III. Gel'fand-Kirillov dimension, Growth of algebras and more

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EACA Summer School on Computer Algebra and its Applications

Filtrations on algebras and modules

Let A be an associative K-algebra, generated by $x_1, \ldots, x_m$.

Degree filtration

Let $V = Kx_1 \oplus \ldots \oplus Kx_m \subset A$ be a vector space.

Set $V_0 = K$, $V_1 = K \oplus V$ and $V_{k+1} = V_k \oplus V^{k+1}$. If

$$V_i \subseteq V_{i+k}, \quad V_i \cdot V_j \subseteq V_{i+j}, \quad A = \bigcup_{k=0}^{\infty} V_k,$$

then $\{V_k : k \in \mathbb{N}\}$ is the **standard (ascending) filtration** of A.

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Let M be a left A-module and let $M_0 \subset M$ be a finite K-vector space, spanned by the generators of M. That is $\dim_K M_0 < \infty$ and $AM_0 = M$.

$\{H_d := V_d M_0 \mid d \in \mathbb{N}\}$ is an induced ascending filtration on M.

The **Gel'fand-Kirillov dimension** of M is defined as follows

$$\operatorname{GKdim}(M) = \limsup_{d\to\infty} (\log_d(\dim_K H_d))$$

In the standard construction one puts $\deg x_i := 1$ and defines $V_d := \{f \in A \mid \deg f \le d\}$.

Conventions: $\operatorname{GKdim}(0) = -\infty$. $\operatorname{GKdim}_{\mathbb{Q}}(\mathbb{Q}) = 0$.

Gel'fand-Kirillov dimension and its properties

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Gel'fand-Kirillov dimension: examples

Lemma

Consider the free associative algebra $T = K\langle x_1, \ldots, x_n\rangle, n \geq 2$. *Then* $\mathrm{GKdim}(T) = \infty$.

Proof: $\dim V_{=d} = n^d, \dim V_d = \dim V_{\leq d} = \frac{n^{d+1}-1}{n-1}$.

Note, that $\frac{n^{d+1}-1}{n-1} > n^d$.

Since $\log_d n^d = d \log_d n = \frac{d}{\log_n d} \to \infty, d \to \infty$, it follows that $\mathrm{GKdim}(T) = \infty$.

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Let A be a K-algebra and a domain. If the standard filtration on A is compatible with the PBW Basis $\{x^\alpha \mid \alpha \in \mathbb{N}^m\}$, then $\mathrm{GKdim}_K(A) = m$.

Proof:

$$\dim V_{=d} = \binom{d+m-1}{m-1}, \ \dim V_d = \binom{d+m}{m} = \frac{d^m}{m!} + \ldots.$$

$$\mathrm{GKdim}(A) = \limsup_{d\to\infty} \log_d \binom{d+m}{m} = m.$$

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G-algebras

For an arbitrary field K
fix the set of variables $X = \{x_1, \dots, x_n\}$, let $c_{ij} \in K \setminus \{0\}$
and $d_{ij} = \sum_\alpha d_{ij}^\alpha x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$.

An associative K-algebra, gen. by X and obeying the relations

$$\forall 1 \leq i < j \leq n \quad x_j x_i = c_{ij} x_i x_j + d_{ij}$$

is called a G-**algebra**, if $\exists$ a monomial ordering $\prec$ on $K[X]$, such that either $d_{ij} = 0$ or $\mathrm{lm}(d_{ij}) \prec x_i x_j$ and certain relations between c_{ij}, d_{ij} are satisfied.

G-algebras are Noetherian integral domains (Weyl, (q-)shift algebras etc are G-algebras).
GKdim as on the previous slide.

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Gel'fand-Kirillov dimension: examples and properties

Properties (similarities and differences)

- $\operatorname{GKdim} M = \sup\{\operatorname{GKdim}(N) : N \in A - mod,\ N \subseteq M\}$,
- $\operatorname{GKdim} A = \sup\{\operatorname{GKdim}(S) : S \subseteq A,\ S$ fin. gen. subalgebra$\}$
- $\operatorname{GKdim} M = \sup\{\operatorname{GKdim}({}_S N) : N \in S - mod,\ S$ f.g. sub.$\}$,

Hence, if $|K| = \infty$, then $\operatorname{GKdim}(K[[x_1, \ldots, x_n]]) = \infty$ for $n \geq 1$.

