Part II. Dimension theory.

From system of equations to modules

Consider Legendre's differential equation (order 2 in ∂_x)

$$(x^2-1){P''}_n(x)^2 + 2x{P'}_n(x) - n(1+n)P_n(x) = 0$$

- x is differentiable with ∂_x as corr. operator
- if $n \in \mathbb{Z}$, n is discretely shiftable with s_n as corr. op.

- then there is a recursive formula of Bonnet (order 2 in shift s_n)

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0.$$

$$\mathfrak{O} := K\langle n, s_n \mid s_n n = n s_n + s_n\rangle \otimes_K K\langle x, \partial_x \mid \partial_x x = x\partial_x + 1\rangle.$$

From the system of equations

$$\begin{array}{rcl} (x^2-1){P''}_n(x)^2 + 2x{P'}_n(x) - n(1+n)P_n(x) & = & 0, \\ (n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) & = & 0. \end{array}$$

one obtains the matrix $P \in \mathfrak{O}^{2\times 1}$; thus $M = \mathfrak{O}/\mathfrak{O}^{1\times 2}P$ and

$$\left[\begin{array}{c} (x^2-1)\partial_x^2 + 2x\partial_x - n(1+n) \\ (n+2)s_n^2 - (2n+3)xs_n + n + 1 \end{array}\right] \bullet P_n(x) = \left[\begin{array}{c} 0 \\ 0 \end{array}\right].$$

With the help of Gröbner bases over $\mathfrak{O}$: a minimal generating set of the left ideal P contains a *compatibility condition*

$$(n+1)s_n\partial_x - (n+1)x\partial_x - (n+1)^2 \equiv (n+1)(s_n\partial_x - x\partial_x + n + 1).$$

From system of equations to modules

Let $f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n)$ be unknown generalized functions, for instance from $C^\infty(\mathbb{R}^n)$.
Then a homogeneous system of linear functional (operator) equations with coefficients from $K[x_1,\ldots,x_n]$ can be presented via the matrix equation in the corresponding operator algebra $\mathfrak{O}$:

$$P \cdot \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad P \in \mathfrak{O}^{\ell\times m}$$

One associates to the system a left $\mathfrak{O}$-module $M = \mathfrak{O}^{1\times m}/\mathfrak{O}^{1\times \ell}P$, saying M **is finitely presented by a matrix** P.

From system of equations to modules

Different matrices P_i can represent the same module M.

For instance, for any unimodular $T \in \mathfrak{O}^{\ell \times \ell}$ one has
$Pf = 0 \Leftrightarrow (TP)f = 0$ and also $\mathfrak{O}^{1\times m}/\mathfrak{O}^{1\times \ell} TP \cong \mathfrak{O}^{1\times m}/\mathfrak{O}^{1\times \ell} P$.

For various purposes we might utilize different presentations of M.
The invariants of a module M, like dimensions, do not depend on the presentation.

Algebraic manipulations from the left on P often need algorithms for left Gröbner bases for a submodule of a free module, generated by rows or columns of P (thus not only GBs of ideals).

From modules to solutions of systems

Let $\mathcal{F}$ be a left $\mathfrak{O}$-module (not necessarily finitely presented), and P a system of equations as before, then

$$\mathrm{Sol}_{\mathfrak{O}}(P,\mathcal{F}) := \{f \in \mathcal{F}^{m\times 1} \ : \ P \bullet f = 0\}.$$

Noether-Malgrange Isomorphism

There exists an isomorphism of K-vector spaces

$$\mathrm{Hom}_{\mathfrak{O}}(M,\mathcal{F}) = \mathrm{Hom}_{\mathfrak{O}}(\mathfrak{O}^{1\times m}/\mathfrak{O}^{1\times \ell}P,\mathcal{F}) \ \cong \ \mathrm{Sol}_{\mathfrak{O}}(P,\mathcal{F}),$$

$$(\phi : M \to \mathcal{F}) \ \mapsto (\phi([e_1]),\ldots,\phi([e_m])) \in \mathcal{F}^{m\times 1}.$$

From functions to modules

Let $\mathcal{F}$ be a left $\mathfrak{D}$-module (not necessarily finitely presented), and $f \in \mathcal{F}$. Consider $\mathfrak{D} f = \{\mathfrak{o} \bullet f \mid \mathfrak{o} \in \mathfrak{D}\}$, which is an $\mathfrak{D}$-submodule of $\mathcal{F}$.

