II. Gröbner bases with SINGULAR:LETTERPLACE

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Letterplace Gröbner Bases Q&A

What? Letterplace Gröbner basis is a special generating set for an ideal in infinitely generated **commutative** ring $K[X|P]$. It is tightly connected to the Gröbner basis of an ideal $I \subset K\langle X\rangle$.

Why? Among other, the Letterplace technique works over a commutative polynomial ring, hence know-how's of last 50 years in commutative computer algebra can be reused. For this reason, we have an implementation in SINGULAR.

How to compute LPGB? The contents of next lectures and exercises.

Example: sneak-peek.tst

```
LIB "eaca.lib";
ring r = 0,(x,y),Dp; // commutative Q[x,y] with Dp = deg lex
int dgb = 7; // degree bound for our computations
def R = makeLetterplaceRing(dgb); // creates a letterplace ring out of r
setring R; // switch into R
option(redSB); option(redTail); // set these options
ideal I = x(1)*x(2) + y(1)*y(2) - 1; // nc sphere in 2D
ideal J = letplaceGBasis(I); // computes a GB of I
J; // print the generators of J
> J[1]=x(1)*x(2) + y(1)*y(2) - 1
> J[2]=x(1)*y(2)*y(3) - y(1)*y(2)*x(3)
```

Thus $J = \{x^2 + y^2 - 1, xy^2 - y^2x\}$ is a Gröbner basis of $I = \langle x^2 + y^2 - 1 \rangle$ up to degree 5. In fact, it is already the whole Gröbner basis of I (Hint: repeat the computation, increasing degree bound to 7).

Letterplace Ring

- X the set of "letters", $P = \mathbb{N} = \{1, 2, \ldots\}$ the set of "places"
- $(x_i|j) = (x_i, j) = x_i(j)$ element of the product set $X \times P$
- $K[X|P]$ the polynomial ring in the (commutative) variables $(x_i|j)$
- X the set of words, $[X|P]$ the set of **letterplace** monomials

* this letterplace is different from the one used in representation theory.

The monoid $\mathbb{N}$ has a natural faithful action on the (places of monomials of) graded ring $K[X|\mathbb{N}]$ given by $s \cdot x_i(j) = x_i(j+s) \;\; \forall s \in \mathbb{N}$.

The **place support** of a monomial $(x_{i_1} \mid j_1) \cdots (x_{i_k} \mid j_k) \in [X|\mathbb{N}]$ is the set $\{j_1, \ldots, j_k\} \subset \mathbb{N}$. A monomial m is **multilinear**, if place support of m is irredundant.

- The **shift** of a monomial $(x_{i_1} \mid j_1) \cdots (x_{i_k} \mid j_k) \in [X|\mathbb{N}]$ is the minimal place in the place support minus one.

  Example: $(x_2 \mid 1)(x_3 \mid 5)$ has shift 0, while $(x_1 \mid 3)(x_4 \mid 5)$ has shift 2.

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Let $V \subset K[X|\mathbb{N}]$ be the K-span of all monomials with place support of the form $(1, 2, \ldots, k, \ldots)$.

Induced multiplication

$$\iota : K\langle X\rangle \to V : \quad x_{i_1} \ldots x_{i_k} \mapsto x_{i_1}(1) \ldots x_{i_k}(k)$$

is an isomorphism of K-vector spaces (mentioned by Feynman, Rota).

Multiplication:

$$\text{for } u, w \in \langle X\rangle, \quad \iota(uw) = \iota(u)(\deg(u) \cdot \iota(w)) \neq \iota(u)\iota(w).$$

Hence, ι is **not** the isomorphism of K-algebras.

Example: $zy \cdot x = zyx$; $\iota(zy \cdot x) = z(1)y(2)(2 \cdot x(1)) = z(1)y(2)x(3)$, while $\iota(zy)\iota(x) = z(1)y(2)x(1)$, which is not multilinear.

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Working with LETTERPLACE: file `arit.tst`

For the convenience of the participants of the summer school, all the LETTERPLACE libraries have been put together in `eaca.lib`. Let me explain you the details at the `sneak-peek.tst` example.

