I. Gröbner bases in free associative algebras

Viktor Levandovskyy

Lehrstuhl D für Mathematik, RWTH Aachen

EACA Summer School on Computer Algebra and its Applications

How general are Gröbner bases?

Let K be a field, $X = \{x_1, x_2, \ldots\}$ be a finite or countable set.
Moreover, let $K\{X\}$ be the free non-associative algebra (magma) and $K\{\{X\}\}$ be the algebra of non-associative (or tree) power series.
Then there is a Gröbner bases theory over both $K\{X\}$ and $K\{\{X\}\}$!

References

Lothar Gerritzen, "Tree polynomials and non-associative Gröbner bases". Journal of Symbolic Computation, 41 (2006), no. 3-4, 297–316.

Serena Cicaló, Willem de Graaf "Non-associative Gröbner bases, finitely-presented Lie rings and the Engel condition", Proceedings of ISSAC 2007 (and follow-ups).

A historical sketch

A K-bilinear map $V \times V \to V$, $(a, b) \mapsto [a, b]$ is a **Lie bracket**, if

$[b, a] = -[a, b]$
$[[a, b], c] + [[c, a], b] + [[b, c], a] = 0$ (Jacobi identity).

$(V, [,])$ is called a Lie algebra. Any associative algebra can be viewed as a Lie algebra by defining $[a, b] := ab - ba$.

References

A.I. Shirshov "Some algorithmic problem for Lie algebras". Sibirsk. Mat. Zh. 3, (2) 292–296 (**1962**);
English translation in SIGSAM Bull. 33, 3–6 (**1999**)
the idea of *composition* (BB: *S-polynomial*) was present already in A.I. Shirshov "On free Lie rings". Mat. Sbornik 45 (87), 2 (**1958**), 178–218.

Adjusting terminology: **Gröbner-Shirshov bases** for non-associative and non-commutative algebras.

Back to the cozy associativity: taxonomy of structures

Notations: $X := \{x_1, \ldots, x_n\}$ is the finite set of **variables** and K is a field.

Semigroup = associative magma

Monoid = semigroup with the neutral element (␣ or ϵ or 1)

Group = monoid, each element of which is invertible

Ring (with 1) $(R, +, 0, \star, 1)$:

- $(R, +, 0)$ is an abelian group with the neutral element 0
- $(R, \star, 1)$ is a monoid with the neutral element 1
- $\star$ is both left and right distributive over $+$, i.e. $a \star (b + c) = a \star b + a \star c$ and $(b + c) \star a = b \star a + c \star a$.

If R is a commutative ring, then an **associative** R**-algebra** is a ring and an R-module, such that $\forall r \in R \ \forall a, b \in A$ one has

$$r \star (a \star b) = (r \star a) \star b = a \star (r \star b) = (a \star b) \star r.$$

Free structures and some taxonomy

The free monoid on $X = \{x_1, \ldots, x_n\}$:

denoted by $\langle X \rangle$

carrier set: all finite words (including the empty word as the neutral element) in the alphabet X

multiplication: $\star$ is the concatenation $x_2 \star x_1 = x_2 x_1 \neq x_1 x_2 = x_1 \star x_2$.

divisibility: a partial relation on the set of words by string inclusion.

The free group on $X = \{x_1, \ldots, x_n\}$:

denoted by $\langle X \rangle$ (arrgh, same as monoid!)

carrier set: all finite reduced words (including the empty word as the neutral element) in the alphabet $X \cup X'$, where $X' = \{x_1^{-1}, \ldots, x_n^{-1}\}$

multiplication: $\star$ is the concatenation taking inverses into account: $x_2 \star x_1 = x_2 x_1$ but $x_1 \star x_1^{-1} = x_1^{-1} \star x_1 = 1$.

divisibility: a partial relation on the set of reduced words by string inclusion.