Consider a homomorphism of left $\mathfrak{D}$-modules $\phi_f : \mathfrak{D} \to \mathcal{F}$, $\mathfrak{o} \mapsto \mathfrak{o} \bullet f$, in other words $\phi_f(1) = f \in \mathcal{F}$. Then

- $Im\phi_f = \mathfrak{D} f$, $\quad \operatorname{Ker} \phi_f = \{\mathfrak{o} \in \mathfrak{D} : \mathfrak{o} \bullet f = 0\} =: \operatorname{Ann}_{\mathfrak{D}} \mathbf{f}$
- as left $\mathfrak{D}$-modules, one has $\quad \mathfrak{D} f \cong \mathfrak{D} / \operatorname{Ann}_{\mathfrak{D}} f$
- hence $\mathfrak{D} f$ is finitely presented left $\mathfrak{D}$-module.

An element $m \in \mathcal{F}$ is called a **torsion element**, if $\operatorname{Ann}_{\mathfrak{D}} m \neq 0$.

Many classical functions in common functional spaces are torsion.

Hence, algorithms for the computation of the left ideal $\operatorname{Ann}_{\mathfrak{D}} m$ (which is finitely generated when $\mathfrak{D}$ is Noetherian) are very important.

From functions to modules

Many classical functions in common functional spaces are torsion. **But not all**.

Example: $f = \tan(x)$ is not a torsion element in a module over Weyl algebra, since there exists **no** system of linear ODEs with variable coefficients, having $\tan(x)$ as solution. However, there is a nonlinear ODE $f' = 1 + f^2$.

Recall: we are able to treat polynomials in the operator $\tan(x)\cdot$ as coefficients in an algebra with differentiation w.r.t x.

From functions to modules

Let $\mathcal{F}$ be a left $\mathfrak{O}$-module, and $f_1, \ldots, f_m \in \mathcal{F}$ be torsion elements. Consider $M = \mathfrak{O} f_1 + \ldots + \mathfrak{O} f_m$. As we know, every $\mathfrak{O} f_i$ is finitely presented $\mathfrak{O}$-submodule of $\mathcal{F}$.

Consider a homomorphism of left $\mathfrak{O}$-modules

$$\phi : \mathfrak{O}^m = \bigoplus_{i=1}^{m} \mathfrak{O} e_i \to \mathcal{F}, \quad \sum \mathfrak{o}_i e_i \mapsto \sum \mathfrak{o}_i \bullet f_i,$$

in other words $\phi(e_i) = f_i \in \mathcal{F}$. Then $Im\ \phi = M = \sum \mathfrak{O} f_i$,

- $\operatorname{Ker} \phi = \{[\mathfrak{o}_1, \ldots, \mathfrak{o}_m] \in \mathfrak{O}^m : \sum \mathfrak{o}_i \bullet f_i = 0\} =: \operatorname{Mann}_{\mathfrak{O}} \mathbf{M}$
- as left $\mathfrak{O}$-modules, one has $\quad M = \sum \mathfrak{O} f_i \cong \mathfrak{O}^m / \operatorname{Mann}_{\mathfrak{O}} M$
- hence $M = \sum_i \mathfrak{O} f_i$ is finitely presented left $\mathfrak{O}$-module.

Clearly $\oplus \operatorname{Ker} \phi_{f_i} e_i \subseteq \operatorname{Mann}_{\mathfrak{O}} M$.

Idea: Model polynomial-exponential signals by linear systems.
Question: What is more precise in such a modeling: operator algebras with constant or with polynomial coefficients?

Answer: algebras with polynomial coefficients.

Theorem (Zerz–L.–Schindelar, 2011)

Let $K = \mathbb{R}$, $p_i \in K[x_1, \ldots, x_n]^\ell$ *and* $V = Kp_1 + \cdots + Kp_m$. *Let* $\mathfrak{D}$ *be the n-th Weyl algebra and* $\mathfrak{D} \supset \operatorname{Ann}_{\mathfrak{D}}(V) := \cap \operatorname{Ann}_{\mathfrak{D}} p_i$ *be the left ideal of operators, simultaneously annihinalting* $p_1, \ldots, p_m$. *Then*

$$\operatorname{Sol}_{\mathfrak{D}}(\ \mathfrak{D}/\operatorname{Ann}_{\mathfrak{D}}(V),\ C^\infty(\mathbb{R}^\ell)) = V.$$

Keywords: **V**ariant **M**ost **P**owerful **U**nfalsified **M**odel, cf. two recent papers by Zerz, L. and Schindelar.