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Lemma (R is commutative, note the difference to Krull dimension)

(i) *Let R be a commutative affine K-algebra. Then (by Noether normalization) $\exists S = K[x_1, \ldots, x_t] \subseteq R$ and R is finitely generated S-module. Then $\operatorname{GKdim}_K R = \operatorname{Krdim} S = t$.*

(ii) *If R is an integral domain, $\operatorname{GKdim}_K R = \operatorname{tr.deg}_K \operatorname{Quot}(R)$.*

For any K-algebra R: $\operatorname{GKdim} R[x_1, \ldots, x_m] = \operatorname{GKdim} R + m$.
Curiosity: $\operatorname{GKdim}(R) \in \{0, 1\} \cup [2, +\infty) \subset \mathbb{R}$.

Proposition (Exactness, J. Gómez-Torrecillas et al.)

Let R be a G-algebra. Then GKdim is exact on short exact sequences of fin. gen. left R-modules. That is,

$$0 \to L \to M \to N \to 0 \ \Rightarrow \ \operatorname{GKdim} M = \sup\{\operatorname{GKdim} L, \operatorname{GKdim} N\}$$

In general, GKdim (unlike Krdim) does not need to be exact.

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Elimination and GK-dimension

Lemma

Let $I \subset A$ be a left ideal and $S \subset A$ be a finitely generated subalgebra. Then

- $I \cap S = 0$ *implies* $\mathrm{GKdim}\, A/I \geq \mathrm{GKdim}\, S$,
- $\mathrm{GKdim}\, A/I < \mathrm{GKdim}\, S$ *implies* $I \cap S \neq 0$.

Definition (V. Bavula)

The **holonomic number** of an algebra A is the number

$$h(A) = \min\{\mathrm{GKdim}(M) : 0 \neq M \in A - mod\}.$$

The module of dimension $h(A)$ is called **holonomic**.

If GKdim is exact on A-mod, then a nonzero sub- and factor-module of a holonomic module are holonomic.

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GKdim and Ore localization

Lemma (Proposition: KL 4.2, McR 8.2.13)

Let S be a multiplicatively closed set of central regular elements of a ring R. Then $\operatorname{GKdim}(S^{-1}R) = \operatorname{GKdim}(R)$.

In order to be able to localize a non-commutative ring with respect to a multiplicatively closed set $S \subset R$ of regular elements, S need to satisfy the **Ore condition**.

If S is an Ore m.c. set in a domain R, in general the GKdim of the localization will increase.

Theorem (Makar-Limanov)

Let $\operatorname{char} K = 0$ and D_n be the n-th Weyl algebra. Then for any $n \geq 1$ one can embed the free algebra $K\langle a, b\rangle$ into $\operatorname{Quot}(D_n)$, hence $\operatorname{GKdim} \operatorname{Quot}(D_n) = \infty$.

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GKdim and Ore localization II

However, Weyl algebras over $K \supseteq \mathbb{Q}$ are very special in this regard.

Proposition (L, reinforced by J. Zhang'97)

Let $D[S]$ be the n-th Weyl algebra, tensored with $K[s_1, \ldots, s_m]$. Then the following sets

1. $\Omega_1 = K[X, S] \setminus \{0\}$,
2. $\Omega_2 = K[X] \setminus \{0\}$,
3. $\Omega_3 = K[S] \setminus \{0\}$
4. $\Omega_4 = K[X] \setminus \mathfrak{p}, \mathfrak{p} \in K[X]$ *prime*

are Ore sets in $D[S]$ and one has for $i = 1 \ldots 4$

$$\operatorname{GKdim} D[S] = \operatorname{GKdim} \Omega_i^{-1} D[S] = 2n + m.$$

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Ufnarovski graph and properties of algebras

For associative K-algebras there is a remarkable algorithm.

Victor Ufnarovski, "A growth criterion for graphs and algebras defined by words", Mat. Zametki, 31(1982), 465–472 (in Russian); English translation: Math. Notes, 37(1982), 238–241.

Given an (infinite) Gröbner basis of $I \subset K\langle X\rangle$.
If it is possible to construct a finite **Ufnarovski graph**, then one can compute from its data

- the **Gel'fand-Kirillov (GK) dimension** of the algebra $K\langle X\rangle/I$,
- the upper bound for the **global (homological) dimension** of the algebra $K\langle X\rangle/I$,
- whether $K\langle X\rangle/I$ is left/right Noetherian,
- whether $K\langle X\rangle/I$ is prime and/or semiprime,
- ...

In the implementation we are limited to finite Gröbner bases.

Ufnarovski graph

Definition and Construction

Given a finite and minimal Gröbner basis G of $I \subset K\langle X\rangle$.
◦ Let d be the maximum degree (length) of elements in G.
◦ Consider the set M of all the monomials in degree $d-1$, which do not belong to $\langle \mathrm{LM}(G)\rangle$. These are the vertices of the graph.
◦ Between any two vertices u, w, we set the arrow $u \to w$ iff $\exists i, j$ such that $ux_i = x_j w \notin \mathrm{LM}(G)$.

