Part IV. Purity.

Dimension function

Let A be a Noetherian algebra. A dimension function δ assigns a value $\delta(M)$ to each finitely generated A-module M and satisfies the following properties:

(i) $\delta(0) = -\infty$.

(ii) If $0 \to M' \to M \to M'' \to 0$ is exact sequence, then $\delta(M) \geq \sup\{\delta(M'), \delta(M'')\}$ with equality if the sequence is split.

(iii) If P is a (two-sided) prime ideal with $P \subseteq \mathrm{Ann}_A(M)$ and M is a torsion module over A/P, then $\delta(M) \leq \delta(A/P) - 1$.

- generalized Krull dimension is an exact dimension function
- Gel'fand-Kirillov dimension is a dimension function, not always exact

Purity w.r.t dimension function

Let A be a K-algebra and δ a dimension function on A-mod.
A module $M \neq 0$ is δ-**pure** (or δ-homogeneous), if

$$\forall 0 \neq N \subseteq M, \quad \delta(N) = \delta(M).$$

- A simple module is pure. Thus, purity is a useful weakening of the concept of simplicity of a module.
- Unlike simplicity, the purity (w.r.t a dimension function) is algorithmically decidable over many common algebras.

M. Barakat, A. Quadrat: Algorithms for the computation of the purity filtration of a module with $\delta =$ homological grade; there are several implementations: in HOMALG, OREMODULES(MAPLE) and SINGULAR:PLURAL.

Purity with respect to a dimension function

Lemma (L.)

Let A be a K-algebra and δ a dimension function on A-mod. Moreover, let $0 \neq M_1, M_2 \subset N$ be two δ-pure modules with $\delta(M_1) = \delta(M_2)$. Then

the set of δ-pure submodules (of the same dimension) of a module is a lattice, i. e.

1. $M_1 \cap M_2$ *is either 0 or it is δ-pure with* $\delta(M_1 \cap M_2) = \delta(M_1)$,
2. $M_1 + M_2$ *is δ-pure with* $\delta(M_1 + M_2) = \delta(M_1)$.

Ubiquity of pure modules

Consider purity with respect to Gel'fand-Kirillov dimension.

Lemma (L.)

Let A be a G-algebra, $S \subset A$ a m. c. Ore set in A. Let $\mathcal{M}$ be a set of left A-modules M, satisfying $S^{-1}M \neq 0$ and having dimension $\operatorname{GKdim} KS$, where KS is the monoid algebra. Then $\mathcal{M}$ consists of pure modules.

Example (Pure modules)

- modules of Krull dimension 0 over $K[x_1, \ldots, x_n]$, i. e. modules M, such that $\dim_K M < \infty$
- any set of modules of smallest possible dimension in A, for instance holonomic modules over the n-th Weyl algebra over a field with char $K = 0$; it is known that they have GK dimension n over K.

Ubiquity of pure modules

Recall

Let A be an operator algebra over $K[x_1, \ldots, x_n]$ and $S = K[x_1, \ldots, x_n] \setminus \{0\} \subset A$ be a m. c. Ore set in A.
A left A-module M is called D-**finite**, if $\dim_{K(x_1,\ldots,x_n)} S^{-1}M < \infty$.

Thus D-finite modules are pure.

Note: we can do much more with the concept of purity

We can consider pure modules of any reasonable dimension, without restricting ourselves to the modules of smallest possible dimension!

Pure functions and operations with them

Let $\mathfrak{O}$ be an operator algebra and $\mathcal{F}$ an $\mathfrak{O}$-module. A torsion element $f \in \mathcal{F}$ (that is a "function" having nonzero annihilator) is called **pure**, is the corresponding left $\mathfrak{O}$-module $\mathfrak{O}f \cong \mathfrak{O}/\operatorname{Ann}_{\mathfrak{O}} f$ is pure.

This definition generalizes both the notion of Zeilberger-*holonomic* or *D-finite* function as well as some other.

Lemma (L.)