At first one defines a commutative polynomial ring *r* in the variables (say x, y), equipped with a monomial ordering:

LIB "eaca.lib";
ring r = 0,(x,y),Dp; // commutative Q[x,y] with Dp = deg lex

Then one defines the degree bound and the procedure
○ `makeLetterplaceRing` constructs the Letterplace ring up to the degree bound, with variables and the ordering, induced from *r*

int dgb = 5; // degree bound for our computations def R = makeLetterplaceRing(dgb); // creates a letterplace ring out of r
setring R; // switch into R

Working with LETTERPLACE II

Encoding of associative word is done via ι; i.e. $yx \mapsto y(1) * x(2)$ and the ideal, generated by $x^2 + y^2 - 1$ is declared as

```
ideal I = x(1)*x(2) + y(1)*y(2) - 1; // nc sphere in 2D
```

Note: SINGULAR orders the polynomials according to the ordering on R.

```
poly p = x(1)*x(2) + y(1)*y(2) + y(1)*x(2) + x(1)*y(2) + y(1);
p; // prints p
> x(1)*x(2)+x(1)*y(2)+y(1)*x(2)+y(1)*y(2)+y(1)
poly q = 3*y(1)*y(2) + x(1)*y(2) - 5*y(1); q;
> x(1)*y(2)+3*y(1)*y(2)-5*y(1)
x(1)*y(2) > y(1)*x(2); // compare monomials acc. to mon. ord.
> 1 // true
y(1)*y(2) > y(1)*x(2);
> 0 // false
```

Working with LETTERPLACE III

◦ Conversion of an ideal of letterplace polynomials into strings
`lpPrint(J,r)`

```
lpPrint(p,r);
> [1]:  x*x+x*y+y*x+y*y+y
lpPrint(q,r);
> [1]:  x*y+3*y*y-5*y
```

◦ Arithmetics: +, -, scalar operations: as usual with SINGULAR;
◦ Shift a letterplace polynomial : `shiftPoly(f,s)`

```
poly t = 3*p - q - 8*y(1); t;
> 3*x(1)*x(2)+2*x(1)*y(2)+3*y(1)*x(2)
lpPrint(t,r);
> [1]:  3*x*x+2*x*y+3*y*x > 3*x(1)*x(2)+2*x(1)*y(2)+3*y(1)*x(2)
shiftPoly(t,2);
> 3*x(3)*x(4)+2*x(3)*y(4)+3*y(3)*x(4)
```

Working with LETTERPLACE IV

- Multiplication of polynomials f, g: `lpMult(f,g)`
- Lie bracket of polynomials f, g: `lieBracket(f,g)`
- Powers of polynomial f^m: `lpPower(f,m)`

```
lpMult(q,t);
> 3*x(1)*y(2)*x(3)*x(4)+2*x(1)*y(2)*x(3)*y(4) + ...
lpMult(t,q);
> 3*x(1)*x(2)*x(3)*y(4)+9*x(1)*x(2)*y(3)*y(4) + ...
lieBracket(t,q);
> -3*x(1)*x(2)*x(3)*y(4)-9*x(1)*x(2)*y(3)*y(4)+ ...
lpPower(t,2);
> 9*x(1)*x(2)*x(3)*x(4)+6*x(1)*x(2)*x(3)*y(4)+...
lpPower(t,3);
should produce a poly of deg 6, but degbound = 5
> ? degree bound violated by the product! ...
```

The last call: increase the degree bound (go back to r, repeat the setup).