Towards FPA

Over an arbitrary ring R and a monoid M we can create

The **monoid algebra**

denoted by RM

carrier set: finite sums $\sum r_i m_i$, where $r_i \in R \setminus \{0\}$ and $m_i \in M$

multiplication: $(\sum r_i m_i) \star (\sum r'_j m'_j) := \sum (r_i r'_j)(m_i m'_j)$

$\mathbf{K}\langle\mathbf{X}\rangle$, for X as above and a field K is called the **free associative algebra over** K = the tensor algebra TV of the vector space $V = K \oplus \bigoplus Kx_i$.

A K-algebra A is a finitely presented associative algebra (**FPA**), if $\exists n \in \mathbb{N}_0$ such that A is a homomorphic image of a free associative algebra over K on the set of n variables, i.e. $A = K\langle X\rangle / I$, where $I \subsetneq K\langle X\rangle$ is a **two-sided ideal**.

Free group is a finitely related (and thus not free!) monoid: generators $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and relations $\{x_i y_i = 1, y_i x_i = 1 \mid 1 \leq i \leq n\}$

Graded structures

A ring R is called **($\mathbb{N}_0$-)graded** if there exist additive subgroups $R_i \subseteq R, i \in \mathbb{N}_0$, such that

- $R = \bigoplus_{i \in \mathbb{N}} R_i$
- $\forall k, j \in \mathbb{N}_0 \; R_k \cdot R_j \subseteq R_{k+j}$, that is $\forall r \in R_k, \forall s \in R_j$ one has $rs \in R_{k+j}$.

$p \in R_i$ is called a **homogeneous** (or a **graded**) element of **degree** i.

Properties

$R_0 \subseteq R$ is a subring, R_i are R_0-bimodules.

We are interested in nontrivial gradings, i. e. those for which $R \neq R_0$.
In general, a grading can be provided by an additive semigroup, most often $\mathbb{N}_0^n, \mathbb{Z}, \mathbb{Z}^n$.

Graded structures

An ideal $I \subset R$ in a graded ring R is called **graded** if $I = \bigoplus_i I_i$, where $I_i = I \cap R_i$.

Properties

◦ If I is graded, then $\forall p \in I \;\; p = p_1 + \ldots + p_k, p_i \in R_i \Rightarrow p_i \in I$.
◦ A graded ideal possesses a generating set, consisting of graded elements.
◦ Any monomial ideal is graded.
◦ For a graded ideal $I \subset R$ in a graded ring R, the factor ring R/I has an induced grading.

Graded modules form a very pleasant subcategory of the category of modules (with morphisms being graded morphisms, i.e. those, which respect the grading).

Some properties of $K\langle X\rangle$

$K\langle x_1\rangle = K[x_1]$ is commutative, so let $n \geq 2$.

$\circ$ $A := K\langle X\rangle$ is naturally $\mathbb{N}_0$-graded: set $\deg(x_i) = 1$, then $A_0 = K$ and for $i \geq 1$ $A_i = \oplus\{Kw : w \in X, \deg(w) = i\}$.

$\circ$ The number of variables of $K\langle X\rangle$ **does not** lead to the nice notion of rank : for $n \geq 3$ there exist embeddings of $K\langle x_1, \ldots, x_n\rangle$ into $K\langle x_1, x_2\rangle$.

$\circ$ $K\langle X\rangle$ is a domain (there are no zero-divisors).

$\circ$ $K\langle X\rangle$ is neither left nor right nor weak Noetherian: there exist infinite strictly ascending chains of ideals; we have to admit infinite generating sets.

Gröbner Bases Q&A

What? Gröbner basis of an ideal $I \subset K\langle X \rangle$ is a generating set for I, possessing many nice properties.
Why? Knowing a Gröbner basis of I, we can answer the following questions about $K\langle X \rangle / I$:

- is $K\langle X \rangle / I = 0$? This happens iff $1 \in I$ iff $1 \in GB(I)$
- is $K\langle X \rangle / I$ finite dimensional algebra? Compute a K-basis of such.
- for $p \in K\langle X \rangle$, is $p \in I$? Ideal membership problem.
- is $K\langle X \rangle / I$ commutative algebra?
- ⋆ is $K\langle X \rangle / I$ left or right or weak Noetherian? Is it prime or semi-prime?
- ⋆ what are the values of various ring-theoretic dimensions of $K\langle X \rangle / I$?
- and many other... ⋆ means that the answer is not always complete

How to compute GB? The contents of next lectures and exercises.