Let $f \in \mathcal{F}$ be a pure function. Then for any $\mathfrak{o} \in \mathfrak{O} \setminus \{0\} \quad h = \mathfrak{o}f$ is pure as well.

Proof: $\mathfrak{O}g = \mathfrak{O}\mathfrak{o}f \subset \mathfrak{O}f$ is a natural submodule, hence it is pure. Moreover, $\operatorname{Ann}_{\mathfrak{O}} \mathfrak{o}f =$

$$\{r \in \mathfrak{O} : r(\mathfrak{o}f) = (r\mathfrak{o})f = 0\} = \{s \in \operatorname{Ann}_{\mathfrak{O}} f : \exists r \in \mathfrak{O}, s = r\mathfrak{o}\} =$$

$$\operatorname{Ann}_{\mathfrak{O}} f : \mathfrak{o} = \operatorname{Ker}_{\mathfrak{O}}(\mathfrak{O} \to \mathfrak{O}/\operatorname{Ann}_{\mathfrak{O}} f,\ 1 \mapsto \mathfrak{o})$$ is computable.

Operations with pure functions

Lemma (L.)

Let $f, g \in \mathcal{F}$ be pure functions. Then for any $\mathfrak{p}, \mathfrak{q} \in \mathfrak{O} \setminus \{0\}$ $h = \mathfrak{p}f + \mathfrak{q}g$ is pure as well.

Proof: by the previous lemma $M_f = \mathfrak{O}\mathfrak{p}f$ and $M_g = \mathfrak{O}\mathfrak{q}g$ are pure modules. By another lemma before $M_f + M_g$ is pure. Hence $\mathfrak{O}h \subseteq M_f + M_g$ is pure as well.
Moreover, $(\mathrm{Ann}_{\mathfrak{O}} f : \mathfrak{p}) \cap (\mathrm{Ann}_{\mathfrak{O}} g : \mathfrak{q}) \subseteq \mathrm{Ann}_{\mathfrak{O}} h$.

More operations, preserving the purity, are under investigation.

Observation : many (but not all) special functions give rise to pure modules.

Identities, Elimination, Purity Filtration

Let $0 \to M_1 \to M_2 \to M_2/M_1 \to 0$ be an exact sequence of fin. pres. $\mathfrak{D}$-modules. Moreover, let $\mathcal{F}$ be an arbitrary $\mathfrak{D}$-module. Then we have that $\mathrm{Sol}_{\mathfrak{D}}(M_2/M_1, \mathcal{F}) \subseteq \mathrm{Sol}_{\mathfrak{D}}(M_2, \mathcal{F})$.

If $\mathcal{F}$ is injective $\mathfrak{D}$-module, the natural map $\mathrm{Sol}_{\mathfrak{D}}(M_2, \mathcal{F}) \to \mathrm{Sol}_{\mathfrak{D}}(M_1, \mathcal{F})$ is surjective (not true for general $\mathcal{F}$).

Purity filtration with $\delta = \mathrm{GKdim}$

Let $\mathfrak{D}$ be a Noetherian domain, being Auslander-regular and Cohen-Macaulay algebra with $\mathrm{GKdim}\,\mathfrak{D} = n$.
Given a fin. pres. $\mathfrak{D}$-module M of dimension $n > d \geq 0$, then the purity filtration of M is the sequence

$$M = M_{n-d} \supset M_{n-d+1} \ldots \supset M_{n-1} \supset M_n = 0.$$

where $\mathrm{GKdim}\, M_{n-(d-i)} = d - i$. Moreover, $M_{n-d+k}/M_{n-d+k+1}$ is either 0 or pure of dimension $d - k$.