Working with LETTERPLACE V

◦ `letplaceGBasis(I)` computes a GB of I up to a fixed degree bound

```
option(redSB); option(redTail); // set these options before GB call
ideal I = x(1)*x(2) + y(1)*y(2) - 1; // nc sphere in 2D
ideal J = letplaceGBasis(I); // computes a GB of I
J; // print the generators of J
> J[1]=x(1)*x(2)+y(1)*y(2)-1
> J[2]=x(1)*y(2)*y(3)-y(1)*y(2)*x(3)
```

◦ `lpNF(f,G)` computes a normal form of f with respect to a GB G

```
poly f = lieBracket(x(1),y(1));        lpNF(f,J);
> x(1)*y(2)-y(1)*x(2)
f - lpNF(f,J);
> 0
poly g = lieBracket(x(1),y(1)*y(2));    lpNF(g,J);
> 0
```

Working with LETTERPLACE VI

◦ `lpNF(f,G)` computes a normal form of f with respect to a GB G
◦ `lpDivision(f,G)` returns a list, where the first element is $\mathrm{NF}(f,G)$ while
the second is the list of expressions $[i, \ell_{ij}, r_{ij}]$ such that
$\sum_{i,j} \ell_{ij} g_i r_{ij} = f - \mathrm{NF}(f,G)$ is a GB presentation of the latter
◦ `lpGBPres2Poly(L,G)` assembles the polynomial from its `lpDivision` Gröbner presentation

```
poly h = lieBracket(x(1)*x(2),y(1));
lpNF(h,J);
> 0
list Lh = lpDivision(h,J);
lpGBPres2Poly(Lh, J); // this is just h itself
> x(1)*x(2)*y(3)-y(1)*x(2)*x(3)
Lh[2];
> [1]: [1, -y(1), 1]    [2]: [1, 1, y(1)]
```

Computing with LETTERPLACE

Given a finite Gröbner basis of $I \subset K\langle X\rangle$, we offer algorithms in SINGULAR:LETTERPLACE to compute

- the **Gel'fand-Kirillov (GK) dimension** of the algebra $K\langle X\rangle/I$,
- the upper bound for the **global (homological) dimension** of the algebra $K\langle X\rangle/I$,
- whether $K\langle X\rangle/I$ is left/right Noetherian,
- whether $K\langle X\rangle/I$ is prime and/or semiprime, ...

Moreover, it is possible to compute

- the K-**dimension** of the algebra $K\langle X\rangle/I$,
- the canonical monomial K-**basis** of the algebra $K\langle X\rangle/I$,
- a finite part of Hilbert series of the algebra $K\langle X\rangle/I$, ...

Exercise: locate the documentation to the procedures above either in the SINGULAR manual or in the `eaca.lib`.

An Algorithm For Graded Ideals

The construction of one-to-one correspondence between **graded ideals** of $K\langle X\rangle$ and Letterplace ideals of $K[X|\mathbb{N}]$ and especially their Gröbner bases was introduced in

R. La Scala and V. Levandovskyy " Letterplace ideals and non-commutative Gröbner bases", Journal of Symbolic Computation, 44(10), 1374–1393, 2009. It was further developed and generalized in

R. La Scala, V. Levandovskyy "Skew polynomial rings, Gröbner bases and the letterplace embedding of the free associative algebra", Journal of Symbolic Computation, 48(0), 110 - 131, 2013.

- Let $I \subset K\langle X\rangle$ be a graded two-sided ideal. Denote $\tilde{\iota}(I)$ the shift-invariant ideal $J \subset K[X|P]$ generated by $\bigcup_{s\in\mathbb{N}} s\cdot\iota(I)$.
- Let $J \subset K[X|P]$ be a shift-invariant place-multigraded ideal. Denote $\tilde{\iota}^{-1}(J)$ the graded two-sided ideal $I = \iota^{-1}(J\cap V) \subset K\langle X\rangle$.

Theorem

The following inclusions hold:

- $\tilde{\iota}^{-1}(\tilde{\iota}(I)) = I,\ \tilde{\iota}(\tilde{\iota}^{-1}(J)) \subseteq J,$
- $\tilde{\iota}(\tilde{\iota}^{-1}(J)) = J$ *if and only if* J *is generated by* $\bigcup_{s\in\mathbb{N}} s\cdot(J\cap V)$.

An ideal, satisfying the latter property, is called a **letterplace ideal**.

Corollary

The map $\iota : K\langle X\rangle \to V$ *induces a 1-to-1 correspondence* $\tilde{\iota}$ *between graded two-sided ideals* I *of the free associative algebra* $K\langle X\rangle$ *and letterplace ideals* J *of the polynomial ring* $K[X|P]$.

