

III. Gel'fand-Kirillov dimension, Growth of algebras and more

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Filtrations on algebras and modules

Let A be an associative K -algebra, generated by x_1, \dots, x_m .

Degree filtration

Let $V = Kx_1 \oplus \dots \oplus Kx_m \subset A$ be a vector space.

Set $V_0 = K$, $V_1 = K \oplus V$ and $V_{k+1} = V_k \oplus V^{k+1}$. If

$$V_i \subseteq V_{i+k}, \quad V_i \cdot V_j \subseteq V_{i+j}, \quad A = \bigcup_{k=0}^{\infty} V_k,$$

then $\{V_k : k \in \mathbb{N}\}$ is the **standard (ascending) filtration** of A .

Gel'fand-Kirillov dimension and its properties

Let M be a left A -module and let $M_0 \subset M$ be a finite K -vector space, spanned by the generators of M . That is $\dim_K M_0 < \infty$ and $AM_0 = M$.

$\{H_d := V_d M_0 \mid d \in \mathbb{N}\}$ is an induced ascending filtration on M .

The **Gel'fand-Kirillov dimension** of M is defined as follows

$$\text{GKdim}(M) = \limsup_{d \rightarrow \infty} (\log_d(\dim_K H_d))$$

In the standard construction one puts $\deg x_i := 1$ and defines

$$V_d := \{f \in A \mid \deg f \leq d\}.$$

Conventions: $\text{GKdim}(0) = -\infty$. $\text{GKdim}_{\mathbb{Q}}(\mathbb{Q}) = 0$.

Gel'fand-Kirillov dimension: examples

Lemma

Consider the free associative algebra $T = K\langle x_1, \dots, x_n \rangle$, $n \geq 2$. Then $\text{GKdim}(T) = \infty$.

Proof: $\dim V_{=d} = n^d$, $\dim V_d = \dim V_{\leq d} = \frac{n^{d+1}-1}{n-1}$.

Note, that $\frac{n^{d+1}-1}{n-1} > n^d$.

Since $\log_d n^d = d \log_d n = \frac{d}{\log_n d} \rightarrow \infty$, $d \rightarrow \infty$, it follows that $\text{GKdim}(T) = \infty$.

Gel'fand-Kirillov dimension: examples

Lemma

Let A be a K -algebra and a domain. If the standard filtration on A is compatible with the PBW Basis $\{x^\alpha \mid \alpha \in \mathbb{N}^m\}$, then $\text{GKdim}_K(A) = m$.

Proof:

$$\dim V_{=d} = \binom{d+m-1}{m-1}, \quad \dim V_d = \binom{d+m}{m} = \frac{d^m}{m!} + \dots$$

$$\text{GKdim}(A) = \limsup_{d \rightarrow \infty} \log_d \binom{d+m}{m} = m.$$

G-algebras

For an arbitrary field K

fix the set of variables $X = \{x_1, \dots, x_n\}$, let $c_{ij} \in K \setminus \{0\}$
and $d_{ij} = \sum_{\alpha} d_{ij}^{\alpha} x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$.

An associative K -algebra, gen. by X and obeying the relations

$$\forall 1 \leq i < j \leq n \quad x_j x_i = c_{ij} x_i x_j + d_{ij}$$

is called a **G-algebra**, if \exists a monomial ordering \prec on $K[X]$, such that either $d_{ij} = 0$ or $\text{lm}(d_{ij}) \prec x_i x_j$ and certain relations between c_{ij}, d_{ij} are satisfied.

G-algebras are Noetherian integral domains (Weyl, $(q-)$ shift algebras etc are G-algebras).

GKdim as on the previous slide.

Gel'fand-Kirillov dimension: examples and properties

Properties (similarities and differences)

- $\text{GKdim } M = \sup\{\text{GKdim}(N) : N \in A - \text{mod}, N \subseteq M\},$
- $\text{GKdim } A = \sup\{\text{GKdim}(S) : S \subseteq A, S \text{ fin. gen. subalgebra}\}$
- $\text{GKdim } M = \sup\{\text{GKdim}({}_S N) : N \in S - \text{mod}, S \text{ f.g. sub.}\},$

Hence, if $|K| = \infty$, then $\text{GKdim}(K[[x_1, \dots, x_n]]) = \infty$ for $n \geq 1$.

Lemma (R is commutative, note the difference to Krull dimension)

- (i) Let R be a commutative affine K -algebra. Then (by Noether normalization) $\exists S = K[x_1, \dots, x_t] \subseteq R$ and R is finitely generated S -module. Then $\text{GKdim}_K R = \text{Krdim } S = t$.
- (ii) If R is an integral domain, $\text{GKdim}_K R = \text{tr. deg}_K \text{Quot}(R)$.

For any K -algebra R : $\text{GKdim } R[x_1, \dots, x_m] = \text{GKdim } R + m$.

Curiosity: $\text{GKdim}(R) \in \{0, 1\} \cup [2, +\infty) \subset \mathbb{R}$.

Proposition (Exactness, J. Gómez-Torrecillas et al.)

