Trends in Computer Algebraic Analysis

Viktor Levandovskyy
Lehrstuhl D für Mathematik, RWTH Aachen, Germany

28.06. ACA 2012, Sofia, Bulgaria

What is computer algebraic Analysis?

Algebraization as a Trend

Algebra: Ideas, Concepts, Methods, Abstractions

Computer algebra works with algebraic concepts in a (semi-)algorithmic way at three levels:

1. Theory: Methods of Algebra in a constructive way
2. Algorithmics: Algorithms (or procedures) and their Correctness, Termination and Complexity results (if possible)
3. Realization: Implementation, Testing, Benchmarking, Challenges; Distribution, Lifecycle, Support and software-technical aspects

What is computer-algebraic Analysis?

Algebraic Analysis

1. As a notion, it arose in 1958 in the group of Mikio Sato (Japan)
2. Main objects: systems of linear partial DEs, generalized functions
3. Main idea: study systems and generalized functions in a coordinate-free way (i. e. by using modules, sheaves, categories, localizations, homological algebra, ...)
4. Keywords: D-Modules, (sub-)holonomic D-Modules, regular resp. irregular holonomic D-Modules
5. Interplay: singularity theory, special functions, ...

Other ingredients: symbolic algorithmic methods for discrete resp. continuous problems (like symbolic summation, symbolic integration etc.)

Some big names in Computer-algebraic Analysis

- W. Gröbner and B. Buchberger: Gröbner bases and constructive ideal/module theory
- O. Ore: Ore Extension and Ore Localization
- I. M. Gel'fand and A. Kirillov: GK-Dimension
- B. Malgrange: M. isomorphism, M. ideal, ...
- J. Bernstein, M. Sato, M. Kashiwara et al.: D-Modules theory
- ...

Operator algebras: partial Classification

Let K be a computable field, that is $(+, -, \cdot, :)$ can be performed algorithmically.
Moreover, let $\mathcal{F}$ be a K-vector space ("function space").

Let x be a local coordinate in $\mathcal{F}$. It induces a K-linear map $X : \mathcal{F} \to \mathcal{F}$, i. e. $X(f) = x \cdot f$ for $f \in \mathcal{F}$.
Moreover, let $\mathfrak{o}_x : \mathcal{F} \to \mathcal{F}$ be a K-linear map.
Then, in general, $\mathfrak{o}_x \circ X \neq X \circ \mathfrak{o}_x$, that is $\mathfrak{o}_x(x \cdot f) \neq x \cdot \mathfrak{o}_x(f)$ for $f \in \mathcal{F}$.

The **non-commutative relation** between $\mathfrak{o}_x$ and X can be often read off by analyzing the properties of $\mathfrak{o}_x$ like, for instance, the product rule.

Classical examples: Weyl algebra

Let $f : \mathbb{C} \to \mathbb{C}$ be a differentiable function and $\partial(f(x)) := \frac{\partial f}{\partial x}$.

Product rule tells us that $\partial(x\ f(x)) = x\ \partial(f(x)) + f(x)$, what is translated into the following relation between operators

$$(\partial \circ x - x \circ \partial - 1)\ (f(x)) = 0.$$

The corresponding operator algebra is the 1st **Weyl algebra**

$$D_1 \ = \ K\langle x, \partial \mid \partial x = x\partial + 1\rangle.$$

Classical examples: shift algebra

Let g be a sequence in discrete argument k and $\mathbf{s}$ is the shift operator $\mathbf{s}(g(k)) = g(k+1)$. Note, that $\mathbf{s}$ is multiplicative.

Thus $\mathbf{s}(kg(k)) = (k+1)g(k+1) = (k+1)\mathbf{s}(g(k))$ holds.

