Computing the Bernstein-Sato polynomial

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Overview

The Weyl algebra

Definition

Let $\mathbb{K}$ be a field of characteristic 0. The $\mathbb{K}$-algebra

$$D = \mathbb{K}\langle x, \partial \rangle = \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \{\partial_i x_j = x_j \partial_i + \delta_{ij}\}\rangle$$

is a G-algebra, the n-th *Weyl algebra* over $\mathbb{K}$.

The initial ideal

Definition

Let $0 \neq w \in \mathbb{R}^n_{\geq 0}$. For $0 \neq p = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha \partial^\beta \in D$ set

$$m := \max_{\alpha,\beta}\{-w\alpha + w\beta \mid c_{\alpha\beta} \neq 0\}.$$

Then

$$\mathrm{in}_{(-w,w)}(p) := \sum_{\alpha,\beta:\ -w\alpha+w\beta=m} c_{\alpha\beta} x^\alpha \partial^\beta$$

is called the *initial form* of p w.r.t. w. Additionally, set $\mathrm{in}_{(-w,w)}(0) := 0$.

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For an ideal $I \subset D$ we call

$$\operatorname{in}_{(-w,w)}(I) := \mathbb{K} \cdot \{\operatorname{in}_{(-w,w)}(p) \mid p \in I\}$$

the *initial ideal* of I w.r.t. w.

b-function and Bernstein-Sato polynomial

Definition

Let $I \subset D$ be an ideal, $0 \neq w \in \mathbb{R}^n_{\geq 0}$ und $s := \sum_{i=1}^n w_i x_i \partial_i$.
The monic generator $b(s)$ of

$$\mathrm{in}_{(-w,w)}(I) \cap \mathbb{K}[s]$$

is called the (global) *b-function* of I w.r.t. w.

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Forr $f \in \mathbb{K}[x] \setminus \mathbb{K}$ definine the *Malgrange-Ideal* I_f of f to be

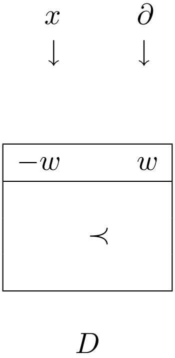
$$I_f := \langle t - f, \partial_i + \frac{\partial f}{\partial x_i}\partial_t \mid i = 1, \ldots, n\rangle \subset D\langle t, \partial_t\rangle.$$

Let $w = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ such that ∂_t gets the weight 1 and let $B(s)$ be the b-function of I_f w.r.t. w. Then

$$b(s) := (-1)^{\deg(B(s))} B(-s-1)$$

is called the (global) *b-function* or the *Bernstein-Sato polynomial* of f.

Let $0 \neq w \in \mathbb{R}^n_{\geq 0}$, $u, v \in \mathbb{R}^n_{>0}$ and let $\prec$ be a global ordering on D.



Computing the initial ideal

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x $\downarrow$	∂ $\downarrow$
$-w$	w
$\prec$	

$\xrightarrow{\text{homogenization}}$

x $\downarrow$	∂ $\downarrow$	h $\downarrow$
u	v	1
$-w$	w	0
$\prec$		

$$D \longrightarrow D^{(h,u,v)} := \mathbb{K}\langle x, \partial, h \mid \{\partial_i x_j = x_j \partial_i + \delta_{ij} \cdot h^{u_j + v_i}\}\rangle$$

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$=: \prec^{(h,u,v)}_{(-w,w)}$

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Algorithm (`initialIdealW`)

Input: $I \subset D$ ideal, $0 \neq w \in \mathbb{R}^n_{\geq 0}$, $\prec$ global ordering on D, $u, v \in \mathbb{R}^n_{>0}$
Output: Gröbner basis G of $\mathrm{in}_{(-w,w)}(I)$ w.r.t. $\prec$

$G^{(h)} :=$ a Gröbner basis of the homogenization of I w.r.t. $\prec^{(h,u,v)}_{(-w,w)}$

return $G = \mathrm{in}_{(-w,w)}(G^{(h)}|_{h=1})$

Statement of the problem

A	associative $\mathbb{K}$-algebra
$J \subset A$	ideal
G	finite Gröbner basis of J w.r.t. an arbitrary global ordering
$s \in A \setminus \mathbb{K}$	arbitrary

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What do we know about $J \cap \mathbb{K}[s]$?

We distinguish between the following situations:

1. $\mathrm{lm}(g) \nmid \mathrm{lm}(s^k)$ for all $g \in G, k \in \mathbb{N}_0$
2. Situation 1 does not apply:
    1. $J \cdot s \subset J$ and $\dim_{\mathbb{K}}(\mathrm{End}_A(A/J)) < \infty$
    2. Situation 2.1 does not apply:
        1. ???

Situations 1 and 2

Lemma (Situation 1)

If there is no $g \in G$ with $\mathrm{lm}(g) \mid \mathrm{lm}(s^k)$ for a $k \in \mathbb{N}_0$, then $J \cap \mathbb{K}[s] = \{0\}$.