Gröbner bases behave well under the correspondence $\tilde{\iota}$.

- Let $I \subset K\langle X\rangle$ be a graded two-sided ideal. Denote $\tilde{\iota}(I)$ the shift-invariant ideal $J \subset K[X|P]$ generated by $\bigcup_{s\in\mathbb{N}} s\cdot\iota(I)$.
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Gröbner bases behave well under the correspondence $\tilde{\iota}$.

When a monoid S acts (by algebra endomorphisms) on a polynomial ring, one has the notion of **Gröbner S-basis**, introduced and applied in V. Drensky and R. La Scala "Gröbner bases of ideals invariant under endomorphisms", J. Symb. Comput., 41(7), 2006.

We utilize this notion for the specific action of $\mathbb{N}$ over $K[X|P]$.

Definition

A monomial ordering on $K[X|P]$ is called **shift-invariant**, when $u \prec v$ if and only if $s \cdot u \prec s \cdot v$ for any $u, v \in [X|P]$ and $s \in \mathbb{N}$. In this case, one has that $\mathrm{lm}(s \cdot f) = s \cdot \mathrm{lm}(f)$ for all $f \in K[X|P] \setminus \{0\}$ and $s \in \mathbb{N}$.

Lemma

Let $f_1, f_2 \in K[X|P] \setminus \{0\}$. Then $S(s \cdot f_1, s \cdot f_2) = s \cdot S(f_1, f_2)$.

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Lemma

Let $f_1, f_2 \in K[X|P] \setminus \{0\}$. *Then* $S(s \cdot f_1, s \cdot f_2) = s \cdot S(f_1, f_2)$.

Definition

Fix the monomial orderings $<$ on $K\langle X\rangle$ and $\prec$ on $K[X|P]$. They are called **compatible with** ι, when $v < w$ holds if and only if $\iota(v) \prec \iota(w)$ for any $v, w \in X$. In this case, it follows that $\mathrm{lm}(\iota(f)) = \iota(\mathrm{lm}(f))$ for all $f \in K\langle X\rangle \setminus \{0\}$.

It follows that we can "reduce by symmetry" the Buchberger's Theorem with respect to the shift action.

NCGBasis

Input: G_0, a homogeneous basis of a graded two-sided ideal $I \subset K\langle X \rangle$.
Output: G, a homogeneous Gröbner basis of I.

$H := \iota(G_0 \setminus \{0\})$;
$P := \{(f, s \cdot g) \mid f, g \in H, s \in \mathbb{N}, f \neq s \cdot g, \gcd(\mathrm{lm}(f), \mathrm{lm}(s \cdot g)) \neq 1,$
$\quad \mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(s \cdot g)) \in V\}$;
while $P \neq \emptyset$ do
  choose $(f, s \cdot g) \in P$;
  $P := P \setminus \{(f, s \cdot g)\}$;
  $h := \mathrm{REDUCE}(S(f, s \cdot g), \bigcup_t t \cdot H)$;
  if $h \neq 0$ then
    $H := H \cup \{h\}$;
    $P := P \cup \{(h, s \cdot g) \mid g \in H, s \in \mathbb{N}, \gcd(\mathrm{lm}(h), \mathrm{lm}(s \cdot g)) \neq 1,$
    $\quad \mathrm{lcm}(\mathrm{lm}(h), \mathrm{lm}(s \cdot g)) \in V\}$;
    $P := P \cup \{(s \cdot h, g) \mid g \in H, s \in \mathbb{N}, \gcd(\mathrm{lm}(s \cdot h), \mathrm{lm}(g)) \neq 1,$
    $\quad \mathrm{lcm}(\mathrm{lm}(s \cdot h), \mathrm{lm}(g)) \in V\}$;
$G := \iota^{-1}(H)$; return G;

Gröbner basis in $V \subset K[X|P]$: Example

Example ($f_1 = x^3 - y^3 = xxx - yyy, f_2 = xyx - yxy$)