The operator algebra, corr. to $\mathbf{s}$ is the 1st **shift algebra**

$$S_1 \;=\; K\langle k, s \mid sk = (k+1)s = ks + s\rangle.$$

Intermezzo

For a function in differentiable argument x and in discrete argument k the natural operator algebra is

$$A = D_1 \otimes_K S_1 = K\langle x, k, \partial_x, s_k \mid \partial_x x = x\partial_x + 1,\ s_k k = ks_k + s_k,$$

$$xk = kx,\ xs_k = s_k x,\ \partial_x k = k\partial_x, \partial_x s_k = s_k\partial_x\rangle.$$

Examples form the q-World

Let $k \subset K$ be fields and $q \in K^*$.
In q-calculus and quantum algebras three situations are common for a fixed k: (a) $q \in k$, (b) q is a root of unity over k, and (c) q is transcendental over k and $k(q) \subseteq K$.

Let $\partial_q(f(x)) = \frac{f(qx)-f(x)}{(q-1)x}$ be a q-differential operator.
The corr. operator algebra is the 1st q-**Weyl algebra**

$$D_1^{(q)} = K\langle x, \partial_q \mid \partial_q x = q \cdot x\partial_q + 1 \rangle.$$

The 1st q-**shift algebra** corresponds to the q-shift operator $\mathbf{s}_q(f(x)) = f(qx)$:

$$K_q[x, s_q] = K\langle x, s_q \mid s_q x = q \cdot x s_q \rangle.$$

Two frameworks for bivariate operator algebras

Algebra with linear (affine) relation

Let $q \in K^*$ and $\alpha, \beta, \gamma \in K$. Define

$$\mathcal{A}^{(1)}(q, \alpha, \beta, \gamma) := K\langle x, y \mid yx - q \cdot xy = \alpha x + \beta y + \gamma \rangle$$

Because of **integration operator** $\mathcal{I}(f(x)) := \int_0^x f(t)dt$, obeying the relation $\mathcal{I}\, x - x\, \mathcal{I} = -\mathcal{I}^2$ we need yet more general framework.

Algebra with nonlinear relation

Let $N \in \mathbb{N}$ and $c_0, \ldots, c_N, \alpha \in K$. Then $\mathcal{A}^{(2)}(q, c_0, \ldots, c_N, \alpha)$ is
$K\langle x, y \mid yx - q \cdot xy = \sum_{i=1}^n c_i y^i + \alpha x + c_0 \rangle$ or
$K\langle x, y \mid yx - q \cdot xy = \sum_{i=1}^n c_i x^i + \alpha y + c_0 \rangle$.

Theorem (L.–Koutschan–Motsak, 2011)

$\mathcal{A}^{(1)}(q,\alpha,\beta,\gamma) = K\langle x,y \mid yx - q\cdot xy = \alpha x + \beta y + \gamma\rangle$,
where $q \in K^*$ *and* $\alpha,\beta,\gamma \in K$
is isomorphic to the 5 following **model algebras***:*

1. $K[x,y]$,
2. *the 1st Weyl algebra* $D_1 = K\langle x,\partial \mid \partial x = x\partial + 1\rangle$,
3. *the 1st shift algebra* $S_1 = K\langle x,s \mid sx = xs + s\rangle$,
4. *the 1st q-commutative algebra* $K_q[x,s] = K\langle x,s \mid sx = q\cdot xs\rangle$,
5. *the 1st q-Weyl algebra* $D_1^{(q)} = K\langle x,\partial \mid \partial x = q\cdot x\partial + 1\rangle$.

Theorem (L.–Makedonsky–Petravchuk, unpublished)

For $N \geq 2$ *and* $c_0, \ldots, c_N, \alpha \in K$, $\mathcal{A}^{(2)}(q, c_0, \ldots, c_N, \alpha)$ $= K\langle x, y \mid yx - q \cdot xy = \sum_{i=1}^{N} c_i y^i + \alpha x + c_0 \rangle$ *is isomorphic to . . .*

1. $K_q[x, s]$ *or* $D_1^{(q)}$, *if* $q \neq 1$,
2. $S_1 = K\langle x, s \mid sx = xs + s \rangle$, *if* $q = 1$ *and* $\alpha \neq 0$,
3. $K\langle x, y \mid yx = xy + f(y) \rangle$, *where* $f \in K[y]$ *with* $\deg(f) = N$, *if* $q = 1$ *and* $\alpha = 0$.

