Elements of Computer-Algebraic Analysis

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Recommended literature (textbooks)

1. J. C. McConnell, J. C. Robson, "Noncommutative Noetherian Rings", Graduate Studies in Mathematics, 30, AMS (2001)
2. G. R. Krause and T. H. Lenagan, "Growth of Algebras and Gelfand-Kirillov Dimension", Graduate Studies in Mathematics, 22, AMS (2000)
3. S. Saito, B. Sturmfels and N. Takayama, "Gröbner Deformations of Hypergeometric Differential Equations", Springer, 2000
4. J. Bueso, J. Gómez–Torrecillas and A. Verschoren, "Algorithmic methods in non-commutative algebra. Applications to quantum groups", Kluwer , 2003
5. H. Kredel, "Solvable polynomial rings", Shaker Verlag, 1993
6. H. Li, "Noncommutative Gröbner bases and filtered-graded transfer", Springer, 2002

Recommended literature (textbooks and PhD theses)

1. V. Ufnarovski, "Combinatorial and Asymptotic Methods of Algebra", Springer, Encyclopedia of Mathematical Sciences 57 (1995)
2. F. Chyzak, "Fonctions holonomes en calcul formel", PhD. Thesis, INRIA, 1998
3. V. Levandovskyy, "Non-commutative Computer Algebra for polynomial algebras: Gröbner bases, applications and implementation" PhD. Thesis, TU Kaiserslautern, 2005
4. C. Koutschan, "Advanced Applications of the Holonomic Systems Approach", PhD. Thesis, RISC Linz, 2009
5. K. Schindelar, "Algorithmic aspects of algebraic system theory", PhD. Thesis, RWTH Aachen, 2010

Software

D-modules and algebraic analysis:

- KAN/SM1 by N. Takayama et al.
- `D-modules` package in MACAULAY2 by A. Leykin and H. Tsai
- RISA/ASIR by M. Noro et al.
- OREMODULES package suite for MAPLE by D. Robertz, A. Quadrat et al.
- SINGULAR:PLURAL with a *D*-module suite; by V. L. et al.

holonomic and *D*-finite functions:

- GROEBNER, ORE ALGEBRA, MGFUN, ... by F. Chyzak
- HOLONOMICFUNCTIONS by C. Koutschan
- SINGULAR:LOCAPAL (partly under development) by V. L. et al.

Overview

- Operator algebras and their partial classification
- More general: G-algebras and Gröbner bases in G-algebras
- Module theory; Dimension theory; Gel'fand-Kirillov dimension
- Linear modeling with variable coefficients
- Elimination of variables and Gel'fand-Kirillov dimension
- Ore localization; smallest Ore localizations
- Solutions via homological algebra
- The complete annihilator program
- Some computational D-module theory, Weyl closure
- Purity; pure modules, pure functions, preservation of purity
- Purity filtration of a module; connection to solutions
- Jacobson normal form

What is computer algebraic Analysis?

Algebraic Analysis

1. As a notion, it arose in 1958 in the group of Mikio Sato (Japan)
2. Main objects: systems of linear partial DEs with variable coefficients, generalized functions
3. Main idea: study systems and generalized functions in a coordinate-free way (i. e. by using modules, sheaves, categories, localizations, homological algebra, ...)
4. Keywords include D-Modules, (sub-)holonomic D-Modules, regular resp. irregular holonomic D-Modules
5. Interplay: singularity theory, special functions, ...

Other ingredients: symbolic algorithmic methods for discrete resp. continuous problems like symbolic summation, symbolic integration etc.

What is computer-algebraic Analysis?

