Trends in Computer Algebraic Analysis

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What is computer algebraic Analysis?

Algebraization as a Trend
Algebra: Ideas, Concepts, Methods, Abstractions

Computer algebra works with algebraic concepts in a (semi-)algorithmic way at three levels:

1. Theory: Methods of Algebra in a constructive way
2. Algorithmics: Algorithms (or procedures) and their Correctness, Termination and Complexity results (if possible)
3. Realization: Implementation, Testing, Benchmarking, Challenges; Distribution, Lifecycle, Support and software-technical aspects
What is computer-algebraic Analysis?

Algebraic Analysis

1. As a notion, it arose in 1958 in the group of Mikio Sato (Japan)
2. Main objects: systems of linear partial DEs, generalized functions
3. Main idea: study systems and generalized functions in a coordinate-free way (i.e. by using modules, sheaves, categories, localizations, homological algebra, . . .)
4. Keywords: $D$-Modules, (sub-)holonomic $D$-Modules, regular resp. irregular holonomic $D$-Modules
5. Interplay: singularity theory, special functions, . . . .

Other ingredients: symbolic algorithmic methods for discrete resp. continuous problems (like symbolic summation, symbolic integration etc.)
Some big names in Computer-algebraic Analysis

- W. Gröbner and B. Buchberger: Gröbner bases and constructive ideal/module theory
- O. Ore: Ore Extension and Ore Localization
- I. M. Gel’fand and A. Kirillov: GK-Dimension
- B. Malgrange: M. isomorphism, M. ideal, ...
- J. Bernstein, M. Sato, M. Kashiwara et al.: $D$-Modules theory
- ...
Let $K$ be a computable field, that is $(+,−,·,:)$ can be performed algorithmically.
Moreover, let $\mathcal{F}$ be a $K$-vector space ("function space").

Let $x$ be a local coordinate in $\mathcal{F}$. It induces a $K$-linear map $X : \mathcal{F} → \mathcal{F}$, i.e. $X(f) = x · f$ for $f ∈ \mathcal{F}$.
Moreover, let $\sigma_x : \mathcal{F} → \mathcal{F}$ be a $K$-linear map.
Then, in general, $\sigma_x ∘ X \neq X ∘ \sigma_x$, that is $\sigma_x(x · f) \neq x · \sigma_x(f)$ for $f ∈ \mathcal{F}$.

The non-commutative relation between $\sigma_x$ and $X$ can be often read off by analyzing the properties of $\sigma_x$ like, for instance, the product rule.
Let $f : \mathbb{C} \to \mathbb{C}$ be a differentiable function and $\partial(f(x)) := \frac{\partial f}{\partial x}$.

Product rule tells us that $\partial(x \, f(x)) = x \, \partial(f(x)) + f(x)$, what is translated into the following relation between operators

$$(\partial \circ x - x \circ \partial - 1)(f(x)) = 0.$$

The corresponding operator algebra is the 1st Weyl algebra

$$D_1 = K\langle x, \partial \mid \partial x = x\partial + 1 \rangle.$$
Classical examples: shift algebra

Let \( g \) be a sequence in discrete argument \( k \) and \( s \) is the shift operator \( s(g(k)) = g(k + 1) \). Note, that \( s \) is multiplicative.

Thus \( s(kg(k)) = (k + 1)g(k + 1) = (k + 1)s(g(k)) \) holds.

The operator algebra, corr. to \( s \) is the 1st **shift algebra**

\[
S_1 = K \langle k, s \mid sk = (k + 1)s = ks + s \rangle.
\]

**Intermezzo**

For a function in differentiable argument \( x \) and in discrete argument \( k \) the natural operator algebra is

\[
A = D_1 \otimes_K S_1 = K \langle x, k, \partial_x, s_k \mid \partial_x x = x\partial_x + 1, \ s_k k = ks_k + s_k, \ xk = kx, \ xs_k = s_k x, \ \partial_x k = k\partial_x, \ \partial_x s_k = s_k \partial_x \rangle.
\]
Examples form the $q$-World

Let $k \subset K$ be fields and $q \in K^*$.