$K\langle x, y \mid yx = xy + f(y) \rangle \cong K\langle z, w \mid wz = zw + g(w) \rangle$ *if and only if* $\exists \lambda, \nu \in K^*$ *and* $\exists \mu \in K$, *such that* $g(t) = \nu f(\lambda t + \mu)$ *(in particular* $\deg(f) = \deg(g)$*).*

Example: Legendre's differential equation

$$(x^2 - 1){P''}_n(x)^2 + 2x{P'}_n(x) - n(1+n)P_n(x) = 0$$

- x is differentiable with ∂_x as corr. operator
- if $n \in \mathbb{Z}$, n is discretely shiftable with s_n as corr. op.

- then there is a recursive formula of Bonnet (order 2 in s_n)

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0.$$

Example: Legendre's differential equation

$$\mathfrak{D} := K\langle n, s_n \mid s_n n = n s_n + s_n\rangle \otimes_K K\langle x, \partial_x \mid \partial_x x = x\partial_x + 1\rangle.$$

From the system of equations

$$\begin{aligned} (x^2-1){P''}_n(x)^2 + 2x{P'}_n(x) - n(1+n)P_n(x) &= 0, \\ (n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) &= 0. \end{aligned}$$

one obtains the matrix $P \in \mathfrak{D}^{2\times 1}$; thus $M = \mathfrak{D}/\mathfrak{D}^{1\times 2}P$ and

$$\begin{bmatrix} (x^2-1)\partial_x^2 + 2x\partial_x - n(1+n) \\ (n+2)s_n^2 - (2n+3)xs_n + n + 1 \end{bmatrix} \bullet P_n(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

With the help of Gröbner bases: a minimal generating set of the left ideal P contains a *compatibility condition*

$$(n+1)s_n\partial_x - (n+1)x\partial_x - (n+1)^2 \equiv (n+1)(s_n\partial_x - x\partial_x + n + 1).$$

Solutions and Malgrange isomorphism

Let $\mathcal{F}$ be K-vector space and a left $\mathfrak{O}$-module, then

$$\mathrm{Sol}_{\mathfrak{O}}(P, \mathcal{F}) := \{f \in \mathcal{F}^{m \times 1} \;:\; Pf = 0\}.$$

Malgrange Isomorphism

There exists an isomorphism of abelian groups (and K-vector spaces)

$$\mathrm{Hom}_{\mathfrak{O}}(M, \mathcal{F}) = \mathrm{Hom}_{\mathfrak{O}}(\mathfrak{O}^{1 \times m}/\mathfrak{O}^{1 \times \ell} P, \mathcal{F}) \;\cong\; \mathrm{Sol}_{\mathfrak{O}}(P, \mathcal{F}),$$

$$(\phi : M \to \mathcal{F}) \;\mapsto (\phi(e_1), \ldots, \phi(e_m)) \in \mathcal{F}^{m \times 1}.$$

Question: What is better to use in modeling: operator algebras with constant or with polynomial coefficients?

Answer: with polynomial coefficients.

Theorem (Zerz–L.–Schindelar, 2011)

Let $K = \mathbb{R}$, $p_i \in K[x_1, \ldots, x_n]^\ell$ *and* $V = Kp_1 + \cdots + Kp_m$. *Let* $\mathfrak{D}$ *be the n-th Weyl algebra and* $\operatorname{Ann}_{\mathfrak{D}}(V) \subset \mathfrak{D}$ *be the minimal left ideal of equations, having* $p_1, \ldots, p_m$ *as solutions. Then*

$$\operatorname{Sol}_{\mathfrak{D}}(\,\mathfrak{D}/\operatorname{Ann}_{\mathfrak{D}}(V),\; C^\infty(\mathbb{R}^\ell)) = V.$$

Ore Localization

Let A be a Noetherian domain and
S a multiplicatively closed set in A, where $0 \notin S$.

