Noncommutative Extensions of SINGULAR

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Outline

1. PLURAL: non-commutative *GR*-algebras
2. LOCAPAL: certain Ore localizations of *G*-algebras
3. LETTERPLACE: free associative algebras

To each subsystem: (draw a pic at this place)

- Description and maths behind
- Current status of the implementation
- Functionality under current development
- *Ambitious* future plans and wishes to SINGULAR

1. SINGULAR:PLURAL. Description I

Mission Statement

Provide most computer algebraic functionality for modules over the ubiqitous class of *GR*-algebras, that is factor algebras by two-sided ideals of *G*-algebras.

Properties

G-algebra is a Noetherian domain with PBW basis, that is any polynomial can be represented in terms of standard monomials $x^\alpha, \alpha \in \mathbb{N}^n$. SINGULAR polynomial data structures are unchanged.

Objects: by default, ideal or module stands for a **left** object. There are some functions for two-sided input like `twostd`, which convert them to left-sided structures.

1. SINGULAR:PLURAL. Description II

Construction of a G-algebra

- exact field $\mathbb{K}$ (available in SINGULAR), variables $x_1, \ldots, x_n$ and a global monomial ordering $\prec$, set $R := \mathbb{K}[x_1, \ldots, x_n]_\prec$
- define $C \in Mat(\mathbb{K}, n \times n)$ for **coefficients** $c_{ij} \neq 0$ for $i < j$
- define strictly upper triangular matrix $D \in Mat(R, n \times n)$ for **rests**
- ensure that $\forall 1 \leq i < j \leq n$, $\mathrm{lm}_\prec d_{ij} < x_i x_j$
- create noncommutative algebra with relations $x_j x_i = c_{ij} x_i x_j + d_{ij}$
- calling `def G = nc_algebra(C,D)` produces a noncommutative algebra with relations $x_j x_i = c_{ij} x_i x_j + d_{ij}$
- for G-algebra we need, that $\forall\ 1 \leq i < j < k \leq n$

$$c_{ik} c_{jk} \cdot d_{ij} x_k - x_k d_{ij} + c_{jk} \cdot x_j d_{ik} - c_{ij} \cdot d_{ik} x_j + d_{jk} x_i - c_{ij} c_{ik} \cdot x_i d_{jk} = 0.$$

1. SINGULAR:PLURAL. Status I

An integral part of SINGULAR since 2005.

Gröbner trinity is implemented:

- left Gröbner basis for a left submodule (`std, slimgb`)
- left syzygy module (`syz`)
- left transformation matrix between two left bases (`lift`)

- two-sided Gröbner basis of an ideal (`twostd`)

1. SINGULAR:PLURAL. Status II

Most of classical Gröbner basics are implemented:

- Ideal (resp. module) membership problem (`NF, reduce`)
- Intersection with subrings (elimination of variables) (`eliminate`)
- Intersection of ideals (resp. submodules) (`intersect`)
- Quotient and saturation of ideals (`quot`)
- Kernel of a module homomorphism (`modulo, moduloSlim`)
- Kernel of a ring homomorphism (`preimage`)
- Algebraic relations between pairwise commuting polynomials
- Gel'fand–Kirillov dimension of a module (`gkdim.lib`)
- $\mathbb{K}$-dimension and $\mathbb{K}$-basis (`vdim, kbase`)

1. SINGULAR:PLURAL. Status II

Extensions of functionality: 12 PLURAL libraries in the release (including the whole package for *D*-modules) and several more under development.

Homology-related functions

- free left resolutions (`res`, `nres`, `mres`, `minres`)
- graded Betti numbers for graded modules over graded algebras (`betti`)
- maps, related to *Hom* and $\otimes$; left-right transfer etc.

Under current development

NCHOMOLOG.LIB: $ncExt\_R(M, R)$, $ncExt(M, N)$, $ncTor(M, N)$, where M is a left module and N a centralizing $R-R$-bimodule.

BFUNVAR.LIB: Bernstein-Sato polynomial, operators, annihilators for affine varieties.