Identities, Elimination, Purity Filtration

Consider the mixed system, annihilating Legendre polynomials

$$\mathfrak{O} = K\langle n, s_n \mid s_n n = n s_n + s_n \rangle \otimes_K K\langle x, \partial_x \mid \partial_x x = x\partial_x + 1\rangle.$$

$$M = \mathfrak{O}/P,$$

$$P = \langle (x^2-1)\partial_x^2 + 2x\partial_x - n(1+n),\ (n+2)s_n^2 - (2n+3)x s_n + n + 1,$$
$$(n+1)(s_n\partial_x - x\partial_x + n + 1)\rangle.$$

$$\operatorname{GKdim} \mathfrak{O} = 4, \quad \operatorname{GKdim} M = 2, \quad t(M) = M = \mathfrak{O}/P.$$

The purity filtration of $M = t(M)$ is $0 \subsetneq M_3 \subsetneq M_2 = M$,

$$M_3 \ \cong \mathfrak{D}/\langle n+1, s_n, \partial_x \rangle \ \text{ with } \ \operatorname{GKdim} M_3 = 1.$$

What are the most general solutions $g(n,x)$ of this system?

Since $\partial_x(g) = 0$, one has $g(n,x) = g(n)$.
however, $g(n)$ should not be identically zero:
in case $n \in \{-1, 0, 1, \ldots\}$, one can select $g(-1) \in K$ arbitrary (step of the jump function).

Localization

The ideal $\langle n+1, s_n \rangle$ is two-sided and maximal. Hence the submodule M_3 vanishes under any nontrivial Ore localization w. r. t $S \subset K\langle n, s_n \ldots \rangle$, for instance when $n \in S$ or $s_n \in S$ (then s_n^{-1} is present and therefore $n \in \mathbb{Z}$ should hold). And $S^{-1}M$ is then a pure module.

The purity filtration of $M = t(M)$ is $0 \subsetneq M_3 \subsetneq M_2 = M$.
The pure part of GK dimension 2 is $t(M)/M_3 \cong$

$$\mathfrak{O}/\langle (x^2-1)\partial_x^2 + 2x\partial_x - n(1+n),\ (n+2)S_n^2 - (2n+3)xS_n + n+1,$$

$$(1-x^2)\partial_x + (n+1)S_n - (n+1)x\rangle.$$

For further investigations of M over localizations w.r.t. n or S_n one should then take the simplified equations from the ideal P' above.

Elimination leads to new identities

The elimination property guarantees, that 1 arbitrary variable of $\mathfrak{O}$ can be eliminated from P and from P'; so one gets for instance

$$\mathbf{x}\textbf{−free}:\ (n+1)(n+2)\cdot\big((S_n^2-1)\partial_x - (2n+3)S_n\big) \bullet P_n(x) = 0,$$

$$\mathbf{n}\textbf{−free}:\ (1-x^2)\cdot\big((S_n^2 - 2xS_n + 1)\partial_x - S_n)\big) \bullet P_n(x) = 0.$$

The hypergeometric series is defined for $|z| < 1$ and $-c \notin \mathbb{N}_0$ as follows:

$$ {}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} $$

We derive two annihilating ideals from the anihilator of ${}_2F_1(a, b, c; z)$:

- J_a which does not contain a,
- J_c which does not contain c,

and analyze corresponding modules for purity.

Case J_a

The ideal in $\mathfrak{O} = K[b,c,z]\langle Sb, Sc, Dz \mid \ldots\rangle$ is generated by:

$$bcSb - czDz - bc$$
$$bSbSc - bSc + cSc - c$$
$$bSb^2 - zSbDz - bSb + Sb^2 - Sb$$
$$b^2Sb - bzDz - b^2 + bSb - zDz - b$$
$$bzSbDz - z^2Dz^2 - bzDz - bSbDz + zDz^2 - bSb + bDz + b + Dz$$

Let $M = M_a = \mathfrak{O}/J_a$. Then $\operatorname{GKdim} \mathfrak{O} = 6, \operatorname{GKdim} M = 4$.