A commutative implies the existence of $S^{-1}A$.
A non-commutative: if S is an Ore set in A, $\exists\ S^{-1}A$.

Ore condition

For all $s_1 \in S,\ r_1 \in A$ there exist $s_2 \in S,\ r_2 \in A$, such that

$$\mathbf{r_1} s_2 = \mathbf{s_1} r_2, \quad \text{that is} \quad s_1^{-1} r_1 = r_2 s_2^{-1}.$$

The **Ore localization** of A w.r.t S is a Ring $A_S := S^{-1}A$ together with an injective homomorphism $\phi : A \to A_S$, such that

(i) for all $s \in S$ $\phi(s)$ is a unit in A_S,

(ii) for all $f \in A_S$, $\exists a \in A, s \in S$ s. t. $f = \phi(s)^{-1}\phi(a)$.

Example

- Let $S = A^* := A \setminus \{0\}$. Then $S^{-1}A \cong \mathrm{Quot}(A)$.
- If $K \subsetneq S \subsetneq A^*$, then $A \to A_S \to \mathrm{Quot}(A)$,
- For any S, $S^{-1}A$ is an A-module (not finitely generated),
- in general A is not an $S^{-1}A$-module.

S^{-1} gives rise to a functor A-mod $\to S^{-1}A$-mod.

Polynomial or rational Coefficients?

With Ore localization we can recognize, that

$$K(X)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_m; \sigma_m, \delta_m] \cong (K[X] \setminus \{0\})^{-1} K\langle X, \partial_1, \ldots, \partial_m \mid \ldots \rangle$$

and the functor S^{-1} connects categories of modules.

Algorithmic aspects

Algorithmic computations over $S^{-1}A$ can be replaced **completely** with computations over A.
Keywords: **integer strategy, fraction-free strategy**.
For instance, a Gröbner basis theory over A induces a Gröbner basis theory over $S^{-1}A$.

There are implementations for the rational localization $K(X)\langle \partial_1, \ldots \rangle$.

Polynomial or rational Coefficients?

Let A be a K-algebra and $S \subset A$ a mult. closed Ore set in A. Moreover, let

- $M \cong A^n / A^m P$, a finitely presented left A-module,
- $\mathcal{F}$ a left A-module,
- $\widetilde{\mathcal{F}}$ a left $S^{-1}A$-module.

- $S^{-1}M \;\cong\; (S^{-1}A)^n / (S^{-1}A)^m P$.

$$\mathrm{Sol}_A(M, \widetilde{\mathcal{F}}) \cong \mathrm{Sol}_{S^{-1}A}(S^{-1}M, \widetilde{\mathcal{F}}),$$

- Assume $\widetilde{\mathcal{F}} \subset \mathcal{F}$ as left A-modules. Then

$$\mathrm{Sol}_A(M, \widetilde{\mathcal{F}}) \subseteq \mathrm{Sol}_A(M, \mathcal{F}),$$

Let $\mathcal{G} \subset \mathcal{F}$ be function spaces, i. e. K-vector spaces and left $\mathfrak{O}$-modules over a fixed operator algebra $\mathfrak{O}$.

Let $f \in \mathcal{F}$, then $\operatorname{Ann}_{\mathfrak{O}}^{\mathcal{F}} f := \{p \in \mathfrak{O} : pf = 0 \in \mathcal{F}\}$ is the **annihilator** of f, which is a left ideal in $\mathfrak{O}$.

Let $I \subsetneq \mathfrak{O}$ be an ideal and suppose, that $\dim_K(\mathcal{G}) < \infty$.
I is called **the complete annihilator of** $\mathcal{G}$ **over** $\mathfrak{O}$, if the following properties hold:
"most powerful": if $\forall g \in \mathcal{G}\ rg = 0$ for $r \in \mathfrak{O}$, then $r \in I$
"unfalsified": $\operatorname{Sol}_{\mathfrak{O}}(\mathfrak{O}/I,\ \mathcal{F}) = \mathcal{G}$.