1. SINGULAR:PLURAL. Wishlist and future plans

1. Hilbert polynomial of graded ideals and modules
2. Hilbert-driven Gröbner basis
3. modular Gröbner basis, Gröbner trinity, Gröbner basics
4. smth like `trinity` command: `std`, `syz`, `lift` at once
5. implement Gel'fand–Kirillov dimension in kernel (now in a library)
6. implement `division` for PLURAL
7. special Gröbner basis for homogenized objects: compute $(I^h) : h^\infty$ by extracting h-content of any new element, entering the basis.
8. bisyzygies and biresolution (algorithms known)
9. Gröbner-free assistance in elimination (linear algebra)
10. standard basis and division algorithm for algebras like $\mathbb{K}(q_{ij})[x]_{\langle x \rangle}\langle \partial \mid \partial_j x_i = q_{ij} x_i \partial_j + \delta_{ij} \rangle$ (local (q-)Weyl algebras)

2. SINGULAR:PLURAL:LOCAPAL. Description

Algebras

Let $A = \mathbb{K}\langle x_1, \ldots, x_n, y_1, \ldots, y_m \mid \ldots \rangle$ be a G-algebra, such that e.g. x_i commute, that is $B = \mathbb{K}[x]$ is a subalgebra of A.
Consider $S = B \setminus \{0\}$, then it is known that S is an Ore set in A. Under some mild assumptions there exists a localization (two-sided ring of fractions) $S^{-1}A$. This ring's objects are computable.

Mission Statement

Provide most computer algebraic functionality for modules over such Ore localizations of G-algebras. With special emphasis on applications to operator algebras (differential, difference etc.) and to special functions.

2. LOCAPAL. Status I

Under beta-testing, preparation for the release.

There are configure flags `--with-ratGB`, `--enable-ratGB`.

Presentation of objects

- common notation $\mathbb{K}(X)\langle Y\rangle \cong (\mathbb{K}[X] \setminus \{0\})^{-1}\mathbb{K}[X]\langle Y\rangle$
- **only** polynomials can be used as coefficients instead of partially rational functions (native in this case)
- trick: due to the PBW properrty, one can show objects in the associated commutative ring $\mathbb{K}(X)[Y]$ (of course, without explicit arithmetics)
- for user-friendliness: two rings are needed (working ring $\mathbb{K}[X]\langle Y\rangle$ and an interface ring $\mathbb{K}(X)[Y]$)

Connected: released `jacobson.lib` of PLURAL, which computes Jacobson (generalized Smith) form for non-commutative PID's like an Ore localization of a G-algebra in $n+1$ variables by S (subalgebra in n variables $\setminus\{0\}$). The algorithm uses fraction-free operations.

2. LOCAPAL. Status II. Functions

- Initialization of Ore localization: `system("ratVar'',var1,vark)`, where var_1, var_k are variable names with $nvar(var_1) \leq nvar(var_k)$. All variables between var_1 and var_k will be of polynomial nature, the rest is treated as invertibles.
- Gröbner basis of a left submodule (hence syzygy can be computed by hands) `std(I)`, internally extracts polynomial content
- Gröbner basis of a left submodule, its $\mathbb{K}(x)$-dimension (integer or ∞) and both polynomial and fake rational presentations of an object: via the procedure `ratstd` of the library RATGB.LIB, calls polynomial `slimgb` with an elimination ordering
- normal form `system("intratNF'', what, with, nratvar)`, where `what` is a polynomial, `with` an ideal, `nratvar` an integer (number of the 1st variable of the polynomial block)

2. LOCAPAL. Future plans and wishes

- finish beta-testing (introduce `redTail` normal form, a kind of `division`, completely reduced Gröbner basis) and release that functionality
- Gröbner trinity: `syz, lift` are needed
- user-friendly interface concept & natural presentation of objects
- need: antiblock ordering (ring def/ringlist like $\omega(<_1, \prec_2)$)
- need: fast and furious linear algebra over $\mathbb{K}(X)$
- **crucial need:** ehnanced gcd over $\mathbb{K}[X]$ (e.g. content)
- Algorithms to be implemented
    1. free resolutions, `modulo`, left-right transfer (for hom. algebra)
    2. closure properties (annihilators of a sum/product/... of functions)
    3. integration (Zeilberger, Takayama, Chyzak and other algorithms with plenty of applications), which is used for summation as well
    4. factorization of operators in $\mathbb{K}(X)\langle Y\rangle$
- SAGE: interface for manipulating rational noncommutative expressions?