Algebraization as a trend

Algebra: Ideas, Concepts, Methods, Abstractions

Computer algebra works with algebraic concepts in a (semi-)algorithmic way at three levels:

1. Theory: Methods of Algebra in a constructive way
2. Algorithmics: Algorithms (or procedures) and their Correctness, Termination and Complexity results (if possible)
3. Realization: Implementation, Testing, Benchmarking, Challenges; Distribution, Lifecycle, Support and software-technical aspects

Some important names in computer-algebraic analysis

- W. Gröbner and B. Buchberger: Gröbner bases and constructive ideal/module theory
- O. Ore: Ore Extension and Ore Localization
- I. M. Gel'fand and A. Kirillov: GK-Dimension
- B. Malgrange: M. isomorphism, M. ideal, ...
- J. Bernstein, M. Sato, M. Kashiwara, C. Sabbah, Z. Mebkhout, B. Malgrange et al.: D-module theory
- N. Takayama, T. Oaku, B. Sturmfels, M. Saito, M. Granger, U. Walther, F. Castro, H. Tsai, A. Leykin et al.: (not only) computational D-module theory
- ...

Part I. Operator algebras and their partial classification.

Operator algebras: partial Classification

Let K be an effective field, that is $(+, -, \cdot, :)$ can be performed algorithmically.
Moreover, let $\mathcal{F}$ be a K-vector space ("function space").

Let x be a local coordinate in $\mathcal{F}$. It induces a K-linear map $X : \mathcal{F} \to \mathcal{F}$, i. e. $X(f) = x \cdot f$ for $f \in \mathcal{F}$. Moreover, let

$\mathfrak{o}_x : \mathcal{F} \to \mathcal{F}$ be a K-linear map.

Then, in general, $\mathfrak{o}_x \circ X \neq X \circ \mathfrak{o}_x$, that is $\mathfrak{o}_x(x \cdot f) \neq x \cdot \mathfrak{o}_x(f)$ for $f \in \mathcal{F}$.

The **non-commutative relation** between $\mathfrak{o}_x$ and X can be often read off by analyzing the properties of $\mathfrak{o}_x$ like, for instance, the product rule.

Classical examples: Weyl algebra

Let $f : \mathbb{C} \to \mathbb{C}$ be a differentiable function and $\partial(f(x)) := \frac{\partial f}{\partial x}$.

Product rule tells us that $\partial(x\ f(x)) = x\ \partial(f(x)) + f(x)$, what is translated into the following relation between operators

$$(\partial \circ x - x \circ \partial - 1)\ (f(x)) = 0.$$

The corresponding operator algebra is the 1st **Weyl algebra**

$$D_1 \ = \ K\langle x, \partial \mid \partial x = x\partial + 1\rangle.$$

Classical examples: shift algebra

Let g be a sequence in discrete argument k and **s** is the shift operator $\mathbf{s}(g(k)) = g(k+1)$. Note, that **s** is multiplicative.

Thus $\mathbf{s}(kg(k)) = (k+1)g(k+1) = (k+1)\mathbf{s}(g(k))$ holds.

The operator algebra, corr. to **s** is the 1st **shift algebra**

$$S_1 \ = \ K\langle k, s \mid sk = (k+1)s = ks + s \rangle.$$

Intermezzo

For a function in differentiable argument x and in discrete argument k the natural operator algebra is

$$A = D_1 \otimes_K S_1 = K\langle x, k, \partial_x, s_k \mid \partial_x x = x\partial_x + 1, \ s_k k = k s_k + s_k,$$

$$xk = kx, \ x s_k = s_k x, \ \partial_x k = k\partial_x, \partial_x s_k = s_k \partial_x \rangle.$$

Examples form the q-World

Let $k \subset K$ be fields and $q \in K^*$.
In q-calculus and in quantum algebra three situations are common for a fixed k: (a) $q \in k$, (b) q is a root of unity over k, and (c) q is transcendental over k and $k(q) \subseteq K$.