The complete annihilator program

There exists no general algorithm, which can compute the complete annihilator program of f over $\mathfrak{O}$ (where $\mathfrak{O}$ is an algebra with polynomial coefficients).

Therefore one investigates some classes of f and develops special methods for the classes.

One of successes is **computational D-module theory**, where among other one can compute the complete annihilators of

$$f(\mathbf{x}, \mathbf{s}) = f_1(x_1, \ldots, x_n)^{s_1} \cdot \ldots \cdot f_m(x_1, \ldots, x_m)^{s_m}, \ f_i(\mathbf{x}) \in K[x_1, \ldots, x_n]$$

$$\text{over } \ \mathfrak{O} = \bigotimes_{i=1}^{n}{}_K K\langle x_i, \partial_i \mid \partial_i x_i = x_i \partial_i + 1 \rangle \otimes_K K[s_1, \ldots, s_m]$$

in an algorithmic way. There are implementations.

Some computational D-module theory

Let $D_n(K) = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \mid \partial_j x_i = x_i \partial_j + \delta_{ij} \rangle$ be the n-th Weyl algebra and $D_n[s] = D_n \otimes_K K[s]$.

Theorem (J. Bernstein, 1971/72)

Let $f(x) \in \mathbb{C}[x_1, \dots, x_n]$. Then there exist

- *an operator $P(s) \in D_n \otimes_{\mathbb{C}} \mathbb{C}[s]$,*
- *a monic polynomial $0 \neq b_f(s) \in \mathbb{C}[s]$ of the smallest degree (called* **the global Bernstein-Sato polynomial***),*

such that for arbitrary s the following functional equation holds

$$P(s) \bullet f^{s+1} = b_f(s) \cdot f^s.$$

Let $\mathrm{Ann}_{D[s]}(f^s) = \{Q(s) \in D[s] \mid Q(s) \bullet f^s = 0\} \subset D[s]$ be the annihilator, then $P(s)f - b_f(s) \in \mathrm{Ann}_{D[s]}(f^s)$ holds.

Dimensions

- Generalized Krull dimension is called Krull-Rentschler-Gabriel dimension; not algorithmic
- global homological dimension (of an algebra) resp. projective dimension (of a module); for modules: algorithmic (relatively expensive), implemented
- Gel'fand-Kirillov Dimension; algorithmic (relatively cheap), implemented; plays the role of Krull dimension in non-commutative case.

GK dimension and its properties

Let A be a K-algebra, generated by $x_1, \ldots, x_m$.

Degree filtration

Let $V = Kx_1 \oplus \ldots \oplus Kx_m$ be a vector space.
Set $V_0 = K$, $V_1 = K \oplus V$ and $V_{k+1} = V_k \oplus V^{k+1}$.
Let $M_0 \subset M$, $\dim_K M_0 < \infty$ and $AM_0 = M$.
An ascending filtration on M is defined via $\{H_d := V_d M_0, d \geq 0\}$.

The **Gel'fand-Kirillov dimension** of M is defined as follows

$$\mathrm{GKdim}(M) \;=\; \limsup_{d \to \infty} \log_d(\dim_K H_d)$$

Gel'fand-Kirillov-Dimension: examples

Let $\deg x_i := 1$, $V_d := \{f \mid \deg f = d\}$ and $V^d := \{f \mid \deg f \leq d\}$.

Lemma

Let A be a K-algebra and a domain. If the standard filtration on A is compatible with the PBW Basis $\{x^\alpha \mid \alpha \in \mathbb{N}^m\}$, then $\mathrm{GKdim}(A) = m$.

$$\dim V_d = \binom{d+m-1}{m-1}, \dim V^d = \binom{d+m}{m}.$$

Thus $\binom{d+m}{m} = \frac{(d+m)\ldots(d+1)}{m!} = \frac{d^m}{m!} + \ldots$ and

$$\mathrm{GKdim}(A) = \limsup_{d\to\infty} \log_d \binom{d+m}{m} = m.$$

Gel'fand-Kirillov-Dimension: examples and properties

$\mathrm{GKdim}(K\langle x_1, \ldots, x_n \rangle) = \infty$ for $n \geq 2$.
$\mathrm{GKdim}(K[[x_1, \ldots, x_n]]) = \infty$ for $n \geq 1$, when $|K| = \infty$.