In $q$-calculus and quantum algebras three situations are common for a fixed $k$: (a) $q \in k$, (b) $q$ is a root of unity over $k$, and (c) $q$ is transcendental over $k$ and $k(q) \subseteq K$.

Let $\partial_q(f(x)) = \frac{f(qx)-f(x)}{(q-1)x}$ be a $q$-differential operator. The corr. operator algebra is the 1st $q$-Weyl algebra

$$D_1^{(q)} = K\langle x, \partial_q \mid \partial_q x = q \cdot x\partial_q + 1 \rangle.$$ 

The 1st $q$-shift algebra corresponds to the $q$-shift operator $s_q(f(x)) = f(qx)$:

$$K_q[x, s_q] = K\langle x, s_q \mid s_q x = q \cdot xs_q \rangle.$$
Two frameworks for bivariate operator algebras

Algebra with linear (affine) relation

Let $q \in K^*$ and $\alpha, \beta, \gamma \in K$. Define

$$\mathcal{A}^{(1)}(q, \alpha, \beta, \gamma) := K\langle x, y \mid yx - q \cdot xy = \alpha x + \beta y + \gamma \rangle$$

Because of integration operator $\mathcal{I}(f(x)) := \int_{0}^{x} f(t)dt$, obeying the relation $\mathcal{I}x - x\mathcal{I} = -\mathcal{I}^2$ we need yet more general framework.

Algebra with nonlinear relation

Let $N \in \mathbb{N}$ and $c_0, \ldots, c_N, \alpha \in K$. Then $\mathcal{A}^{(2)}(q, c_0, \ldots, c_N, \alpha)$ is

$K\langle x, y \mid yx - q \cdot xy = \sum_{i=1}^{n} c_i y_i + \alpha x + c_0 \rangle$ or

$K\langle x, y \mid yx - q \cdot xy = \sum_{i=1}^{n} c_i x_i + \alpha y + c_0 \rangle$. 
Theorem (L.–Koutschan–Motsak, 2011)

\[ A^{(1)}(q, \alpha, \beta, \gamma) = K\langle x, y \mid yx - q \cdot xy = \alpha x + \beta y + \gamma \rangle, \]

where \( q \in K^\ast \) and \( \alpha, \beta, \gamma \in K \)

is isomorphic to the 5 following \textbf{model algebras}:

1. \( K[x, y] \),
2. the 1st Weyl algebra \( D_1 = K\langle x, \partial \mid \partial x = x\partial + 1 \rangle \),
3. the 1st shift algebra \( S_1 = K\langle x, s \mid sx = xs + s \rangle \),
4. the 1st q-commutative algebra \( K_q[x, s] = K\langle x, s \mid sx = q \cdot xs \rangle \),
5. the 1st q-Weyl algebra \( D_1^{(q)} = K\langle x, \partial \mid \partial x = q \cdot x\partial + 1 \rangle \).
Theorem (L.–Makedonsky–Petravchuk, unpublished)

For \( N \geq 2 \) and \( c_0, \ldots, c_N, \alpha \in K \), \( \mathcal{A}^{(2)}(q, c_0, \ldots, c_N, \alpha) \)
\( = K \langle x, y \mid yx - q \cdot xy = \sum_{i=1}^{N} c_i y^i + \alpha x + c_0 \rangle \) is isomorphic to . . .

1. \( K_q[x, s] \) or \( D_1^{(q)} \), if \( q \neq 1 \),
2. \( S_1 = K \langle x, s \mid sx = xs + s \rangle \), if \( q = 1 \) and \( \alpha \neq 0 \),
3. \( K \langle x, y \mid yx = xy + f(y) \rangle \), where \( f \in K[y] \) with \( \deg(f) = N \), if \( q = 1 \) and \( \alpha = 0 \).