Practically, we describe this graph via its incidence matrix, see `lpUfGraph`.

Exercises

Describe Ufnarovski graphs for $\langle yx\rangle, \langle yx, x^2\rangle, \langle y^2, x^2\rangle$ and $\langle yx, xy\rangle$ in $K\langle x, y\rangle$.

Growth of algebras and GK dimension

Let $M \subset \langle X \rangle$ be a finite and minimal set of monomials.
Denote by $\Gamma(M)$ the Ufnarovski graph of $\langle M \rangle$.

Theorem (Ufnarovski)

(i) $K\langle X \rangle / \langle M \rangle$ *is finite dimensional over* K *iff* $\Gamma(M)$ *does not contain any cycle*

(ii) $K\langle X \rangle / \langle M \rangle$ *has exponential growth (i.e.* $\mathrm{GKdim} = \infty$*) iff there are two different cycles with a common vertex in* $\Gamma(M)$

(iii) otherwise, $K\langle X \rangle / \langle M \rangle$ *has polynomial growth (i.e.* $\mathrm{GKdim} = m$*) iff* m *is, among all routes of* $\Gamma(M)$*, the largest number of distinct cycles occuring in a single route.*

Theorem (Ufnarovski)

If there is no edge entering (resp. leaving) any cycle of $\Gamma(M)$*, then* $K\langle X \rangle / \langle M \rangle$ *is left (resp. right) Noetherian.*

The Gel'fand-Kirillov dimension can be computed with `lpGkDim`.

Let A be an FPA. The **projective dimension** of an A-module M is defined to be

$$\operatorname{proj.dim} M = \sup\{i \in \mathbb{Z} : Ext^i_A(M, A) \neq 0\} \in \mathbb{Z} \cup \{\pm\infty\}.$$

The **global homological dimension of an algebra** A is defined to be $\operatorname{gl.dim} A = \sup\{\operatorname{proj.dim} M : M \in A\text{-mod}\}$.

The upper bound for the global homological dimension can be computed with `lpGlDimBound`.

Dimension functions

Definition

A dimension function δ on an algebra A assigns a value $\delta(M)$ to each finitely generated A-module M and satisfies the following properties:

(i) $\delta(0) = -\infty$.

(ii) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence in A-mod, then $\delta(M) \geq \sup\{\delta(M'), \delta(M'')\}$ with equality if the sequence splits.

(iii) If P is a (two-sided) prime ideal with $PM = 0$ and M is a torsion module over A/P, then $\delta(M) \leq \delta(A/P) - 1$.

Moreover, δ is called an **exact** dimension function, if in (ii) one always has $\delta(M) = \sup\{\delta(M'), \delta(M'')\}$.

The following dimensions are known to be dimension functions:

- Krull-Rentschler-Gabriel dimension is an exact dimension function.
- Gel'fand-Kirillov dimension is a dimension function, which is not always exact.

Projective dimension is **not** a dimension function.

Properties of dimension functions

From now on, A and R are associative K-algebras over a field K. By A-Mod resp. A-mod we denote the category of left A-modules resp. of fin.gen. left A-modules.

Proposition

Let δ be an exact dimension function on A-mod. For the exact sequence

$$0 \to L \to M \to N \to 0$$

the following three situations are possible:

I. $\delta(L) = \delta(M) = \delta(N)$,
II. $\delta(L) < \delta(M) = \delta(N)$,
III. $\delta(L) = \delta(M) > \delta(N)$.

Inexact: only $\delta(L) < \delta(M) > \delta(N)$ is possible in addition.

Let δ be a dimension function on R-mod. Then

Proposition

1. $\delta(M \oplus N) = \sup\{\delta(M), \delta(N)\}$.
2. *If* $M = \sum_{i=1}^{n} N_i$*, for* $N_i \in R$*-mod, then* $\delta(M) = \sup_i\{\delta(N_i)\}$.
3. *If* $\cap N_i = 0$*, then for* $M = \sum_{i=1}^{n} N_i$ *holds* $\delta(M) = \sup_i\{\delta(M/N_i)\}$.
4. $\delta(M) \leq \delta(R)$.

Corollary

1. $\delta(M) = \sup\{\delta(N) : 0 \neq N \subseteq M, N \in R\text{-mod}\}$.
2. $\delta(M) = \sup\{\delta(C) : 0 \neq C \subseteq M, C \text{ cyclic}\}$.
3. *If* $I \subset J \subset R$ *are (left, right) ideals, then* $\delta(R/I) \geq \delta(R/J)$.
4. $\delta(R) < \infty$ *implies one can replace* sup *by* max *above.*
5. $\delta(R) = \sup\{\delta(M) : M \in R\text{-mod}\}$.