Properties

- $\mathrm{GKdim}\, M = \sup\{\mathrm{GKdim}(N) : N \in A - mod,\ N \subseteq M\}$,
- $\mathrm{GKdim}\, A = \sup\{\mathrm{GKdim}(S) : S \subseteq A,\ S \text{ fin. gen. subalgebra}\}$

Over G-algebras (and even more) there are algorithms and implementations for the computation of the GK dimension of finitely presented modules.

Elimination and dimension

Lemma

Let $I \subset A$ be a left ideal and $S \subset A$ be a finitely generated subalgebra. Then

- $I \cap S = 0$ *implies* $\operatorname{GKdim} A/I \geq \operatorname{GKdim} S$,
- $\operatorname{GKdim} A/I < \operatorname{GKdim} S$ *implies* $I \cap S \neq 0$.

Recall: Bernstein's inequality

Let A be the n-th Weyl algebra over K with $\operatorname{char} K = 0$ (thus $\operatorname{GKdim}(A) = 2n$) and $0 \neq M$ be an A-module, then $\operatorname{GKdim} M \geq n$.

Elimination and dimension

Classically: for a function f, $\operatorname{Ann}_{\mathfrak{D}} f \cap K[x_1, \ldots, x_n] = 0$. Hence $\operatorname{GKdim} \mathfrak{D} / \operatorname{Ann}_{\mathfrak{D}} f \geq n$.

Proposition

Let $\mathfrak{D} = \bigotimes_{i=1}^{n} \mathfrak{D}_i$, $\mathfrak{D}_i = K\langle x_i, \mathfrak{o}_i \mid \ldots \rangle$. Moreover, let $I \subset \mathfrak{D}$ and $\operatorname{GKdim} \mathfrak{D}/I = m$. Then for any finitely generated subalgebra $S \subset \mathfrak{D}$ of GK dimension $\geq m+1$ one has $I \cap S \neq 0$.

Application: For I such that $\operatorname{GKdim} \mathfrak{D}/I = n$ we guarantee that $2n - (n+1) = n-1$ variables can be eliminated from I, for instance

- all but one operators,
- all but one coordinate variables.

Elimination, dimension and localization

Suppose that $I, S \subset \mathfrak{D}$ are such that

- S is an Ore set in $\mathfrak{D}$ (so $S^{-1}\mathfrak{D}$ exists)
- $S^{-1}\mathfrak{D}I \neq S^{-1}\mathfrak{D}$ (i. e. I is proper in the localized algebra).

Then $I \cap S = 0$, what implies $\operatorname{GKdim} \mathfrak{D}/I \geq \operatorname{GKdim} S$.

Note, that for every $J \in S^{-1}\mathfrak{D}$ there exists $I \in \mathfrak{D}$ such that $S^{-1}\mathfrak{D}I = S^{-1}\mathfrak{D}J$. In general

$\operatorname{GKdim} S^{-1}\mathfrak{D}/S^{-1}\mathfrak{D}L \geq \operatorname{GKdim} \mathfrak{D}/L$.

Dimension function

Let A be a Noetherian algebra. A dimension function δ assigns a value $\delta(M)$ to each finitely generated A-module M and satisfies the following properties:

(i) $\delta(0) = -\infty$.

(ii) If $0 \to M' \to M \to M'' \to 0$ is exact sequence, then $\delta(M) \geq \sup\{\delta(M'), \delta(M'')\}$ with equality if the sequence is split.

(iii) If P is a (two-sided) prime ideal with $PM = 0$ and M is a torsion module over A/P, then $\delta(M) \leq \delta(A/P) - 1$.