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Remark (Situation 2)

The converse is not true. Consider $J = \langle y^2 + x \rangle \subset \mathbb{K}[x, y]$.
Then $J \cap \mathbb{K}[y] = \{0\}$, but $\{y^2 + x\}$ is a Gröbner basis of J for any ordering.

Situation 2.1

Lemma (Situation 2.1)

Let $J \cdot s \subset J$ and $\mathrm{End}_A(A/J)$ (A/J viewed as A-module) finite dimensional as $\mathbb{K}$-vector space. Then $J \cap \mathbb{K}[s] \neq \{0\}$.

Proof:

- $\cdot s : A/J \to A/J, \ [a] \mapsto [a \cdot s]$ is a well-defined A-module endomorphism.
- $\cdot s$ has a minimal polynomial μ.
- $\mu \in \mathbb{K}[s] \cap J$ and $\mu \neq 0$ (even $\mu \notin \mathbb{K}$). □

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Remark

Especially, the second condition holds, if

- A/J itself is a finite dimensional A-module, or
- A is a Weyl algebra and A/J is a holonomic module.

Back to the b-function

Theorem

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The b-function of a holonomic ideal $I \subset D$ is non-constant.

Proof: Let $0 \neq w \in \mathbb{R}^n_{\geq 0}$, $J := \mathrm{in}_{(-w,w)}(I)$ for a holonomic ideal $I \subset D$ and $s := \sum_{i=1}^n w_i x_i \partial_i$.

- D/J is a holonomic D-module.
- Endomorphism spaces of holonomic modules are finite dimensional.
- For $(-w,w)$-homogeneous $p \in J$ it holds that

$$p \cdot s = (s+m) \cdot p \in J, \qquad \text{where } m = \deg_{(-w,w)}(p).$$

□

Algorithm (`pIntersect`, *to be changed to* `uniIntersect`)

Input: $s \in A \setminus \mathbb{K}$, $J \subset A$ such that $J \cap \mathbb{K}[s] \neq \{0\}$
Output: $b \in \mathbb{K}[s]$ monic such that $J \cap \mathbb{K}[s] = \langle b \rangle$

$G :=$ finite Gröbner basis of J w.r.t. an *arbitrary* global ordering
$i := 1$
loop
if there exist $a_0, \ldots, a_{i-1} \in \mathbb{K}$ satisfying

$$\mathrm{NF}(s^i, G) + \sum_{j=0}^{i-1} a_j \, \mathrm{NF}(s^j, G) = 0 \text{ **then**}$$

return $b := s^i + \sum_{j=0}^{i-1} a_j s^j$
else
$i := i + 1$
end if
end loop

Tuning of the normal form

Lemma

Let A be a G-algebra of Lie type (e.g. a Weyl algebra), $J \subset A$ an ideal and $s \in A$. For $i \in \mathbb{N}$ put $r_i = \mathrm{NF}(s^i, J)$ and $q_i = s^i - r_i \in J$. Then it holds for all $i \in \mathbb{N}$ that

$$r_{i+1} = \mathrm{NF}([s^i - r_i, r_1] + r_i r_1, J).$$

Proof:

$$\begin{aligned} s^{i+1} &= q_i s + r_i s = q_i(q_1 + r_1) + r_i(q_1 + r_1) \\ &= q_i q_1 + q_i r_1 + r_i q_1 + r_i r_1 \to q_i r_1 + r_i r_1 \\ &\to [q_i, r_1] + r_i r_1 = [s^i - r_i, r_1] + r_i r_1 \end{aligned}$$

□

Applications

- Computation of b-functions
- Solving of zerodimensional systems
- Computation of central characters

Example (Computation of central characters)

Universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$:

$$A = U(\mathfrak{sl}_2, \mathbb{K}) = \mathbb{K}\langle e, f, h \mid [e, f] = h, [h, e] = 2e, [h, f] = -2f\rangle$$

Center of A:

$$Z(A) = \mathbb{K}[4ef + h^2 - 2h]$$

$F = \{e^{11}, f^{12}, h^5 - 10h^3 + 9h\} \subset A$

$L = {}_A\langle F\rangle$ as left ideal,

$T = {}_A\langle F\rangle_A$ as two-sided ideal

Algorithm (`intersectUpTo`, *to be changed to* `multiIntersect`)

Input: $s_1, \ldots, s_r \in A \setminus \mathbb{K}$ pairw. comm., $J \subset A$ ideal, $k \in \mathbb{N}$
Output: Gröbner basis B of $J \cap \mathbb{K}[s_1, \ldots, s_r]$ up to degree k

$G :=$ Gröbner basis of J w.r.t. an arbitrary ordering
$d := 0, \qquad B := \emptyset$
while $d \leq k$ **do**
  $M_d := \{s^\alpha \mid |\alpha| \leq d\}$
  if $\exists\, a_m \in \mathbb{K}$ with $\sum_{m \in M_d} a_m \operatorname{NF}(m, G) = 0$ **then**
    $f := \sum_{m \in M_d} a_m m$
    if $f \neq 0$ and $f \notin \langle B \rangle$ **then**
      $B := B \cup \{f\}$
    end if
  end if
  $d := d + 1$
end while
return B