Let $\partial_q(f(x)) = \frac{f(qx)-f(x)}{(q-1)x}$ be a q-differential operator.
The corr. operator algebra is the 1st q-**Weyl algebra**

$$D_1^{(q)} \ = \ K\langle x, \partial_q \mid \partial_q x = q \cdot x\partial_q + 1\rangle.$$

The 1st q-**shift algebra** corresponds to the q-shift operator $\mathbf{s}_q(f(x)) = f(qx)$:

$$K_q[x, s_q] \ = \ K\langle x, s_q \mid s_q x = q \cdot x s_q\rangle.$$

Two frameworks for bivariate operator algebras

Algebra with linear (affine) relation

Let $q \in K^*$ and $\alpha, \beta, \gamma \in K$. Define

$$\mathcal{A}^{(1)}(q, \alpha, \beta, \gamma) := K\langle x, y \mid yx - q \cdot xy = \alpha x + \beta y + \gamma \rangle$$

Because of **integration operator** $\mathcal{I}(f(x)) := \int_a^x f(t)dt$ for $a \in \mathbb{R}$, obeying the relation $\mathcal{I}\ x - x\ \mathcal{I} = -\mathcal{I}^2$ we need yet more general framework.

Algebra with nonlinear relation

Let $N \in \mathbb{N}$ and $c_0, \ldots, c_N, \alpha \in K$. Then $\mathcal{A}^{(2)}(q, c_0, \ldots, c_N, \alpha)$ is
$K\langle x, y \mid yx - q \cdot xy = \sum_{i=1}^n c_i y^i + \alpha x + c_0 \rangle$ or
$K\langle x, y \mid yx - q \cdot xy = \sum_{i=1}^n c_i x^i + \alpha y + c_0 \rangle$.

Theorem (L.–Koutschan–Motsak, 2011)

*$\mathcal{A}^{(1)}(q,\alpha,\beta,\gamma) = K\langle x, y \mid yx - q\cdot xy = \alpha x + \beta y + \gamma\rangle$,
where $q \in K^*$ and $\alpha, \beta, \gamma \in K$
is isomorphic to the 5 following* **model algebras***:*

1. $K[x, y]$,
2. *the 1st Weyl algebra* $D_1 = K\langle x, \partial \mid \partial x = x\partial + 1\rangle$,
3. *the 1st shift algebra* $S_1 = K\langle x, s \mid sx = xs + s\rangle$,
4. *the 1st q-commutative algebra* $K_q[x, s] = K\langle x, s \mid sx = q\cdot xs\rangle$,
5. *the 1st q-Weyl algebra* $D_1^{(q)} = K\langle x, \partial \mid \partial x = q\cdot x\partial + 1\rangle$.

Theorem (L.–Makedonsky–Petravchuk, new)

For $N \geq 2$ and $c_0, \ldots, c_N, \alpha \in K$, $\mathcal{A}^{(2)}(q, c_0, \ldots, c_N, \alpha)$ $= K\langle x, y \mid yx - q \cdot xy = \sum_{i=1}^{N} c_i y^i + \alpha x + c_0 \rangle$ is isomorphic to ...

1. $K_q[x, s]$ *or* $D_1^{(q)}$, **if** $q \neq 1$,
2. $S_1 = K\langle x, s \mid sx = xs + s \rangle$, **if** $q = 1$ **and** $\alpha \neq 0$,
3. $K\langle x, y \mid yx = xy + f(y) \rangle$, *where* $f \in K[y]$ *with* $\deg(f) = N$, **if** $q = 1$ **and** $\alpha = 0$.

Application

Given a system of equations S in terms of other operators,

one can look up a concrete isomorphism of K-algebras (e. g. from the mentioned papers)

and rewrite S as S' in terms of the operators above.

Further results on S' after performing computations can be transferred back to original operators.

Example: difference and divided difference operators $\Delta_n = S_n - 1$, $\Delta_n^{(q)} = S_n^{(q)} - 1$ etc.