Back to the cozy associativity

From now on, all algebras will be considered associative.

A Gröbner bases theory for (free) assoc. algebras builds on top of
G. M. Bergman, "The diamond lemma for ring theory", Adv. in Math., 29 (**1978**), 178–218.

However, L. A. Bokut in "Imbeddings into simple associative algebras", Algebra Logika, 15 (**1976**), 117–142 has already specialized Gröbner-Shirshov bases for the associative case.

More systematic approach to Gröbner bases (also for free algebras) was performed by Teo Mora in

"Seven variations on standard bases", **1988**, preprint
"Groebner bases in non-commutative algebras", Proc. ISSAC'88 (**1989**), 150–161

Higmans' lemma

Definitions

- A **quasi-ordering** is a binary relation $\preceq$, which is reflexive ($a \preceq a$) and transitive ($a \preceq b,\ b \preceq c \Rightarrow a \preceq c$).
- An ordering is **well-founded**, if every nonempty set has a minimal element.
- A **well-quasi-ordering** is a well-founded quasi-ordering, such that there is no infinite sequence $\{x_i\}$ with $x_i \not\preceq x_j$ for all $i < j$

Higmans' lemma (1952)

The set of finite sequences over a well-quasi-ordered set of labels is itself well-quasi-ordered.

Now, we enter the realm of Gröbner bases.

- $A = K\langle X\rangle$, the free associative algebra over K.
- $M = \langle X\rangle$ is the free monoid (with 1 as the empty word)

A **monomial ordering** $\prec$ on A is a total ordering on M which is compatible with multiplication. Precisely one has:

(i) either $u \prec v$ or $v \prec u$, for any $u, v \in M, u \neq v$;

(ii) if $u \prec v$ then $wu \prec wv$ and $uw \prec vw$, for all $u, v, w \in M$;

Moreover, if every non-empty subset of M has a minimal element wrt $\prec$ (that is, $\prec$ is well-founded), one says that $\prec$ **is a monomial well-ordering**.

Remark

By Higman's lemma, any total ordering on M (even if the number of variables of the polynomial algebra A is infinite), which is compatible with multiplication and such that $1 \prec x_0 \prec x_1 \prec \ldots$ holds, is a monomial well-ordering.

(Monomial) orderings

Let $\langle X \rangle = \langle x_1, \ldots, x_n \rangle$. We always impose a *linear preordering* $x_1 > x_2 > \ldots > x_n > 1$ first.

- For $\mu = x_{j_1} x_{j_2} \cdots x_{j_k}$ and $\nu = x_{l_1} x_{l_2} \cdots x_{l_{\tilde{k}}}$ from $\langle X \rangle$

$$\mu <_{\text{llex}} \nu \iff \exists 1 \leq i \leq \min\{k, \tilde{k}\} : \; x_{j_w} = x_{l_w} \; \forall w < i \; \wedge \; x_{j_i} < x_{l_i}$$
$$\text{or } \nu = \mu\tilde{\nu} \quad \text{for some } \tilde{\nu} \in \langle X \rangle.$$

This is called the **left lexicographical ordering**.

Analogously one can define the **right lexicographical ordering** rlex.

Houston, we've got a problem!

Neither llex nor rlex are monomial orderings.

Hint: $x_2 x_1 <_{llex} x_1$, but this is a contradiction (why?) to $1 < x_2$.

Monomial degree orderings

- Take μ, ν as before. We define:

$$\mu <_{\text{degllex}} \nu \quad \Longleftrightarrow \begin{cases} k < \tilde{k} & \text{, or} \\ k = \tilde{k} \text{ and } \mu <_{\text{llex}} \nu. \end{cases}$$

  This is called the **degree (left) lexicographical ordering**.