\( K \langle x, y \mid yx = xy + f(y) \rangle \cong K \langle z, w \mid wz = zw + g(w) \rangle \) if and only if \( \exists \lambda, \nu \in K^* \) and \( \exists \mu \in K \), such that \( g(t) = \nu f(\lambda t + \mu) \) (in particular \( \deg(f) = \deg(g) \)).
Example: Legendre’s differential equation

\[(x^2 - 1)P''_n(x)^2 + 2xP'_n(x) - n(1 + n)P_n(x) = 0\]

- \(x\) is differentiable with \(\partial_x\) as corr. operator
- if \(n \in \mathbb{Z}\), \(n\) is discretely shiftable with \(s_n\) as corr. op.

then there is a recursive formula of Bonnet (order 2 in \(s_n\))

\[(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0.\]
Example: Legendre’s differential equation

\[ \mathcal{O} := K\langle n, s_n \mid s_n n = ns_n + s_n \rangle \otimes_K K\langle x, \partial_x \mid \partial_x x = x\partial_x + 1 \rangle. \]

From the system of equations

\[
\begin{align*}
(x^2 - 1)P''_n(x)^2 + 2xP'_n(x) - n(1 + n)P_n(x) &= 0, \\
(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) &= 0.
\end{align*}
\]

one obtains the matrix \( P \in \mathcal{O}^{2 \times 1} \); thus \( M = \mathcal{O}/\mathcal{O}^{1 \times 2}P \) and

\[
\begin{bmatrix}
(x^2 - 1)\partial_x^2 + 2x\partial_x - n(1 + n) \\
(n + 2)s_n^2 - (2n + 3)xs_n + n + 1
\end{bmatrix} \cdot P_n(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

With the help of Gröbner bases: a minimal generating set of the left ideal \( P \) contains a compatibility condition

\[
(n + 1)s_n\partial_x - (n + 1)x\partial_x - (n + 1)^2 \equiv (n + 1)(s_n\partial_x - x\partial_x + n + 1).
\]
Solutions and Malgrange isomorphism

Let $\mathcal{F}$ be $K$-vector space and a left $\mathcal{O}$-module, then

$$\text{Sol}_{\mathcal{O}}(P, \mathcal{F}) := \{ f \in \mathcal{F}^{m \times 1} : Pf = 0 \}.$$ 

**Malgrange Isomorphism**

There exists an isomorphism of abelian groups (and $K$-vector spaces)

$$\text{Hom}_{\mathcal{O}}(M, \mathcal{F}) = \text{Hom}_{\mathcal{O}}(\mathcal{O}^{1 \times m}/\mathcal{O}^{1 \times \ell} P, \mathcal{F}) \cong \text{Sol}_{\mathcal{O}}(P, \mathcal{F}),$$

$$(\phi : M \to \mathcal{F}) \mapsto (\phi(e_1), \ldots, \phi(e_m)) \in \mathcal{F}^{m \times 1}.$$
Question: What is better to use in modeling: operator algebras with constant or with polynomial coefficients?

Answer: with polynomial coefficients.

Theorem (Zerz–L.–Schindelar, 2011)

Let $K = \mathbb{R}$, $p_i \in K[x_1, \ldots, x_n]^\ell$ and $V = Kp_1 + \cdots + Kp_m$. Let $\mathcal{O}$ be the $n$-th Weyl algebra and $\text{Ann}_\mathcal{O}(V) \subset \mathcal{O}$ be the minimal left ideal of equations, having $p_1, \ldots, p_m$ as solutions. Then

$$\text{Sol}_\mathcal{O}(\mathcal{O}/\text{Ann}_\mathcal{O}(V), C^\infty(\mathbb{R}^\ell)) = V.$$
Let $A$ be a Noetherian domain and $S$ a multiplicatively closed set in $A$, where $0 \notin S$.

A commutative implies the existence of $S^{-1}A$. A non-commutative: if $S$ is an Ore set in $A$, $\exists S^{-1}A$.

**Ore condition**

For all $s_1 \in S$, $r_1 \in A$ there exist $s_2 \in S$, $r_2 \in A$, such that

$$r_1s_2 = s_1r_2,$$

that is

$$s_1^{-1}r_1 = r_2s_2^{-1}.$$
The **Ore localization** of $A$ w.r.t $S$ is a Ring $A_S := S^{-1}A$ together with an injective homomorphism $\phi : A \to A_S$, such that

(i) for all $s \in S$ $\phi(s)$ is a unit in $A_S$, 
(ii) for all $f \in A_S$, $\exists a \in A$, $s \in S$ s. t. $f = \phi(s)^{-1}\phi(a)$.