2. LOCAPAL. Future plans and wishes II

Generalization

Let $A = \mathbb{K}\langle x_1, \ldots, x_n, y_1, \ldots, y_m \mid \ldots \rangle$ be a G-algebra as before, but now we do not assume that x_i commutes with x_j. Instead we suppose that $B = \mathbb{K}\langle X \mid \ldots \rangle$ is a sub-G-algebra of A. Consider $S = B \setminus \{0\}$, then under some assumptions S is an Ore set in A and there exists a localization (two-sided ring of fractions) $S^{-1}A$.

Comments: even simple arithmetics in $S^{-1}A$ requires Gröbner bases, like e.g. the rewriting of a left fraction into a right one (basic multiplication). Ore property guarantees that such other-side fraction exists. Hence we look for $b \in A, t \in S$, such that $s^{-1}a = bt^{-1} \Leftrightarrow a\,\mathbf{t} = s\,\mathbf{b}$. Hence we need a right syzygy of (a, s) of a special form (since $t \in S$).

3. LETTERPLACE. Description I

Finitely presented algebra (f.p.a) $:= \mathbb{K}\langle X\rangle / T$, T is twosided ideal.

Mission Statement

There is a need of a computer algebra system for finitely presented algebras, which is able to be analogously rich in functionality as commutative systems are. That is one needs for a left submodule over an f.p.a: Gröbner trinity+basics, homological algebra and so on.

Letterplace paradigm uses special computations in **commutative** polynomial rings and allows to compute noncommutative Gröbner basis via Letterplace Gröbner basis algorithm. Hence it is possible to build the whole functionality on it.

3. LETTERPLACE. Description II

Notations

- X the set of "letters", $P = \mathbb{N} = \{0, 1, \ldots\}$ the set of "places"
- $K\langle X\rangle = \mathbb{K}\langle x_1, \ldots, x_n\rangle$ free associative algebra
- $(x_i|j) = (x_i, j)$ element of the product set $X \times P$
- $K[X|P]$ the polynomial ring in the (commutative) variables $(x_i|j)$ (inf. gen.)

There's one-to-one correspondence
$\langle X\rangle \ni x_{i_1} \cdots x_{i_n} \leftrightarrow (x_{i_1}|0) \cdots (x_{i_n}|n-1) \in V \subset K[X|P]$.

Theorem (La Scala, L., 2009)

The map $\iota : K\langle X\rangle \to V$ induces a 1-to-1 correspondence $\tilde{\iota}$ between graded two-sided ideals $I \subset K\langle X\rangle$ and letterplace ideals $J \subset K[X|P]$.

3. LETTERPLACE. Status

Released since 3-0-1

Gröbner basis of two-sided homogeneous ideal in free associative algebra $\mathbb{K}\langle x_1, \ldots, x_n \rangle$. The implementation consists of

- kernel part of SINGULAR, command `system("freegb",I,d,n))` (*I* ideal, *d* degbound, $n = nvars$
- the library `freegb.lib`: format converters (like `lp2lstr` and `lst2str`, auxiliary procedures `makeLetterplaceRing`, some user-friendliness like `freeGBasis` (uses vector presentation instead of letterplace)

Under beta-testing (the same command call)

Gröbner basis of arbitrary two-sided ideal in the free algebra

Under development

`freeqa.lib` by Grischa Studzinski: dimensions and tools

3. LETTERPLACE. Future plans and wishes

Finitely presented algebra (f.p.a) $:= \mathbb{K}\langle X\rangle / T$, T is twosided ideal.

Plans

- One-sided Gröbner bases over finitely presented algebras
- Gröbner basics for one- and two-sided modules over f.p.a
- One- and two-sided syzygies and resolutions over f.p.a.
- Hilbert function and dimension (like Gel'fand-Kirillov)
- Homological algebra (need resolution, modulo, opposite structure)

SAGE: interface for manipulating expressions containing words in a finite alphabet with coefficients, LETTERPLACE as back-engine.

Distant future: vector enumeration (code by Steve Linton) under SAGE umbrella?

Since letterplace computations take place in a commutative ring indeed, when will it be possible to incorporate coefficients over a ring like $\mathbb{Z}[a, b, c]$?

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