- Take $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n_+$ and again let $\mu, \nu \in \langle X \rangle$ as before.

$$\mu <_{\omega} \nu \quad \Longleftrightarrow \begin{cases} \sum_{i=1}^{k} \omega_{j_i} < \sum_{i=1}^{\tilde{k}} \omega_{l_i} & \text{or} \\ \sum_{i=1}^{k} \omega_{j_i} = \sum_{i=1}^{\tilde{k}} \omega_{l_i} \text{ and } \mu <_{\text{llex}} \nu. \end{cases}$$

  This is called the **weighted degree left lexicographical ordering** with weight vector ω.

  Both `degllex` and ω-`degllex` are monomial orderings.

Notations

- $\mathrm{lm}(f) \in \langle X \rangle$ the leading (greatest) monomial of $f \in K\langle X \rangle \setminus \{0\}$
- $\mathrm{lc}(f) \in K \setminus \{0\}$ the leading coefficient of $f \in K\langle X \rangle \setminus \{0\}$
- $\mathrm{lm}(G) = \{\mathrm{lm}(g) \,|\, g \in G \setminus \{0\}\}$ with $\emptyset \neq G \subset K\langle X \rangle$
- $\mathrm{LM}(G)$ the two-sided ideal generated by $\mathrm{lm}(G)$

Definition

Let I be a left (right, two-sided) ideal of $K\langle X \rangle$ and $G \subset I$.

If $\mathrm{LM}(G) = \mathrm{LM}(I)$ as left (right, two-sided) monoid ideals, then G is called a **left (right, two-sided) Gröbner basis** of I.

In other words, for all $f \in I \setminus \{0\}$ $\exists g \in G \setminus \{0\}$ and

Left GB: $\exists w_L \in \langle X \rangle \ : \ \mathrm{lm}(f) = w_L \cdot \mathrm{lm}(g)$.

Two-sided GB: $\exists w_L, w_R \in \langle X \rangle \ : \ \mathrm{lm}(f) = w_L \cdot \mathrm{lm}(g) \cdot w_R$.

Gröbner representation

Definition

Let $G \subset K\langle X \rangle, f \in K\langle X \rangle$. We say that f has a **two-sided Gröbner representation** with respect to G if $f = 0$ or there is a finite index set I, $\lambda_i, \rho_i \in K\langle X \rangle, g_i \in G$ such that

$$f = \sum_{i \in I} \lambda_i g_i \rho_i$$

with either $\lambda_i g_i \rho_i = 0$ or $\mathrm{lm}(f) \succeq \mathrm{lm}(\lambda_i)\mathrm{lm}(g_i)\mathrm{lm}(\rho_i)$ holds.

Lemma

Let $\prec$ be a well ordering. Then G is a Gröbner basis (of $\langle G \rangle$) if and only if every $f \in \langle G \rangle \setminus \{0\}$ has a Gröbner representation.

Intuition: given an ordering and a generating set G of an ideal, we want to produce new polynomials, which do not possess a Gröbner representation with respect to G, and enlarge G by those.

Divisibility and overlaps

Let $u, w \in \langle X \rangle$ be two monomials.

- We say that u **divides** w (or w **is divisible by** u), if there exist $p, q \in \langle X \rangle$ such that $\mathbf{w} = p \cdot \mathbf{u} \cdot q$.
- If $w = pu$, then w **is divisible by** u **from the left**.
- The set G is called **minimal**, if $\forall g_1, g_2 \in G$, $\mathrm{lm}(g_1)$ does not divide $\mathrm{lm}(g_2)$ and vice versa.

Two monomials $u, w \in \langle X \rangle$ have an **overlap** at a monomial o, if $w = ow'$ and $u = u'o$. We denote the overlapping by $u' \cdot o \cdot w'$. If $o = 1$, the overlap is trivial.