The purity filtration of $M = t(M)$

$0 \subsetneq M_5 = M_4 \subsetneq M_3 = M_2 = M$, where

$M/M_5 \cong \mathfrak{O}/\langle bSb - zDz - b, zDzSc + cSc - c\rangle$, $\operatorname{GKdim} M/M_5 = 4$

The purity filtration of $M = t(M)$

... and

$$M_5 \cong \mathfrak{D}/\langle c, Sb, b+1, zDz - Dz - 1\rangle, \ \text{GKdim}\, M_5 = 2.$$

The solutions can be read off:

$$\delta_{c,0} \cdot \delta_{b,-1} \cdot (\ln(z-1) + k_0), \ k_0 \in K$$

Case J_c

The ideal in $\mathfrak{D} = K[b,c,z]\langle Sb, Sc, Dz \mid \ldots\rangle$ is generated by:

$$aSa - bSb - a + b$$
$$bSb^2 - SbzDz - bSb + Sb^2 - Sb$$
$$b^2Sb - bzDz - b^2 + bSb - zDz - b$$
$$abSb - azDz - ab + bSb - zDz - b$$
$$bSbzDz - z^2Dz^2 - bSbDz - bzDz + zDz^2 - bSb + bDz + b + Dz$$

Let $M = M_c = \mathfrak{D}/J_c$. Then $\operatorname{GKdim} \mathfrak{D} = 6, \operatorname{GKdim} M = 4$.

The purity filtration of $M = t(M)$

$0 \subsetneq M_6 = M_5 = M_4 \subsetneq M_3 = M_2 = M$, where

$$M/M_6 \cong \mathfrak{D}/\langle bSb - zDz - b, aSa - zDz - a\rangle, \ \operatorname{GKdim} M/M_6 = 4.$$

The purity filtration of $M = t(M)$

... and

$$M_6 \cong \mathfrak{D}/\langle Sb, b+1, Sa, a+1, zDz - Dz - 1\rangle,\ \operatorname{GKdim} M_6 = 2.$$

The solutions:

$$\delta_{a,-1} \cdot \delta_{b,-1} \cdot (\ln(z-1) + k_0),\ k_0 \in K$$

Part V. Jacobson normal form.

One of the most important questions in algebra is undecidable in general:

Let A be a (Noetherian) K-algebra and M, N are two finitely presented A-modules. Can we decide, whether $M \cong N$ as A-modules?

Yet another application of localization as a functor:

Let $S \subset A$ be a m. c. Ore set, then $S^{-1}A$ exists.
Given an A-module homomorphism $\varphi : M \to N$ (M, N are finitely presented). Then there is an induced homomorphism of $S^{-1}A$-modules $S^{-1}\varphi : S^{-1}M \to S^{-1}N$.

Application to the isomorphism problem

If there exists such m. c. Ore set $\tilde{S} \subset A$, that $\tilde{S}^{-1}\varphi$ is not an isomorphism, then φ is not an isomorphism.

Invariants

Above we have seen several dimensions of modules, some of them are computable. What can one achieve with the help of localization?

- Let $S = A \setminus \{0\}$. Then the **rank** of f. g. A-module M is defined to be $\dim_{S^{-1}A} S^{-1}M$.

- Let $R = A[\partial; \sigma, \delta]$ for an integral domain A and $S = A \setminus \{0\}$. Then $S^{-1}M$ is a vector space over $\mathrm{Quot}(A) = S^{-1}A$ and $\dim_{S^{-1}R} S^{-1}M$ is an invariant of the module.

Jacobson, Teichmüller, Cohn

Let R be a non-commutative Euclidean domain and $M \in R^{m \times n}$. Then there exist

- unimodular matrices $U \in R^{m \times m}$, $V \in R^{n \times n}$;
- a matrix $D \in R^{m \times n}$ with elements $d_1, \ldots, d_r$ on the main diagonal and 0 outside of the main diagonal ...
- such that $d_i || d_{i+1}$ (total divisibility), meaning $\mathfrak{O}\langle d_{i+1}\rangle\mathfrak{O} \subseteq \mathfrak{O}\langle d_i\rangle \cap \langle d_i\rangle\mathfrak{O}$

such that $U \cdot M \cdot V = D$.