In the case $d = 5$, one considers then the polynomial ring $K[X|P_5] = K[x(1), y(1), \ldots, x(5), y(5)]$ and the polynomials

$$
\begin{aligned}
f_1 &= x(1)x(2)x(3) - y(1)y(2)y(3),\\
f_2 &= 1 \cdot f_1 = x(2)x(3)x(4) - y(2)y(3)y(4),\\
f_3 &= 2 \cdot f_1 = x(3)x(4)x(5) - y(3)y(4)y(5),\\
f_4 &= x(1)y(2)x(3) - y(1)x(2)y(3),\\
f_5 &= 1 \cdot f_4 = x(2)y(3)x(4) - y(2)x(3)y(4),\\
f_6 &= 2 \cdot f_4 = x(3)y(4)x(5) - y(3)x(4)y(5).
\end{aligned}
$$

We write (i, j) for the pair (i, j). There are 15 pairs to consider.

Gröbner basis in $V \subset K[X|P]$: Example continued

Criteria

- the pairs $(1,4),(1,5),(2,4),(4,5)$ are discarded by the V-criterion, since lcm's of their lm's are non-multilinear in places.
- the pairs $(2,3),(2,5),(2,6),(3,5),(3,6),(5,6)$ are discarded by the V-criterion, since lcm's of their lm's have nonzero shift.
- $(4,5)$ and $(5,6)$ can also be discarded by the product criterion.

Thus it remains to consider $P = \{(1,2),(1,3),(1,6),(3,4),(4,6)\}$.
Following the Algorithm `NCGBasis`, $H := \{f_1, f_4\}$.

Gröbner basis in $K[X|P]$: Example continued

$(\mathbf{1},\mathbf{2})$: $\mathrm{spoly}(1,2) = f_1 x(4) - x(1) f_2 =$
$x(1)y(2)y(3)y(4) - y(1)y(2)y(3)x(4) =: g_1$, hence $H := H \cup \{g_1\}$ and $g_2 := 1 \cdot \mathrm{spoly}(1,2) = x(2)y(3)y(4)y(5) - y(2)y(3)y(4)x(5)$ is the only admissible shift of g_1.

$(\mathbf{1},\mathbf{3})$: $\mathrm{spoly}(1,3) = f_1 x(4)x(5) - x(1)x(2) f_3 =$
$x(1)x(2)y(3)y(4)y(5) - y(1)y(2)y(3)x(4)x(5) = x(1)g_2 + g_1 x(5) \to 0$.
Note, that since $\mathrm{lm}(f_2) \mid \mathrm{lcm}(\mathrm{lm}(f_1), \mathrm{lm}(f_3))$, by the Chain criterion we can skip the pair $(1,3)$ from the pairset $(1,2),(1,3),(2,3)$. The pair $(2,3)$ is skipped by the V-criterion above.

$(\mathbf{1},\mathbf{6})$: $\mathrm{spoly}(1,6) = f_1 y(4)x(5) - x(1)x(2) f_6 =$
$x(1)x(2)y(3)x(4)x(5) - x(1)x(2)x(3)y(4)x(5)$. Indeed,
$\mathrm{spoly}(1,6) = x(1) f_5 y(5) + f_4 y(4)y(5) + y(1) g_2 \to 0$.

Gröbner basis in $K[X|P]$: Example continued

$(\mathbf{3},\mathbf{4})$: $\mathrm{spoly}(3,4) = f_4 x(4)x(5) - x(1)y(2)f_3 =$
$x(1)y(2)y(3)y(4)y(5) - y(1)x(2)y(3)x(4)x(5)$ and
$\mathrm{spoly}(3,4) = g_1 y(5) - y(1)f_5 x(5) - y(1)y(2)f_6 \to 0$.

$(\mathbf{4},\mathbf{6})$: $\mathrm{spoly}(4,6) = f_4 y(4)x(5) - x(1)y(2)f_6 =$
$x(1)y(2)y(3)x(4)y(5) - y(1)x(2)y(3)y(4)x(5)$ cannot be reduced, hence $g_3 := \mathrm{spoly}(4,6)$ and H becomes $\{f_1, f_4, g_1, g_3\}$. Note, that there are no admissible shifts for g_3.