Quadratic algebras

Lemma (L.–Makedonsky–Petravchuk, new)

$K\langle x, y \mid yx = xy + f(y)\rangle \cong K\langle z, w \mid wz = zw + g(w)\rangle$

if and only if

$\exists \lambda, \nu \in K^*$ *and* $\exists \mu \in K$, *such that* $g(t) = \nu f(\lambda t + \mu)$ *(in particular* $\deg(f) = \deg(g)$*).*

Lemma (L.–Makedonsky–Petravchuk, new)

For any algebra of the type $B = K\langle a, b \mid ba = ab + f(a)\rangle$ *for* $f \neq 0$ *there exists an injective homomorphism into the 1st Weyl algebra.*

Quadratic algebras

Let $N = \deg f(y) = 2$ and K be algebraicaly closed field of char $K > 2$. Then there are precisely two classes of non-isomorphic algebras of the type $K\langle x, y \mid yx = xy + f(y)\rangle$:

$K\langle x, y \mid yx = xy + y^2\rangle$ type

- integration algebra $K\langle x, \mathcal{I} \mid \mathcal{I}\, x = x\, \mathcal{I} - \mathcal{I}^2\rangle$,
- the algebra $K\langle x^{-1}, \partial = \frac{d}{dx} \mid \partial x^{-1} = x^{-1}\partial - (x^{-1})^2\rangle$,
- the algebra $K\langle x, \partial^{-1} \mid \partial^{-1}x = x\partial^{-1} - (\partial^{-1})^2\rangle$ etc.

$K\langle x, y \mid yx = xy + y^2 + 1\rangle$ type

- tangent algebra $K\langle \tan, \partial \mid \partial \cdot \tan = \tan \cdot\partial + \tan^2 + 1\rangle$ (take $y = \tan$, $x = -\partial$)
- the subalgebra of the 1st Weyl algebra, generated by $Y = -x$ and $X = (x^2+1)\partial$; then $YX = XY + Y^2 + 1$ etc.

Open problems for the Part 1

Let A be a bivariate algebra as before.

- If S is a multiplicatively closed Ore set (see next parts), then there exists localization $S^{-1}A$, such that $A \subset S^{-1}A$ holds.
- Problem: establish isomorphy classes for the localized algebras $S^{-1}A$, depending on the type of S.
- Example: in the part on localization.

More general framework: G-algebras

Let $R = K[x_1, \ldots, x_n]$. The standard **monomials** $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, $\alpha_i \in \mathbb{N}$, form a K-basis of R.

$$\mathrm{Mon}(R) \ni x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \mapsto (\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha \in \mathbb{N}^n.$$

1. a total ordering $\prec$ on $\mathbb{N}^n$ is called a **well–ordering**, if $\forall F \subseteq \mathbb{N}^n$ there exists a minimal element of F, in particular $\forall\ a \in \mathbb{N}^n,\ 0 \prec a$
2. an ordering $\prec$ is called a **monomial ordering on** R, if
   - $\forall \alpha, \beta \in \mathbb{N}^n\ \alpha \prec \beta \ \Rightarrow\ x^\alpha \prec x^\beta$
   - $\forall \alpha, \beta, \gamma \in \mathbb{N}^n$ such that $x^\alpha \prec x^\beta$ we have $x^{\alpha+\gamma} \prec x^{\beta+\gamma}$.
3. Any $f \in R \setminus \{0\}$ can be written uniquely as $f = cx^\alpha + f'$, with $c \in K^*$ and $x^{\alpha'} \prec x^\alpha$ for any non–zero term $c'x^{\alpha'}$ of f'.
   $\mathrm{lm}(f) = x^\alpha$, the **leading monomial** of f
   $\mathrm{lc}(f) = c$, the **leading coefficient** of f.

Towards G-algebras

Suppose we are given the following data

1. a field K and a commutative ring $R = K[x_1, \ldots, x_n]$,
2. a set $C = \{c_{ij}\} \subset K^*$, $1 \le i < j \le n$
3. a set $D = \{d_{ij}\} \subset R$, $\quad 1 \le i < j \le n$

Assume, that there is a monomial well–ordering $\prec$ on R such that

$$\forall 1 \le i < j \le n, \; \operatorname{lm}(d_{ij}) \prec x_i x_j.$$

To the data $(R, C, D, \prec)$ we associate an algebra

$$A = K\langle x_1, \ldots, x_n \mid \{x_j x_i = c_{ij} \cdot x_i x_j + d_{ij}\} \; \forall 1 \le i < j \le n \rangle.$$