Improvements

- If $p \in B$ mit $\operatorname{lm}(p) = m$ has been found, ignore all multiples of m in the following steps.
- If G is a Gröbner basis of J w.r.t. $\prec$, forget about

$$\{m \in M_d \mid \max_{\prec}(m' \in L(G) \cap M_d) \prec m\},$$

  since $p \in J \cap \mathbb{K}[s_1, \ldots, s_r] \Leftrightarrow \operatorname{lm}(p) \in L(G) \cap \mathbb{K}[s_1, \ldots, s_r]$.
- $\mathrm{NF}(m, G) = m$, if $m \notin L(G) \cap \mathbb{K}[s_1, \ldots, s_r]$.
- General stop criterion without degree bound?
  Only clear, if $J \cap \mathbb{K}[s_1, \ldots, s_r]$ is
  - zerodimensional, or
  - known to be a principal ideal.

s-parametric annihilator

The ring $\mathbb{K}[s, x, f^{-1}, f^s]$ becomes a $D \otimes_{\mathbb{K}} \mathbb{K}[s] = D[s]$-module via

$$\begin{aligned} x_i \bullet g(s,x) f^{s+j} &:= x_i \cdot g(s,x) f^{s+j}, \\ s \bullet g(s,x) f^{s+j} &:= s \cdot g(s,x) f^{s+j}, \\ \partial_i \bullet g(s,x) f^{s+j} &:= \frac{\partial g}{\partial x_i} f^{s+j} + (s+j) g \frac{\partial f}{\partial x_i} f^{s+j-1}. \end{aligned}$$

Definition

The ideal

$$\mathrm{Ann}(f^s) = \{ p \in D[s] \mid p \bullet f^s = 0 \}$$

is called the *(s-parametric) annihilator* of f^s.

Bernstein's Theorem

Theorem (Bernstein)

The Bernstein-Sato polynomial $b(s)$ of $f \in \mathbb{K}[x_1, \ldots, x_n] \setminus \mathbb{K}$ is the uniquely determined, monic polynomial of smallest degree in $\mathbb{K}[s]$, satisfying the identity

$$P \bullet f^{s+1} = b(s) \cdot f^s$$

for some $P \in D[s]$.

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Thus, we obtain an alternate algorithm to compute $b(s)$:

$$\begin{aligned} & (P \cdot f - b(s)) \bullet f^s = 0, \\ \text{i.e.} \quad & P \cdot f - b(s) \in \mathrm{Ann}(f^s), \\ \text{hence} \quad & \langle b(s) \rangle \subseteq \langle f \rangle + \mathrm{Ann}(f^s). \end{aligned}$$

Two algorithms

Algorithm (bfctAnn)

Input: $f \in \mathbb{K}[x_1, \ldots, x_n] \setminus \mathbb{K}$
Output: The Bernstein-Sato polynomial $b(s) \in \mathbb{K}[s]$ of f

$G :=$ Gröbner basis of $\langle f \rangle + \mathrm{Ann}(f^s)$ $\quad \triangleright G \subset D[s]$
$b(s) :=$ `pIntersect`(s, G) $\quad \triangleright s$ indeterminate
return $b(s)$

Algorithm (bfct)

Input: $f \in \mathbb{K}[x_1, \ldots, x_n] \setminus \mathbb{K}$
Output: The Bernstein-Sato polynomial $b(s) \in \mathbb{K}[s]$ of f

$w := (1, 0, \ldots, 0), \qquad s := t\partial_t$
$G :=$ `initialIdealW`(I_f, w) $\quad \triangleright G \subset D\langle t, \partial_t \rangle$
$B(s) :=$ `pIntersect`(s, G) $\quad \triangleright s$ polynomial
return $b(s) = (-1)^{\deg(B(s))} \cdot B(-s-1)$

Improvements to `bfctAnn`

- Since

$$\left(\mathrm{Ann}(f^s) + \langle f \rangle + \langle \frac{\partial f}{\partial x_i} \mid i = 1, \ldots, n \rangle\right) \cap \mathbb{K}[s] = \langle \frac{b(s)}{s+1} \rangle,$$

one saves one NF computation.

- For every

$$(y_0, y_1, \ldots, y_n) \in \mathrm{syz}_{\mathbb{K}[x]}(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$$

it holds that

$$y_0 \cdot s + \sum_{i=1}^{n} y_i \cdot \partial_i \in \mathrm{Ann}(f^s),$$

yielding elements of smallest possible degree in ∂_i.

SINGULAR wish list

- modular methods for G-algebras (Gröbner basis and intersection)
- Hilbert-driven Gröbner basis computation in G-algebras
- Gröbner walk techniques for G-algebras
- computation of $\mathrm{Ann}(f^s)$ in the kernel, making use of the identity $\mathrm{Ann}(f^s) \cap \mathbb{K}[x, s] = \{0\}$ in the involved Gröbner basis computation
- more efficient computation of NF of powers of polynomials
- generalization of `finduni` to polynomials and G-algebras
- fast Gaussian elimination
- generalization of (parts of) FGLM?

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Thank you!