- generalized Krull dimension is an exact dimension function
- Gel'fand-Kirillov dimension is a dimension function, not always exact

Purity w.r.t dimension function

Let A be a K-algebra and δ a dimension function on A-mod.
A module $M \neq 0$ is δ-**pure**, if $\forall 0 \neq N \subseteq M$, $\delta(N) = \delta(M)$.

- Purity is a useful weakening of the concept of simplicity of a module.
- Unlike simplicity, the purity (w.r.t a dimension function) is algorithmically decidable over many common algebras.

M. Barakat, A. Quadrat: Algorithms for the computation of the purity filtration of a module with $\delta =$ homological co-grade; there are several implementations.

Purity with respect to a dimension function

Lemma (L.)

Let A be a K-algebra and δ a dimension function on A-mod. Moreover, let $0 \neq M_1, M_2 \subset N$ be two δ-pure modules with $\delta(M_1) = \delta(M_2)$. Then

the set of δ-pure submodules (of the same dimension) of a module is a lattice, i. e.

1. $M_1 \cap M_2$ *is either 0 or it is δ-pure with* $\delta(M_1 \cap M_2) = \delta(M_1)$,
2. $M_1 + M_2$ *is δ-pure with* $\delta(M_1 + M_2) = \delta(M_1)$.

Identities, Elimination, Purity Filtration

Consider the mixed system, annihilating Legendre polynomials

$$\mathfrak{O} = K\langle n, s_n \mid s_n n = n s_n + s_n\rangle \otimes_K K\langle x, \partial_x \mid \partial_x x = x\partial_x + 1\rangle.$$

$$M = \mathfrak{O}/P,$$

$$P = \langle (x^2-1)\partial_x^2 + 2x\partial_x - n(1+n),\ (n+2)s_n^2 - (2n+3)x s_n + n + 1,$$
$$(n+1)(s_n\partial_x - x\partial_x + n + 1)\rangle.$$

$$\operatorname{GKdim} \mathfrak{O} = 4, \quad \operatorname{GKdim} M = 2, \quad t(M) = M = \mathfrak{O}/P.$$

The purity filtration of $M = t(M)$ is $0 \subsetneq M_3 \subsetneq M_2 = M$,

$$M_3 \;\cong \mathfrak{O}/\langle n+1, s_n, \partial_x\rangle \quad \text{with} \quad \operatorname{GKdim} M_3 = 1.$$

What are the solutions $g(n,x)$ of this system?

Since $\partial_x(g) = 0$, one has $g(n,x) = g(n)$.
however, $g(n)$ should not be identically zero:
in case $n \in \{-1, 0, 1, \ldots\}$, one can select $g(-1) \in K$ arbitrary (step of the jump function).

Localization

The ideal $\langle n+1, s_n\rangle$ is two-sided and maximal. Hence the submodule M_3 vanishes under any Ore localization in $K\langle n, s_n \ldots\rangle$, for instance when n invertible or s_n invertible (then s_n^{-1} is present and therefore should $n \in \mathbb{Z}$ hold).

The purity filtration of $M = t(M)$ is $0 \subsetneq M_3 \subsetneq M_2 = M$.
The pure part of GK dimension 2 is $t(M)/M_3 \cong$

$$\mathfrak{D}/\langle (x^2-1)\partial_x^2 + 2x\partial_x - n(1+n),\ (n+2)S_n^2 - (2n+3)xS_n + n + 1,$$

$$(1-x^2)\partial_x + (n+1)S_n - (n+1)x\rangle.$$

For further investigations of M over localizations w.r.t. n or S_n one should then take the simplified equations.

Elimination leads to new identities

The elimination property guarantees, that 1 arbitrary variable can be eliminated; so one gets for instance

$$(n+1)(n+2) \cdot \big((S_n^2 - 1)\partial_x - (2n+3)S_n\big) \bullet P_n(x) = 0,$$

$$(1-x^2) \cdot \big((S_n^2 - 2xS_n + 1)\partial_x - S_n)\big) \bullet P_n(x) = 0.$$

The hypergeometric series is defined for $|z| < 1$ and $-c \notin \mathbb{N}_0$ as follows:

$$_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

We derive two annihilating ideals from $_2F_1(a, b, c; z)$:

- J_a which does not contain a,
- J_c which does not contain c,

and analyze corresponding modules for purity.

