Computations over the free algebra

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Goal

Compute the $\mathbb{K}$-dimension and a $\mathbb{K}$-basis of a factor algebra of the free associative algebra

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The free algebra is realized as a letterplace-ring, a (finite) Gröbner basis of an ideal is given

Overview of the Procedures

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- Check, if the factor algebra is of finite dimension

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Using the Ufnarovskij graph this procedure returns yes, if the dimension of the factor algebra is infinite and no otherwise.

There is no need to compute the dimension or a basis here.

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Procedures may be combined to save computation time.

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The monomials are stored as integer vectors, where every variable corresponds to a number

Example

$$x \mapsto 1 \qquad y \mapsto 2 \qquad z \mapsto 3$$

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$$xyxzx \sim x(1)y(2)x(3)z(4)x(5) \mapsto (1,2,1,3,1)$$

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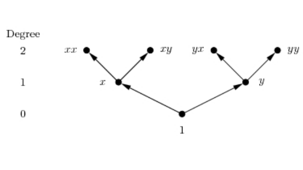
Also the procedure **sickle** allows to invoke all main procedures at once or partially. At the moment this is only possible if the input is an letterplace-ideal.

The Basis-Tree

The basis is build degree-wise: Starting in 1 we get all monomials with total degree 1 by right multiplication with all variables. All vertices representing normal words will be extended by right multiplication again. This procedure will give us a monomial basis.

Example

A basis for $\mathbb{K}\langle x, y\rangle$ up to degree 2



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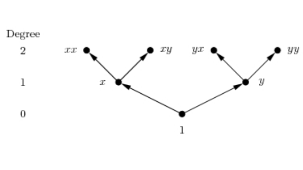
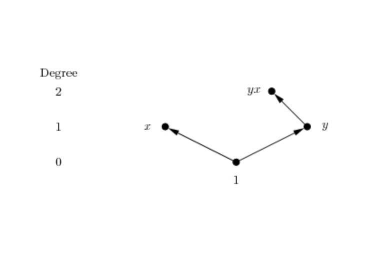


Note that every vertex has a unique path leading to it!
This is a difference to the commutative case.

Example

A basis for $\mathbb{K}\langle x, y\rangle / \langle xx, yy, xy\rangle$



Mistletoes

Observation

To each maximal vertex there is a unique path leading from 1 to this vertex and every vertex can be extended to a maximal one. Hence the whole tree is completely described by those vertices.

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They are called **mistletoes**.

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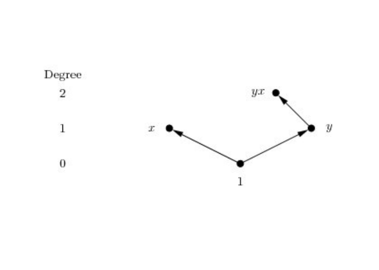

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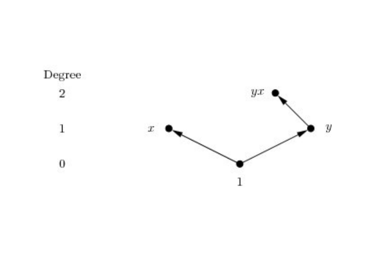

The mistletoes are $\{x, yx\}$.

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It is easy to get the $\mathbb{K}$-basis and$\backslash$or the $\mathbb{K}$-dimension out of the mistletoes: If ordered lexicographically one only needs to compare the mistletoes that are side by side. This is due to the fact that the lexicographically order is the natural order for the basis-tree.

Example

The Symmetric Group Algebra S_3

$\mathbb{K}\langle x, y \rangle$, $\boldsymbol{G} = \{x^2 - 1, y^2 - 1, xyx - yxy\}$, $x > y$

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$\dim_{\mathbb{K}}(\mathbb{K}\langle x, y\rangle / \langle \boldsymbol{G}\rangle) = 1 + \deg(xy) + \deg(yxy) = 6.$

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Basis

$B = \{1, x, y, xy, yx, yxy\}$

Possible Improvements

Improve the look-ups in the Gröbner basis:

- At the moment a list of intmats is searched for a common subword.
- The usage of a pointer would be very helpful to exploit the tree-structure.
- Comparison to analogues commutative procedures would be helpful.

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By including the procedure **reduce** we can perform arithmetic operations in a factor algebra. So the user should be able to set it as a basering.

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We want to include integer rings as coefficient rings in the letterplace-structure, for example $\mathbb{Z}[a, b, c]$, or rings in general. This can be applied to cyclotomic Hecke-algebras to prove certain conjectures.

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Improve the letterplace structure to an effective, fast and user-friendly subsystem of Singular