A is called **a G–algebra** in n variables, if

$$c_{ik} c_{jk} \cdot d_{ij} x_k - x_k d_{ij} + c_{jk} \cdot x_j d_{ik} - c_{ij} \cdot d_{ik} x_j + d_{jk} x_i - c_{ij} c_{ik} \cdot x_i d_{jk} = 0.$$

G-algebras

Theorem (Properties of G-algebras)

Let A be a G-algebra in n variables. Then

- *A is left and right Noetherian,*
- *A is an integral domain,*
- *the Gel'fand-Kirillov dimension over K is* $\operatorname{GKdim}(A) = n$,
- *the global homological dimension* $\operatorname{gl.dim}(A) \leq n$,
- *the generalized Krull dimension* $\operatorname{Kr.dim}(A) \leq n$.
- *A is Auslander-regular and a Cohen-Macaulay algebra.*

Classical examples: full shift algebra

Adjoining the backwards shift $s^{-1} : f(x) \mapsto f(x-1)$ to the shift algebra, we incorporate several more relations, which define a so-called **full shift algebra**:

$$K\langle x, s, s^{-1} \mid sx = (x+1)s,\ s^{-1}x = (x-1)s^{-1}, s^{-1}s = s \cdot s^{-1} = 1\rangle$$

Note: full shift algebra is **not** a G-algebra, due to the relation $s \cdot s^{-1} = 1$. But it can be realized as a factor algebra of a G-algebra
$A = K\langle x, s, s^{-1} \mid sx = (x+1)s,\ s^{-1}x = (x-1)s^{-1}, s^{-1}s = ss^{-1}\rangle$
modulo the two-sided ideal $\langle s^{-1}s - 1\rangle$.

We can also realize this algebra as an Ore localization of the shift algebra, see next parts.

Gröbner Bases in G-algebras

Let A be a G-algebra in $x_1, \dots, x_n$. From now on, we assume that a given ordering is a **well-ordering**.

Definition

We say that $x^\alpha \mid x^\beta$, i. e. monomial x^α **divides** monomial x^β, if $\alpha_i \le \beta_i \; \forall i = 1 \dots n$.

It means that x^β is **reducible** by x^α, that is there exists $\gamma \in \mathbb{N}^n$, such that $\beta = \alpha + \gamma$. Then $\operatorname{lm}(x^\alpha x^\gamma) = x^\beta$, hence $x^\alpha x^\gamma = c_{\alpha\gamma} x^\beta +$ lower order terms.

Definition

Let $\prec$ be a monomial ordering on A, $I \subset A$ be a left ideal and $G \subset I$ be a finite subset. G is called a **(left) Gröbner basis** of I, if $\forall \; f \in I \setminus \{0\}$ there exists a $g \in G$ satisfying $\operatorname{lm}(g) \mid \operatorname{lm}(f)$.

Gröbner Bases in G-algebras

- There exists a generalized Buchberger's algorithm (as well as other generalized algorithms for Gröbner bases), which works along the lines of the classical commutative algorithm.
- There exist algorithms for computing a two-sided Gröbner basis, which has no analogon in the commutative case.
- G-algebras are fully implemented in the actual system SINGULAR:PLURAL, as well as in older systems MAS, FELIX.
- In SINGULAR:PLURAL there are many thorougly implemented functions, including Gröbner bases, Gröbner basics (module arithmetics) and numerous useful tools.

Gröbner Technology = Gröbner trinity + Gröbner basics

Gröbner trinity:

- left Gröbner basis of a submodule of a free module
- left syzygy module of a given set of generators
- left transformation matrix, expressing elements of Gröbner basis in terms of original generators

Gröbner basics (Buchberger, Sturmfels, ...)

- Ideal (resp. module) membership problem (NF, REDUCE)
- Intersection with subrings (ELIMINATE)
- Intersection and quotient of ideals (INTERSECT, QUOT)
- Kernel of a module homomorphism (MODULO)
- Kernel of a ring homomorphism (PREIMAGE)
- Algebraic dependencies of commuting polynomials
- Hilbert polynomial of graded ideals and modules ...