In particular there is an isomorphism of R-modules

$$R^{1\times n}/R^{1\times m}M \cong R^{1\times n}/R^{1\times m}D.$$

Recognizing the localization

L.–Schindelar (2011, 2012) presented two algorithms, computing matrices U, V, D by using Gröbner bases.

A fraction-free algorithm performs only operations over polynomial (i.e. unlocalized) algebra. A minor modification allows to produce matrices U, V, D with polynomial entries.

Theorem (L.–Schindelar)

Let A be a G-algebra in variables $x_1, \ldots, x_n, \partial$ and assume that $\{x_1, \ldots, x_n\}$ generate a G-algebra $B \subsetneq A$. Suppose, there exists an admissible monomial ordering $\prec$ on A, satisfying $x_k \prec \partial$ for all $1 \leq k \leq n$. Then the following holds

- *B^* is multiplicatively closed Ore set in A.*
- *$(B^*)^{-1}A$ can be presented as an Ore extension of $\mathrm{Quot}(B)$ by the variable ∂.*

Example

Let A_1 be the polynomial and $B_1 = (K[x] \setminus \{0\})^{-1} A_1$ the rational Weyl algebra. Consider the matrix

$$M = \begin{bmatrix} \partial^2 - 1 & \partial + 1 \\ \partial^2 + 1 & \partial - x \end{bmatrix}.$$

The algorithm returns

$$D = \begin{bmatrix} x^2\partial^2 + 2x\partial^2 + \partial^2 - 2x\partial - 2\partial - x^2 - 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} -x\partial - \partial + x^2 + x + 1 & x\partial + \partial + x \\ \partial - x & -\partial - 1 \end{bmatrix},$$

$$V = \begin{bmatrix} 1 & 0 \\ x\partial^2 + \partial^2 + 2\partial - x + 1 & 1 \end{bmatrix}.$$

Unimodularity of Matrices

Let us analyze, under which localizations U, V will be invertible.

Indeed, V is unimodular over A_1, since it admits an inverse:

$$V^{-1} = \begin{bmatrix} 1 & 0 \\ -(x+1)\partial^2 + x - 2\partial - 1 & 1 \end{bmatrix}$$

On the contrary, U is NOT unimodular over A_1, since $U \cdot Z = W$ and W is first invertible in the localization:

$$Z = \begin{bmatrix} 2\partial + 2 & (x+1)\partial + x - 2 \\ 2(\partial - x) & (x+1)\partial - x^2 - x - 3 \end{bmatrix}, W = \begin{bmatrix} 0 & -4x^2 - 8x - 4 \\ 2 & 5x + 5 \end{bmatrix}$$

For the invertibility of W we need only to divide by $x + 1 =: f$.

Lifting the isomorphism

Let $f = x + 1$. Then U from above will be unimodular over any localization, where f is invertible. In particular, the smallest one, as we know, is $C_1 := S_f^{-1} A_1$, where $S_f = \{f^i : i \in \mathbb{N}\}$.

Thus the isomorphism of B_1-modules, provided by the Jacobson form, holds not only over $B_1 = (K[x] \setminus \{0\})^{-1} A_1$, but also over C_1.

General strategy: depending on the concrete questions, analyze U resp. V for unimodularity over localizations, less greedy than the rational one.

Note: the steps of such an analysis are algorithmic.

Recognize and lift localized problems

Strategical remarks for conclusion.

- use the information from the localized situation - for instance, implementations of numerous good algorithms - for the analysis of the unlocalized, "global" situation;
- in algorithms:

  perform fraction-free computations, if possible

  or keep track of operations, requiring localized computations
- use this tracking information and determine a smaller localization, where desired properties still hold. Lift the obtained results to that smaller localization.
- study obstructions to the lifting: this provides several cases, which again hints at the treatment of the problem at a global level by using local ones.
- obtain new powerful and useful results!

Merci beaucoup

pour votre attention!

http://www.singular.uni-kl.de/