**Example**

- Let $S = A^* := A \setminus \{0\}$. Then $S^{-1}A \cong \text{Quot}(A)$.
- If $K \varsubsetneq S \varsubsetneq A^*$, then $A \to A_S \to \text{Quot}(A)$,
- For any $S$, $S^{-1}A$ is an $A$-module (not finitely generated),
- in general $A$ is not an $S^{-1}A$-module.

$S^{-1}$ gives rise to a functor $A\text{-mod} \to S^{-1}A\text{-mod}$.
With Ore localization we can recognize, that

\[ K(X)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_m; \sigma_m, \delta_m] \cong (K[X]\{0\})^{-1}K\langle X, \partial_1, \ldots, \partial_m \mid \ldots \rangle \]

and the functor \( S^{-1} \) connects categories of modules.

**Algorithmic aspects**

Algorithmic computations over \( S^{-1}A \) can be replaced **completely** with computations over \( A \).

Keywords: **integer strategy, fraction-free strategy**.

For instance, a Gröbner basis theory over \( A \) induces a Gröbner basis theory over \( S^{-1}A \).

There are implementations for the rational localization \( K(X)\langle \partial_1, \ldots \rangle \).
Polynomial or rational Coefficients?

Let $A$ be a $K$-algebra and $S \subset A$ a mult. closed Ore set in $A$. Moreover, let
- $M \cong A^n / A^m P$, a finitely presented left $A$-module,
- $\mathcal{F}$ a left $A$-module,
- $\tilde{\mathcal{F}}$ a left $S^{-1}A$-module.

- $S^{-1}M \cong (S^{-1}A)^n / (S^{-1}A)^m P$.

$$\text{Sol}_A(M, \tilde{\mathcal{F}}) \cong \text{Sol}_{S^{-1}A}(S^{-1}M, \tilde{\mathcal{F}}),$$

Assume $\tilde{\mathcal{F}} \subset \mathcal{F}$ as left $A$-modules. Then
$$\text{Sol}_A(M, \tilde{\mathcal{F}}) \subseteq \text{Sol}_A(M, \mathcal{F}),$$
Let $\mathcal{G} \subset \mathcal{F}$ be function spaces, i.e. $K$-vector spaces and left $\mathcal{O}$-modules over a fixed operator algebra $\mathcal{O}$.

Let $f \in \mathcal{F}$, then $\text{Ann}_{\mathcal{O}}^\mathcal{F} f := \{ p \in \mathcal{O} : pf = 0 \in \mathcal{F} \}$ is the annihilator of $f$, which is a left ideal in $\mathcal{O}$.

Let $I \subsetneq \mathcal{O}$ be an ideal and suppose, that $\dim_K(\mathcal{G}) < \infty$. $I$ is called the complete annihilator of $\mathcal{G}$ over $\mathcal{O}$, if the following properties hold:

"most powerful": if $\forall g \in \mathcal{G} \rg = 0$ for $r \in \mathcal{O}$, then $r \in I$

"unfalsified": $\text{Sol}_\mathcal{O}(\mathcal{O}/I, \mathcal{F}) = \mathcal{G}$. 
The complete annihilator program

There exists no general algorithm, which can compute the complete annihilator program of \( f \) over \( \mathcal{O} \) (where \( \mathcal{O} \) is an algebra with polynomial coefficients).

Therefore one investigates some classes of \( f \) and develops special methods for the classes.

One of successes is computational \textit{D-module theory}, where among other one can compute the complete annihilators of

\[
f(x, s) = f_1(x_1, \ldots, x_n)^{s_1} \cdots f_m(x_1, \ldots, x_m)^{s_m}, \quad f_i(x) \in K[x_1, \ldots, x_n]
\]

over

\[
\mathcal{O} = \bigotimes_{i=1}^{n} K \langle x_i, \partial_i \mid \partial_i x_i = x_i \partial_i + 1 \rangle \otimes_K K[s_1, \ldots, s_m]
\]

in an algorithmic way. There are implementations.
Some computational $D$-module theory

Let $D_n(K) = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \partial_j x_i = x_i \partial_j + \delta_{ij} \rangle$ be the $n$-th Weyl algebra and $D_n[s] = D_n \otimes_K K[s]$.