Exercise: for a fixed $u, w \in \langle X \rangle$ there are finitely many overlaps (u, w, o_i).
Observation: Working with left ideals, the only divisibility from the left can be achieved by proper submonomials.

Normal form

Let $\mathcal{G}$ be the set of all finite and ordered subsets of $K\langle X\rangle$.
A map $\mathrm{NF} : K\langle X\rangle \times \mathcal{G} \to K\langle X\rangle,\ (f, G) \mapsto \mathrm{NF}(f|G)$ is called a **(two-sided) normal form** on $K\langle X\rangle$ if

(i) $\mathrm{NF}(0 \mid G) = 0$,

(ii) $\mathrm{NF}(f|G) \neq 0 \ \Rightarrow\ \mathrm{lm}(\mathrm{NF}(f|G)) \notin LM(G)$, and

(iii) $f - \mathrm{NF}(f|G) \in \langle G\rangle$, for all $f \in K\langle X\rangle$ and $G \in \mathcal{G}$.

Let $f, g \in K\langle X\rangle$. Suppose that there are $p, q \in \langle X\rangle$ such that

- $\mathrm{lm}(f)\, q = p\, \mathrm{lm}(g)$,
- $\mathrm{lm}(f)$ does not divide p and $\mathrm{lm}(g)$ does not divide q.

Then the **overlap polynomial (relation)** of f, g by p, q is defined as

$$o(f, g, p, q) = \frac{1}{\mathrm{lc}(f)} fq - \frac{1}{\mathrm{lc}(g)} pg.$$

Division algorithm and Normal form

Algorithm NF

Input: $f \in K\langle x_1, \ldots, x_n\rangle,\ G \in \mathcal{G}$;
Output: h, a normal form of f with respect to G.

$h := f$;
while ($(h \neq 0)$ **and** $(G_h = \{g \in G :\ \mathrm{lm}(g) \text{ divides } \mathrm{lm}(h)\} \neq \emptyset)$) do
choose **any** $g \in G_h$;
compute $w_L, w_R \in \langle X\rangle$ such that $\mathrm{lm}(h) = w_L \cdot \mathrm{lm}(g) \cdot w_R$;
$h := h - \frac{lc(h)}{lc(g)} \cdot w_L \cdot g \cdot w_R$;
return h.

Lemma

NF(h, G) *always terminates. (Key: monomial ordering!)*

A useful isomorphism and K-basis

Lemma

Let $\prec$ be a well-ordering on $K\langle X\rangle$ and $G \subset K\langle X\rangle$ a Gröbner basis of $I = \langle G\rangle$. Then there is the following isomorphism of K-vector spaces

$$K\langle X\rangle \cong K\langle X\rangle/\mathrm{LM}(I) \oplus I, \quad f \mapsto (\mathrm{NF}(f,G),\ f - \mathrm{NF}(f,G)).$$

Since G is a GB of I, $\mathrm{LM}(I) = \mathrm{LM}(G)$. Note, that $K\langle X\rangle/\mathrm{LM}(I)$ is a monomial algebra.

Corollary

$\circ$ $K\langle X\rangle/\mathrm{LM}(I) \cong K\langle X\rangle/I$ as K-vector spaces

$\circ$ $\{w \in \langle X\rangle : w \notin \mathrm{LM}(I)\}$ is the canonical (with respect to $\prec$) monomial K-basis of $K\langle X\rangle/I$.

Generalized Buchberger's Criterion

Theorem

Let $\prec$ be a well-ordering on $K\langle X\rangle$ and $G \subset K\langle X\rangle$.
Then the following conditions are equivalent:

1. *G is a (two-sided) Gröbner basis of $\langle G\rangle$*
2. *$\forall g_1, g_2 \in G$, for every overlap polynomial holds*
$$\mathrm{NF}(\ o(g_1, g_2, p, q)\ \mid\ G) = 0.$$
3. *$\forall g_1, g_2 \in G$, every overlap polynomial $o(g_1, g_2, p, q)$ has a Gröbner representation with respect to G.*

Note: infinite Gröbner bases exist (even monomial ones).