Dimensions

- Generalized Krull dimension (for an algebra or a module, $\mathrm{Kr.dim}\, M$) is called Krull-Rentschler-Gabriel dimension; not algorithmic

- projective dimension of a module, $\mathrm{p.dim}\, M$; algorithmic (relatively expensive), implemented
- global homological dimension of an algebra, $\mathrm{gl.dim}\, A = \sup\{\mathrm{p.dim}\, M : M \in A - mod\}$, in general not algorithmic

- homological grade of a module, $j(M)$; algorithmic (a little less expensive than $\mathrm{p.dim}\, M$), implemented

- Gel'fand-Kirillov Dimension; algorithmic (relatively cheap), implemented; intuition: similar to usual Krull dimension

Filtration on algebras and modules

Let A be a K-algebra, generated by $x_1, \dots, x_m$.

Degree filtration

Let $V = Kx_1 \oplus \dots \oplus Kx_m$ be a vector space.

Set $V_0 = K$, $V_1 = K \oplus V$ and $V_{k+1} = V_k \oplus V^{k+1}$. If

$$V_i \subseteq V_{i+k}, \quad V_i \cdot V_j \subseteq V_{i+j}, \quad A = \bigcup_{k=0}^{\infty} V_k,$$

then $\{V_k \mid k \in \mathbb{N}\}$ is the **standard (ascending) filtration** of A.

Gel'fand-Kirillov dimension and its properties

Let $M_0 \subset M$ be a finite K-vector space, spanned by the generators of M. That is $\dim_K M_0 < \infty$ and $AM_0 = M$.

$\{H_d := V_d M_0, d \in \mathbb{N}\}$ is an induced ascending filtration on M.

The **Gel'fand-Kirillov dimension** of M is defined as follows

$$\mathrm{GKdim}(M) \ = \ \limsup_{d\to\infty} \left(\log_d(\dim_K H_d)\right)$$

In the standard construction one puts $\deg x_i := 1$ and defines $V_d := \{f \mid \deg f = d\}$ and $V^d := \{f \mid \deg f \leq d\}$.

Conventions: $\mathrm{GKdim}(0) = -\infty$. $\mathrm{GKdim}_{\mathbb{Q}}(\mathbb{Q}) = 0$.

Lemma

Let A be a K-algebra and a domain. If the standard filtration on A is compatible with the PBW Basis $\{x^\alpha \mid \alpha \in \mathbb{N}^m\}$, then $\mathrm{GKdim}_K(A) = m$.

$$\dim V_d = \binom{d+m-1}{m-1}, \; \dim V^d = \binom{d+m}{m}.$$

Thus $\binom{d+m}{m} = \frac{(d+m)\ldots(d+1)}{m!} = \frac{d^m}{m!} + \ldots$ and

$$\mathrm{GKdim}(A) = \limsup_{d\to\infty} \log_d \binom{d+m}{m} = m.$$

Hence for any G-algebra A in n variables has $\mathrm{GKdim}_K(A) = n$.

Gel'fand-Kirillov dimension: examples and properties

Free associative algebra $T = K\langle x_1, \ldots, x_n\rangle, n \geq 2$

$\dim V_d = n^d, \dim V^d = \frac{n^{d+1}-1}{n-1}$. Note, that $\frac{n^{d+1}-1}{n-1} > n^d$.
Since $\log_d n^d = d \log_d n = \frac{d}{\log_n d} \to \infty, d \to \infty$, it follows that $\mathrm{GKdim}(T) = \infty$.

Properties

- $\mathrm{GKdim}\, M = \sup\{\mathrm{GKdim}(N) : N \in A - mod,\ N \subseteq M\}$,
- $\mathrm{GKdim}\, A = \sup\{\mathrm{GKdim}(S) : S \subseteq A,\ S$ fin. gen. subalgebra$\}$

Hence, if $|K| = \infty$, then $\mathrm{GKdim}(K[[x_1, \ldots, x_n]]) = \infty$ for $n \geq 1$.

Lemma (R is commutative)

(i) *Let R be a commutative affine K-algebra. Then (by Noether normalization) $\exists S = K[x_1, \ldots, x_t] \subseteq R$ and R is finitely generated S-module. Then $\operatorname{GKdim}_K R = \operatorname{Kr.dim} S = t$.*

(ii) *If R is an integral domain, $\operatorname{GKdim}_K R = \operatorname{tr.deg}_K \operatorname{Quot}(R)$.*

For any K-algebra R: $\operatorname{GKdim} R[x_1, \ldots, x_m] = \operatorname{GKdim} R + m$.
Curiosity: $\operatorname{GKdim}(R) \in \{0, 1\} \cup [2, +\infty)$.

