partial + 8, p_{B_2} = 3\partial^4 + 8\partial^3 + 12\partial^2 + 9\partial\}$, thus the initial ideal of $Cl_0(L)$ is

$$in_0(Cl_0(L)) = \langle in_0(L), x^{-1}\partial in_0(L), x^{-2}\partial^3 in_0(L)\rangle = \langle x^2\partial^2, x\partial^3, \partial^6\rangle$$

Properties of the global closure

Lemma

Let $L = p_n(x)\partial^n + \cdots + p_0(x)$ *and let* $p(x) = \sqrt{p_n(x)}$ *be the squarefree part of* p_n. *Then*

$$Cl(L) = K[x, p^{-1}]\langle\partial\rangle L \cap D$$

We will use this Lemma to proove the following theorem:

Theorem

Let $L = p_n(x)\partial^n + \cdots + p_0(x)$ *and let* $\{\lambda_1, \ldots, \lambda_k\}$ *be the distinct roots of* p_n. *Then*

$$Cl(L) = Cl_{\lambda_1}(L) + \ldots Cl_{\lambda_k}(L)$$

Proof of the Lemma

Write $T \in Cl(L)$ as $T = SL$, $S \in K(x)\langle\partial\rangle$, with

$$S = \frac{1}{h(x)}(g_m(x)\partial^m + \cdots + g_0(x))$$

Thus one can write

$$T = \frac{1}{h(x)}(g_m(x)\partial^m + \cdots + g_0(x))(p_n(x)\partial^n + \cdots + p_0(x))$$

Proof of the Lemma

By expanding the right hand side, one gets that $h(x)$ divides all of the following terms:

$$
\begin{aligned}
& g_m(x)p_n(x) \\
& g_m(x)(\dots) + g_{m-1}(x)p_n(x) \\
& g_m(x)(\dots) + g_{m-1}(x)(\dots) + g_{m-2}(x)p_n(x) \\
& \quad \vdots \\
& g_m(x)(\dots) + \dots + g_0(x)p_n(x)
\end{aligned}
$$

Factor $h(x) = a(x)b(x)$ such that $gcd(a(x), p_n(x)) = 1$ and $\sqrt{b(x)}|p(x)$. Then $a(x)$ divides $g_i(x)$ for all i. Thus we can write S as $b(x)^{-1}\widetilde{S}$ with $\widetilde{S} \in D$. This implies $T = b(x)^{-1}\widetilde{S}L \in K[x, p^{-1}]\langle\partial\rangle L \cap D$.

Weyl closure of L, assuming knowledge of singular points

- Input: $L = p_n(x)\partial^n + \cdots + p_0(x)$, $\{\lambda_1, \ldots, \lambda_t\}$ the distinct roots of $p_n(x)$.
- Local Closures: Let S_i be the set of generators of $Cl_{\lambda_i}(L)$, $1 \leq i \leq t$.
- Output: Set of generators of $Cl(L)$: $\cup_{i=1}^{t} S_i$

Disadvantage: All roots of $p_n(x)$ must be known!

Weyl closure of L without knowledge of singular points

- Input: $L = p_n(x)\partial^n + \cdots + p_0(x) \in \mathbb{Q}[x]\langle\partial\rangle$, $p(x) = \sqrt{p_n(x)} = \prod_{k=1}^{t} f_k(x)$ with $f_k(x)$ irreducible over $\mathbb{Q}[x]$
- For each $1 \leq k \leq t$, let $\theta_\alpha = (x-\alpha)\partial$ and rewrite L as

$$L = \sum_{i=r_k}^{s_k} \zeta_i q_i(\theta_\alpha) \in (Q[\alpha]/_{f_k(\alpha)})[x]\langle\partial\rangle \quad \zeta_i = \begin{cases} \partial^{-i} & i \leq 0 \\ (x-\alpha)^i & i > 0 \end{cases}$$

- For each $1 \leq k \leq t$, set m_k to the k-th critical exponent, that is the largest integer root of q_{r_k} or 0. Set $m := max_k\{m_k + r_k\}$
- Let $W := \langle x^i\partial^j | 0 \leq i \leq deg(p), 0 \leq j \leq m\rangle_K \leq D/_{p(x)D}$. Compute a basis B of

$$ker(W \xrightarrow{\circ L} D/_{p(x)D})$$

- Output: A set $\{L, p(x)^{-1}vL | v \in B\}$ of generators of $Cl(L)$

Example

Consider $L = (x^3+2)\partial - 3x^2$. x^3+2 is already irreducible in $\mathbb{Q}[x]$, thus we write

$$\begin{aligned} L &= (((x-\alpha)+\alpha)^3+2)\partial - 3((x-\alpha)+\alpha)^2 \\ &= (3\alpha^2\theta_\alpha - 3\alpha^2) + (x-\alpha)(3\alpha\theta_\alpha - 6\alpha) + (x-\alpha)^2(\theta_\alpha - 3) \end{aligned}$$

The only and thus maximum integer root of $q_0(\theta) = 3\alpha^2(\theta-1)$ is obviously 1, thus we set $m = r+1 = 0+1 = 1$. We now compute

$$ker\left(\langle 1, x, x^2, \partial, x\partial, x^2\partial\rangle_K \xrightarrow{\circ L} D/_{(x^3+2)D}\right)$$

which is the span of $\{\partial + x^2, x\partial - 2, x^2\partial - 2x\}$. We can now write down the generators of $Cl(L)$.

Definition

Definition

Let $T = p_n(x)\partial^n + \cdots + p_0(x)$ and $I \trianglelefteq D$, define

$$in_{(0,1)}(T) := p_n(x)\partial^n \in K[x,\partial]$$

$$in_{(0,1)}(I) := \langle in_{(0,1)}(T) | T \in I \rangle_K \trianglelefteq K[x,\partial]$$

Calculating the initial ideal

Theorem (without proof)

Let $L = p_n(x)\partial^n + \cdots + p_0(x) \in K[x]\langle\partial\rangle$ *and let* $\{\lambda_1, \ldots, \lambda_k\}$ *be the distinct roots of* $p_n(x)$. *For each* $1 \leq k \leq t$, *let* $V_k \leq ker\left(D/_{(x-\lambda_k)D} \xrightarrow{\circ L} D/_{(x-\lambda_k)D}\right)$ *be a linear subspace with a basis* $\{f_{k,0}, \ldots, f_{k,s_k}\}$ *with the property* $deg(f_{k,i}) < deg(fk, i+1)$. *Furthermore, let*

$$I := I(V_1) + \cdots + I(V_t) = D\{L, (x-\lambda_k)^{-1}vL | v \in V_k, 1 \leq k \leq t\}$$

Then

- $in_{\lambda_k}(I) = in_{\lambda_k}(I(V_k))$
- $in_{(0,1)}(I) = \left\langle \left(\prod_{k=1}^{t}(x-\lambda_k)^{j_k}\right)\partial^m | (x-\lambda_k)^{j_k}\partial^m \in in_{\lambda_k}(I) \right\rangle$

Recall that $I = Cl(I)$ if $V_k = ker\left(D/_{(x-\lambda_k)D} \xrightarrow{\circ L} D/_{(x-\lambda_k)D}\right)$ for all k.

The end.

You may wake up and go home now!