Let R be a G -algebra. Then GKdim is exact on short exact sequences of fin. gen. left R -modules. That is,

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \Rightarrow \text{GKdim } M = \sup\{\text{GKdim } L, \text{GKdim } N\}$$

In general, GKdim (unlike Krdim) does not need to be exact.

Elimination and GK-dimension

Lemma

Let $I \subset A$ be a left ideal and $S \subset A$ be a finitely generated subalgebra. Then

- $I \cap S = 0$ implies $\text{GKdim } A/I \geq \text{GKdim } S$,
- $\text{GKdim } A/I < \text{GKdim } S$ implies $I \cap S \neq 0$.

Definition (V. Bavula)

The **holonomic number** of an algebra A is the number

$$h(A) = \min\{\text{GKdim}(M) : 0 \neq M \in A - \text{mod}\}.$$

The module of dimension $h(A)$ is called **holonomic**.

If GKdim is exact on A -mod, then a nonzero sub- and factor-module of a holonomic module are holonomic.

GKdim and Ore localization

Lemma (Proposition: KL 4.2, McR 8.2.13)

Let S be a multiplicatively closed set of central regular elements of a ring R . Then $\text{GKdim}(S^{-1}R) = \text{GKdim}(R)$.

In order to be able to localize a non-commutative ring with respect to a multiplicatively closed set $S \subset R$ of regular elements, S need to satisfy the **Ore condition**.

If S is an Ore m.c. set in a domain R , in general the GKdim of the localization will increase.

Theorem (Makar-Limanov)

Let $\text{char}K = 0$ and D_n be the n -th Weyl algebra. Then for any $n \geq 1$ one can embed the free algebra $K\langle a, b \rangle$ into $\text{Quot}(D_n)$, hence $\text{GKdim} \text{Quot}(D_n) = \infty$.

GKdim and Ore localization II

However, Weyl algebras over $K \supseteq \mathbb{Q}$ are very special in this regard.

Proposition (L, reinforced by J. Zhang'97)

Let $D[S]$ be the n -th Weyl algebra, tensored with $K[s_1, \dots, s_m]$.

Then the following sets

- 1 $\Omega_1 = K[X, S] \setminus \{0\}$,
- 2 $\Omega_2 = K[X] \setminus \{0\}$,
- 3 $\Omega_3 = K[S] \setminus \{0\}$
- 4 $\Omega_4 = K[X] \setminus \mathfrak{p}, \mathfrak{p} \in K[X] \text{ prime}$

are Ore sets in $D[S]$ and one has for $i = 1 \dots 4$

$$\text{GKdim } D[S] = \text{GKdim } \Omega_i^{-1} D[S] = 2n + m.$$

Recently I have given a new proof of the existence of Bernstein-Sato ideals by using Ore localization and GK dimension and multiplicity.

Ufnarovski graph and properties of algebras

For associative K -algebras there is a remarkable algorithm.

Victor Ufnarovski, “A growth criterion for graphs and algebras defined by words”, Mat. Zametki, 31(1982), 465–472 (in Russian); English translation: Math. Notes, 37(1982), 238–241.

Given an (infinite) Gröbner basis of a **monomial** ideal $I \subset K\langle X \rangle$.

If it is possible to construct a finite **Ufnarovski graph**, then one can compute from its data

- the **Gel'fand-Kirillov (GK) dimension** of the algebra $K\langle X \rangle/I$,
- the upper bound for the **global (homological) dimension** of the algebra $K\langle X \rangle/I$,
- whether $K\langle X \rangle/I$ is left/right/weak Noetherian,
- whether $K\langle X \rangle/I$ is prime and/or semiprime,
- ...

In the implementation we are limited to finite Gröbner bases.

Properties of algebras beyond monomial

Any monomial ordering induces the filtration by monomial ordering: let $w \in \langle X \rangle$, then $\mathcal{F}_w := \bigoplus \{Ku : u \prec w\}$ with $\mathcal{F}_1 = K$. (Exercise: prove that this is indeed a filtration).

Therefore, let G be a Gröbner basis of an ideal $I \subset K\langle X \rangle$. Then the associated graded algebra of $K\langle X \rangle/I$ wrt the filtration \mathcal{F} is the graded monomial algebra $K\langle X \rangle/\text{LM}(G)$.

Theorem (Filtered-Graded Transfer or Jacobson-Okiniski Theorem's)

Let A be a K -algebra, \mathcal{F} a filtration on A and $\text{gr}_{\mathcal{F}}(A)$ the associated graded algebra of A wrt \mathcal{F} . If $\text{gr}_{\mathcal{F}}(A)$ is

- left/right/weak Noetherian,
- domain/prime/semiprime,
- of Gel'fand-Kirillov (GK) dimension d ,

then the same properties hold for A .

Moreover, the global (homological) dimension of the algebra $\text{gr}_{\mathcal{F}}(A)$ is the upper bound for the corresponding dimension of A .

Ufnarovski graph

Definition and Construction

Given a finite and minimal Gröbner basis G of $I \subset K\langle X \rangle$.