Theorem (J. Bernstein, 1971/72)

Let $f(x) \in \mathbb{C}[x_1, \ldots, x_n]$. Then there exist

- an operator $P(s) \in D_n \otimes_{\mathbb{C}} \mathbb{C}[s]$,
- a monic polynomial $0 \neq b_f(s) \in \mathbb{C}[s]$ of the smallest degree (called the global Bernstein-Sato polynomial),

such that for arbitrary $s$ the following functional equation holds

$$P(s) \cdot f^{s+1} = b_f(s) \cdot f^s.$$ 

Let $\text{Ann}_{D[s]}(f^s) = \{ Q(s) \in D[s] \mid Q(s) \cdot f^s = 0 \} \subset D[s]$ be the annihilator, then

$$P(s)f - b_f(s) \in \text{Ann}_{D[s]}(f^s)$$ holds.
Dimensions

- Generalized Krull dimension is called Krull-Rentschler-Gabriel dimension; not algorithmic

- Global homological dimension (of an algebra) resp. projective dimension (of a module); for modules: algorithmic (relatively expensive), implemented

- Gel’fand-Kirillov Dimension; algorithmic (relatively cheap), implemented; plays the role of Krull dimension in non-commutative case.
GK dimension and its properties

Let $A$ be a $K$-algebra, generated by $x_1, \ldots, x_m$.

**Degree filtration**

Let $V = Kx_1 \oplus \ldots \oplus Kx_m$ be a vector space. 
Set $V_0 = K$, $V_1 = K \oplus V$ and $V_{k+1} = V_k \oplus V^{k+1}$.
Let $M_0 \subset M$, $\dim_K M_0 < \infty$ and $AM_0 = M$.
An ascending filtration on $M$ is defined via \( \{ H_d := V_d M_0, d \geq 0 \} \).

The **Gel’fand-Kirillov dimension** of $M$ is defined as follows

\[
\text{GKdim}(M) = \limsup_{d \to \infty} \log_d(\dim_K H_d)
\]
Let $\deg x_i := 1$, $V_d := \{ f \mid \deg f = d \}$ and $V^d := \{ f \mid \deg f \leq d \}$.

**Lemma**

Let $A$ be a $K$-algebra and a domain. If the standard filtration on $A$ is compatible with the PBW Basis $\{x^\alpha \mid \alpha \in \mathbb{N}^m\}$, then $\text{GKdim}(A) = m$.

$$\dim V_d = \binom{d + m - 1}{m - 1}, \dim V^d = \binom{d + m}{m}.$$  

Thus $\binom{d+m}{m} = \frac{(d+m)\ldots(d+1)}{m!} = \frac{d^m}{m!} + \ldots$ and

$$\text{GKdim}(A) = \limsup_{d \to \infty} \log_d \binom{d + m}{m} = m.$$
Gel’fand-Kirillov-Dimension: examples and properties

\[
\text{GKdim}(K\langle x_1, \ldots, x_n \rangle) = \infty \text{ for } n \geq 2.
\]
\[
\text{GKdim}(K[[x_1, \ldots, x_n]]) = \infty \text{ for } n \geq 1, \text{ when } |K| = \infty.
\]

Properties

- \( \text{GKdim } M = \sup\{\text{GKdim}(N) : N \in A - \text{mod}, \ N \subseteq M\} \),
- \( \text{GKdim } A = \sup\{\text{GKdim}(S) : S \subseteq A, \ S \text{ fin. gen. subalgebra}\} \)

Over \( G \)-algebras (and even more) there are algorithms and implementations for the computation of the GK dimension of finitely presented modules.
Elimination and dimension

**Lemma**

Let $I \subseteq A$ be a left ideal and $S \subseteq A$ be a finitely generated subalgebra. Then

- $I \cap S = 0$ implies $\text{GKdim } A/I \geq \text{GKdim } S$,
- $\text{GKdim } A/I < \text{GKdim } S$ implies $I \cap S \neq 0$.