The pairs $(g_1, f_1), \ldots, (g_1, f_6), (f_1, g_2), (f_4, g_2)$, which appear when g_1 enters H and $(g_3, f_1), \ldots, (g_3, f_6), (g_3, g_1), (g_3, g_2)$, which appear when g_3 enters H (note, that $(f_1, g_3), (f_4, g_3), (g_1, g_3)$ are already included in the latter) are discarded by the V-criterion.
Thus $\iota^{-1}(\{f_1, f_4, g_1, g_3\})$ is a truncated Gröbner basis up to deg 5.

General, non-graded Letterplace approach

It has been developed independently in

R. La Scala, "Extended letterplace correspondence for nongraded noncommutative ideals and related algorithms", Int. Journal of Alg. and Comp. 24(08), 2012.
and
V. Levandovskyy, G. Studzinski, B. Schnitzler "Enhanced computations of Gröbner bases in free algebras as a new application of the letterplace paradigm", Proc. ISSAC'13 and the Ph.D. thesis of G. Studzinski.

The latter approach has two different implementations in SINGULAR.

Graded ideals

Example

Free algebra: Consider $I = \langle\{xx - yh\}\rangle \subset K\langle x, y, h\rangle$ with the degree left lexicographical ordering $x > y > h$. The non-commutative Gröbner basis theory tells us we have to consider the only s-polynomial

$$(xx - yh)x - x(xx - yh) = xyh - yhx.$$

Letterplace algebra: With the Letterplace approach we consider an infinite set $\{x(1)x(2) - y(1)h(2), x(2)x(3) - y(2)h(3), \ldots\}$. We have to consider the infinite orbit of shifted s-polynomials

$$x(1)y(2)h(3) - y(1)h(2)x(3), \ldots,$$

$$\ldots, x(i+1)y(i+2)h(i+3) - y(i+1)h(i+2)x(i+3), \ldots$$

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Beyond graded ideals

Free algebra: Consider $\{xx - y\} \subset K\langle x, y\rangle$ with the degree lexicographical ordering $x > y$. The only s-polynomial we get is

$$(xx - y)x - x(xx - y) = xy - yx.$$

Letterplace algebra: In $\{\dots, x(i+1)x(i+2) - y(i+1), \dots\}$ the s-polynomials are

$$x(1)y(2) - \mathbf{y(1)x(3)}, \dots, x(i+1)y(i+2) - \mathbf{y(i+1)x(i+3)}, \dots$$

Monomials in bold do not correspond to monomials in the free algebra!

We say that monomials $y(1)x(3), y(i+1)x(i+3)$, which are still multilinear wrt places, have **holes** in their place supports.

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Beyond graded ideals II

How we deal with the holes? We introduce the **shrinking**, which produces a standard letterplace monomial out of the multilinear one and the **place degree** additionally to the usual degree of letters. It leads to the general theory and, in particular,

Theorem (Levandovskyy, Studzinski, Schnitzler)

There is a K-bilinear associative operation $\star : V \times V \to V$: $(v_1, v_2) \mapsto \mathrm{shrink}(v_1(\mathrm{placedeg}(v_1) \cdot v_2))$, such that there is an isomorphism of K-algebras

$$\vartheta : (K\langle X\rangle, \cdot) \longrightarrow (V, \star).$$

Implementation

We have developed an implementation of the letterplace algorithm in the computer algebra system SINGULAR.
Our implementation, called LETTERPLACE consists of

- kernel part of SINGULAR (two flavors indeed),
- the libraries `freegb.lib`, `fpadim.lib`, `fpaprops.lib` (united into `eaca.lib`), `fpalgebras.lib` in the SINGULAR language.

Other implementations: GAP, MAGMA (classical and Alan Steel's generalization of F4), SAGE (Simon King, Jena, Germany), APCOCOA (Xingqiang Xiu, Passau, Germany).

Thank you for your attention!

RWTH AACHEN UNIVERSITY

SINGULAR letterplace

Please visit the SINGULAR homepage

http://www.singular.uni-kl.de/