- Let d be the maximum degree (length) of elements in G .
- Consider the set M of all the monomials in degree $d - 1$, which do not belong to $\langle \text{LM}(G) \rangle$. These are the vertices of the graph.
- Between any two vertices u, w , we set the arrow $u \rightarrow w$ iff $\exists i, j$ such that $ux_i = x_jw \notin \text{LM}(G)$.

Practically, we describe this graph via its incidence matrix, see `lpUfGraph`.

Exercises

Describe Ufnarovski graphs for $\langle yx \rangle$, $\langle yx, x^2 \rangle$, $\langle y^2, x^2 \rangle$ and $\langle yx, xy \rangle$ in $K\langle x, y \rangle$.

Growth of algebras and GK dimension

Let $M \subset \langle X \rangle$ be a finite and minimal set of monomials.

Denote by $\Gamma(M)$ the Ufnarovski graph of $\langle M \rangle$.

Theorem (Ufnarovski)

- (i) $K\langle X \rangle / \langle M \rangle$ is finite dim. over K iff $\Gamma(M)$ does not contain any cycle
- (ii) $K\langle X \rangle / \langle M \rangle$ has exponential growth (i.e. $\text{GKdim} = \infty$) iff there are two different cycles with a common vertex in $\Gamma(M)$
- (iii) otherwise, $K\langle X \rangle / \langle M \rangle$ has polynomial growth (i.e. $\text{GKdim} = m$) iff m is, among all routes of $\Gamma(M)$, the largest number of distinct cycles occuring in a single route.

Theorem (Ufnarovski)

If there is no edge entering (resp. leaving) any cycle of $\Gamma(M)$, then $K\langle X \rangle / \langle M \rangle$ is left (resp. right) Noetherian.

fpaprops.lib: properties of algebras

The Gel'fand-Kirillov dimension can be computed with `lpGkDim`.

Let A be an FPA and M be an A -module.

The **projective dimension** of M is defined to be

$$\text{proj. dim } M = \sup\{i \in \mathbb{Z} : \text{Ext}_A^i(M, A) \neq 0\} \in \mathbb{Z} \cup \{\pm\infty\}.$$

The **global homological dimension of an algebra** A is defined to be

$$\text{gl. dim } A = \sup\{\text{proj. dim } M : M \in A\text{-mod}\}.$$

- The upper bound for the global homological dimension can be computed with `lpGldDimBound`.
- With `lpNoetherian` one can check whether a monomial algebra is left/right/weak Noetherian.
- `lpIsPrime`, `lpIsSemiPrime` check whether a monomial algebra is prime or semiprime.

Dimension functions

Definition

A dimension function δ on an algebra A assigns a value $\delta(M)$ to each finitely generated A -module M and satisfies the following properties:

- (i) $\delta(0) = -\infty$.
- (ii) If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence in $A\text{-mod}$, then $\delta(M) \geq \sup\{\delta(M'), \delta(M'')\}$ with equality if the sequence splits.
- (iii) If P is a (two-sided) prime ideal with $PM = 0$ and M is a torsion module over A/P , then $\delta(M) \leq \delta(A/P) - 1$.

Moreover, δ is called an **exact** dimension function, if in (ii) one always has $\delta(M) = \sup\{\delta(M'), \delta(M'')\}$.

The following dimensions are known to be dimension functions:

- Krull-Rentschler-Gabriel dimension is an exact dimension function.
- Gel'fand-Kirillov dimension is a dimension function, which is not always exact.

Projective dimension is **not** a dimension function.

Properties of dimension functions

From now on, A and R are associative K -algebras over a field K . By A -mod we denote the category of fin. gen. left A -modules.

Proposition

Let δ be an exact dimension function on A -mod. For the exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

the following three situations are possible:

- I. $\delta(L) = \delta(M) = \delta(N)$,
- II. $\delta(L) < \delta(M) = \delta(N)$,
- III. $\delta(L) = \delta(M) > \delta(N)$.

Inexact: only $\delta(L) < \delta(M) > \delta(N)$ is possible in addition.

Let δ be a dimension function on $R\text{-mod}$. Then

Proposition

- 1 $\delta(M \oplus N) = \sup\{\delta(M), \delta(N)\}$.
- 2 If $M = \sum_{i=1}^n N_i$, for $N_i \in R\text{-mod}$, then $\delta(M) = \sup_i\{\delta(N_i)\}$.
- 3 If $\cap N_i = 0$, then for $M = \sum_{i=1}^n N_i$ holds $\delta(M) = \sup_i\{\delta(M/N_i)\}$.
- 4 $\delta(M) \leq \delta(R)$.

Corollary

- 1 $\delta(M) = \sup\{\delta(N) : 0 \neq N \subseteq M, N \in R\text{-mod}\}$.
- 2 $\delta(M) = \sup\{\delta(C) : 0 \neq C \subseteq M, C \text{ cyclic}\}$.
- 3 If $I \subset J \subset R$ are (left, right) ideals, then $\delta(R/I) \geq \delta(R/J)$.
- 4 $\delta(R) < \infty$ implies one can replace sup by max above.
- 5 $\delta(R) = \sup\{\delta(M) : M \in R\text{-mod}\}$.