Case J_a

The ideal in $\mathfrak{O} = K[b,c,z]\langle s_b, s_c, \partial_z \mid \ldots \rangle$ is generated by:

$$bcSb - czDz - bc$$
$$bSbSc - bSc + cSc - c$$
$$bSb^2 - zSbDz - bSb + Sb^2 - Sb$$
$$b^2Sb - bzDz - b^2 + bSb - zDz - b$$
$$bzSbDz - z^2Dz^2 - bzDz - bSbDz + zDz^2 - bSb + bDz + b + Dz$$

Let $M = M_a = \mathfrak{O}/J_a$. Then $\operatorname{GKdim} \mathfrak{O} = 6, \operatorname{GKdim} M = 4$.

The purity filtration of $M = t(M)$

$0 \subsetneq M_5 = M_4 \subsetneq M_3 = M_2 = M_1 = M$, where

$M/M_5 \cong \mathfrak{O}/\langle bSb - zDz - b, zDzSc + cSc - c \rangle$, $\operatorname{GKdim} M/M_5 = 4$

The purity filtration of $M = t(M)$

. . . and

$$M_5 \cong \mathfrak{O}/\langle c, Sb, b+1, zDz - Dz - 1\rangle, \ \text{GKdim}\, M_5 = 2.$$

The solutions can be read off:

$$\delta_{c,0} \cdot \delta_{b,-1} \cdot (\ln(z-1) + k_0), \ k_0 \in K$$

Case J_c

The ideal in $\mathfrak{D} = K[a,b,z]\langle s_a, s_b, \partial_z \mid \ldots\rangle$ is generated by:

$$
\begin{aligned}
aSa - bSb - a + b \\
bSb^2 - SbzDz - bSb + Sb^2 - Sb \\
b^2Sb - bzDz - b^2 + bSb - zDz - b \\
abSb - azDz - ab + bSb - zDz - b \\
bSbzDz - z^2Dz^2 - bSbDz - bzDz + zDz^2 - bSb + bDz + b + Dz
\end{aligned}
$$

Let $M = M_c = \mathfrak{D}/J_c$. Then $\mathrm{GKdim}\,\mathfrak{D} = 6, \mathrm{GKdim}\,M = 4.$

The purity filtration of $M = t(M)$

$0 \subsetneq M_6 = M_5 = M_4 \subsetneq M_3 = M_2 = M_1 = M$, where

$M/M_6 \cong \mathfrak{D}/\langle bSb - zDz - b, aSa - zDz - a\rangle$, $\mathrm{GKdim}\,M/M_6 = 4.$

The purity filtration of $M = t(M)$

... and

$$M_6 \cong \mathfrak{O}/\langle Sb, b+1, Sa, a+1, zDz - Dz - 1\rangle,\ \text{GKdim}\, M_6 = 2.$$

The solutions:

$$\delta_{a,-1} \cdot \delta_{b,-1} \cdot (\ln(z-1) + k_0),\ k_0 \in K$$

Software

D-modules and algebraic analysis:

- KAN/SM1 by N. Takayama et al.
- `D-modules` package in MACAULAY2 by A. Leykin and H. Tsai
- RISA/ASIR by M. Noro et al.
- OREMODULES package suite for MAPLE by D. Robertz, A. Quadrat et al.
- SINGULAR:PLURAL with a *D*-module suite; by V. L. et al.

holonomic and *D*-finite functions:

- GROEBNER, ORE ALGEBRA, MGFUN, ... by F. Chyzak
- HOLONOMICFUNCTIONS by C. Koutschan
- SINGULAR:LOCAPAL (under development) by V. L. et al.

Thanks for your attention!

RWTH AACHEN UNIVERSITY

SINGULAR PLURAL

http://www.singular.uni-kl.de/