Procedure GroebnerBasis

Input: $G \in \mathcal{G}$.
Output: H, a (two-sided) Gröbner basis of $\langle G \rangle$.

$H := G \setminus \{0\}$;
$P := \{(f,g) \mid f,g \in H\}$; (note: (f,g) and (g,f) are different pairs!)
while $P \neq \emptyset$ do
  choose $(f,g) \in P$;
  $P := P \setminus \{(f,g)\}$;
  $O := \{o(f,g,p,q)\}$; (the set of all overlap polynomials between f,g)
  for $o \in O$ do
    $h := \mathrm{NF}(o,H)$;
    if $h \neq 0$ then
      $H := H \cup \{h\}$;
      $P := P \cup \{(f,h) \mid f \in H\}$; (note: (h,h) are added as well)
  end if; end for; end while;
return H.

Word problem and ideal membership

Lemma

Let $\prec$ be a monomial ordering on $K\langle X \rangle$ and G a Gröbner basis of I wrt $\prec$. Then $f \in I \Leftrightarrow \mathrm{NF}(f, G) = 0$.

Applications

triviality: $K\langle X \rangle / I = 0 \Leftrightarrow 1 \in I \Leftrightarrow 1 \in GB(I)$

commutativity: $K\langle X \rangle / I$ is commutative $\Leftrightarrow \{[x_j, x_i]\} \subseteq I$

algebraicity: $p \in K\langle X \rangle / I$ is algebraic $\Leftrightarrow \exists k \geq 1, c_i \in K : \sum_i^k c_i p^i \in I$

Houston, we've got a problem!

We can check the above properties and many more, if a Gröbner basis of I wrt $\prec$ is finite.

Trying various orderings heuristically might sometimes help.

But there are plenty of ideals, which do not have any finite Gröbner basis!

Finiteness of Gröbner bases

Lemma (T. Mora)

If $\dim_K(K\langle X\rangle/I) < \infty$*, then every minimal Gröbner basis of* I *is finite.*

Proof.

Having a finite K-basis B (wlog monomial) of $K\langle X\rangle/I$ implies, that the set of monomials "below the staircase"

$$\{w \in \mathrm{LM}(I) \mid \exists i \in [1, n]\ \exists b \in B : w = bx_i \text{ or } w = x_i b\}$$

is finite. The same set clearly generates $\mathrm{LM}(I)$, and hence for any Gröbner basis G of I the monoid ideal $\mathrm{LM}(G) = \mathrm{LM}(I)$ is finitely generated, so a minimal G is finite. □

Fine, but what can we do with infinite dimensional algebras?

Finiteness of Gröbner bases II

Proposition

Let $I \subset K\langle X\rangle$ be a **graded** *two-sided ideal and $d > 0$ an integer. If I has a finite number of graded generators F of degree $\leq d$ then the algorithm* NCGBasis *computes in a finite number of steps all elements of degree $\leq d$ of a graded Gröbner basis of I.*

Proof.

Exercise: (a) any overlap polynomial between the elements from F is homogeneous of higher degree,
(b) the normal form of a homogeneous g wrt F is either zero or homogeneous of same degree as g.
This means, that as soon as we process all pairs of polynomials of degree $\leq d$, reduction on overlap polynomials of degree $\geq d+1$ does not have impact on the degrees $\leq d$.
Yet another explanation: since F is a set of graded polynomials, $I = \langle F\rangle$ is a graded ideal $I = \bigoplus I_i$. □

Finiteness of Gröbner bases III and the word problem

The word problem for finitely presented **graded** associative algebras is solvable! If $f \in K\langle X \rangle$ is homogeneous of degree d, compute a Gröbner basis of $I_{\leq d}$ (which is finite) and $NF(f, I_{\leq d})$.