Exactness

Let R be an affine algebra with finite standard fin.-dim. filtration, such that $\operatorname{Gr} R$ is left Noetherian. Then GKdim is exact on short exact sequences of fin. gen. left R-modules. That is,

$$0 \to L \to M \to N \to 0 \;\Rightarrow\; \operatorname{GKdim} M = \sup\{\operatorname{GKdim} L, \operatorname{GKdim} N\}$$

Gel'fand-Kirillov dimension for modules

There is an algorithm by Gomez-Torrecillas et. al., which computes Gel'fand-Kirillov dimension for finitely presented modules over G-algebras over ground field K. It is implemented e. g. in SINGULAR:PLURAL.

$\text{GKDIM}_K(F)$

Let A be a G-algebra in variables $x_1, \ldots, x_n$.

- Input: Left generating set $F = \{f_1, \ldots, f_m\} \subset A^r$
- Output: $k \in \mathbb{N}$, $k = \text{GKdim}(A^r/M)$, where $M = {}_A\langle F \rangle \subseteq A^r$.

- $G = \text{LEFTGRÖBNERBASIS}(F) = \{g_1, \ldots, g_t\}$;
- $L = \{\text{lm}(g_i) = x^{\alpha_i} e_s \mid 1 \leq i \leq t\}$;
- $N = {}_{K[x_1,\ldots,x_n]}\langle L \rangle$;
- **return** $\text{Kr.dim}(K[x_1, \ldots, x_n]^r/N)$;

Gel'fand-Kirillov dimension for modules: example

Recall Legendre's example:

$$\mathfrak{O} := K\langle n, s_n \mid s_n n = n s_n + s_n\rangle \otimes_K K\langle x, \partial_x \mid \partial_x x = x\partial_x + 1\rangle.$$

Then $\operatorname{GKdim}_K \mathfrak{O} = 4$.

The Gröbner basis of the ideal P is

$$(x^2-1)\partial_x^2 + 2x\partial_x - n(1+n),\ (n+2)s_n^2 - (2n+3)xs_n + n + 1,$$
$$(n+1)s_n\partial_x - (n+1)x\partial_x - (n+1)^2.$$

The leading monomials are $x^2\partial_x^2, ns_n^2, ns_n\partial_x$. Hence

$$\operatorname{GKdim}_K \mathfrak{O}/P = \operatorname{Kr.dim} K[n, s_n, x, \partial_x]/\langle x^2\partial_x^2, ns_n^2, ns_n\partial_x\rangle = 2.$$

Elimination and GK-dimension

Lemma (MR, KL)

Let $I \subset A$ be a left ideal and $S \subset A$ be a subalgebra. Then

- $I \cap S = 0$ *implies* $\operatorname{GKdim} A/I \geq \operatorname{GKdim} S$,
- $\operatorname{GKdim} A/I < \operatorname{GKdim} S$ *implies* $I \cap S \neq 0$.

Recall: Bernstein's inequality

Let A be the n-th Weyl algebra over K with char $K = 0 = \operatorname{GKdim} K$, then $\operatorname{GKdim}(A) = 2n$.

Let $0 \neq M$ be an A-module, then $\operatorname{GKdim}_K M \geq n$.

Elimination and GK-dimension

Let $f \in \mathcal{F}$, such that $\operatorname{Ann}_{\mathfrak{D}} f \cap K[x_1, \ldots, x_n] = 0$. Then $\operatorname{GKdim}_K \mathfrak{D} / \operatorname{Ann}_{\mathfrak{D}} f \geq n$.

Proposition (Existence of elimination via dimension)

Let $\mathfrak{D} = \bigotimes_{i=1}^{n} \mathfrak{D}_i$, $\mathfrak{D}_i = K\langle x_i, \mathfrak{o}_i \mid \ldots \rangle$. Moreover, let $I \subset \mathfrak{D}$ and $\operatorname{GKdim} \mathfrak{D}/I = m$. Then for any subalgebra $S \subset \mathfrak{D}$, such that $\operatorname{GKdim} S \geq m+1$ one has $I \cap S \neq 0$.

Application: For I such that $\operatorname{GKdim} \mathfrak{D}/I = m$ we guarantee that $2n - (m+1) = 2n - m - 1$ variables can be eliminated from I, for instance, if $m = n$, we can eliminate

- all but one operators,
- all but one coordinate variables.

More applications will follow ... in the parts, which follow.