If an ideal is not graded, then the word problem is **unsolvable in general**. The truncation of a non-graded ideal up to a given degree is not well-defined, since reduction of overlap polynomials of degree $\geq d+1$ might add new elements of degree $\leq d$.

Models of computation

- we always work up to a fixed degree bound d
- homogeneous input allows to use **truncated** Gröbner basis up to degree d, where $\forall k \in \mathbb{N}\ G_d \subseteq G_{d+k}$ holds (adaptive)
- inhomogeneous input: either compute a Gröbner basis up to degree d (approximation) or homogenize the input and proceed as before
- problems: Gröbner basis of a homogenized set is rather infinite, ...

Gröbner basis computation in $K\langle X \rangle$: Example

Let $X = \{x, y\}$. Consider $f_1 = x^3 - y^3 = xxx - yyy, f_2 = xyx - yxy$ and $I = \langle f_1, f_2 \rangle \subset K\langle X \rangle$ with respect to the degree left lexicographical ordering. We compute truncated Gröbner basis up to degree $d = 5$.
Let $G = \{f_1, f_2\}$. $(\mathbf{f_1}, \mathbf{f_1})$: $\mathrm{lm}(f_1) = xxx$, so there are two self-overlaps

$$o_1 := o_{1,1} = f_1 x - x f_1 = xy^3 - y^3 x,\ o_{1,2} = f_1 x^2 - x^2 f_1 = x^2 y^3 - y^3 x^2.$$

Moreover, $o_{1,2} - x o_{1,1} = xy^3 x - y^3 x^2 = o_{1,1} x$, so $o_{1,2}$ reduces to 0. Hence $G := G \cup \{o_1\} = \{\mathbf{f_1}, \mathbf{f_2}, \mathbf{o_1}\}$.
$(\mathbf{f_2}, \mathbf{f_2})$: $\mathrm{lm}(f_2) = xyx$, there are two self-overlaps. Symmetry implies that both of them originate from the overlap $xy \cdot x \cdot yx$ of $\mathrm{lm}(f_2)$. Then

$$o_2 = f_2 yx - xy f_2 = xyyxy - yxyyx. \text{ So } G := G \cup \{o_2\} = \{\mathbf{f_1}, \mathbf{f_2}, \mathbf{o_1}, \mathbf{o_2}\}.$$

Gröbner basis in $K\langle X \rangle$: Example continued

$(\mathbf{f_1}, \mathbf{f_2})$: $\operatorname{lm}(f_1)$ and $\operatorname{lm}(f_2)$ have two overlaps $xx \cdot x \cdot yx$ and $xy \cdot x \cdot xx$, hence

$$o_{3,1} = f_1 yx - xxf_2 = xxyxy - y^4 x \text{ and } o_{3,2} = f_2 xx - xyf_1 = xy^4 - yxyxx.$$

Performing reductions, we see that $o_{3,1} - xf_2 y - f_2 yy - yo_1 = 0$ and $o_{32} - o_1 y + yf_2 x + yyf_2 = yyyxy - yyyxy = 0$.
$(\mathbf{f_1}, \mathbf{o_1})$ has overlap $xx \cdot x \cdot yyy$, $(\mathbf{f_2}, \mathbf{o_1})$ has overlap $xy \cdot x \cdot yyy$,
$(\mathbf{f_1}, \mathbf{o_2})$ has overlap $xx \cdot x \cdot yyxy$, $(\mathbf{o_1}, \mathbf{o_2})$ has overlap $xyy \cdot xy \cdot yy$,
$\mathbf{o_2}$ has a self-overlap $xyy \cdot xy \cdot yxy$ and $(\mathbf{f_2}, \mathbf{o_2})$ has two overlaps $xy \cdot x \cdot yyxy$ and $xyy \cdot xy \cdot x$. Since all these elements are of degree ≥ 6 and we are in the graded case, we conclude that

$G = \{f_1, f_2, o_1, o_2\}$ is truncated Gröbner basis up to degree 5.