**Recall: Bernstein’s inequality**

Let $A$ be the $n$-th Weyl algebra over $K$ with $\text{char } K = 0$ (thus $\text{GKdim } (A) = 2n$) and $0 \neq M$ be an $A$-module, then $\text{GKdim } M \geq n$. 

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**Operator algebras, partial classification**

**Dimensions and purity**

**Gel'fand-Kirillov dimension**

**Purity**
Elimination and dimension

Classically: for a function $f$, $\text{Ann}_\mathcal{O} f \cap K[x_1, \ldots, x_n] = 0$. Hence $\text{GKdim } \mathcal{O}/\text{Ann}_\mathcal{O} f \geq n$.

**Proposition**

Let $\mathcal{O} = \bigotimes_{i=1}^n \mathcal{O}_i$, $\mathcal{O}_i = K\langle x_i, o_i \mid \ldots \rangle$. Moreover, let $I \subset \mathcal{O}$ and $\text{GKdim } \mathcal{O}/I = m$. Then for any finitely generated subalgebra $S \subset \mathcal{O}$ of GK dimension $\geq m + 1$ one has $I \cap S \neq 0$.

Application: For $I$ such that $\text{GKdim } \mathcal{O}/I = n$ we guarantee that $2n - (n + 1) = n - 1$ variables can be eliminated from $I$, for instance

- all but one operators,
- all but one coordinate variables.
Elimination, dimension and localization

Suppose that \( I, S \subset \mathcal{O} \) are such that
- \( S \) is an Ore set in \( \mathcal{O} \) (so \( S^{-1}\mathcal{O} \) exists)
- \( S^{-1}\mathcal{O}I \neq S^{-1}\mathcal{O} \) (i.e. \( I \) is proper in the localized algebra).

Then \( I \cap S = 0 \), what implies \( \text{GKdim} \mathcal{O}/I \geq \text{GKdim} S \).

Note, that for every \( J \in S^{-1}\mathcal{O} \) there exists \( I \in \mathcal{O} \) such that \( S^{-1}\mathcal{O}I = S^{-1}\mathcal{O}J \). In general

\[
\text{GKdim} S^{-1}\mathcal{O}/S^{-1}\mathcal{O}L \geq \text{GKdim} \mathcal{O}/L.
\]
Dimension function

Let $A$ be a Noetherian algebra. A dimension function $\delta$ assigns a value $\delta(M)$ to each finitely generated $A$-module $M$ and satisfies the following properties:

(i) $\delta(0) = -\infty$.

(ii) If $0 \to M' \to M \to M'' \to 0$ is exact sequence, then $\delta(M) \geq \sup\{\delta(M'), \delta(M'')\}$ with equality if the sequence is split.

(iii) If $P$ is a (two-sided) prime ideal with $PM = 0$ and $M$ is a torsion module over $A/P$, then $\delta(M) \leq \delta(A/P) - 1$.

- generalized Krull dimension is an exact dimension function
- Gel’fand-Kirillov dimension is a dimension function, not always exact
Let $A$ be a $K$-algebra and $\delta$ a dimension function on $A$-mod. A module $M \neq 0$ is $\delta$-pure, if $\forall 0 \neq N \subseteq M$, $\delta(N) = \delta(M)$.

- Purity is a useful weakening of the concept of simplicity of a module.
- Unlike simplicity, the purity (w.r.t a dimension function) is algorithmically decidable over many common algebras.

M. Barakat, A. Quadrat: Algorithms for the computation of the purity filtration of a module with $\delta = \text{homological co-grade}$; there are several implementations.
Lemma (L.)

Let $A$ be a $K$-algebra and $\delta$ a dimension function on $A$-mod. Moreover, let $0 \neq M_1, M_2 \subset N$ be two $\delta$-pure modules with $\delta(M_1) = \delta(M_2)$. Then

the set of $\delta$-pure submodules (of the same dimension) of a module is a lattice, i.e.

1. $M_1 \cap M_2$ is either 0 or it is $\delta$-pure with $\delta(M_1 \cap M_2) = \delta(M_1)$,
2. $M_1 + M_2$ is $\delta$-pure with $\delta(M_1 + M_2) = \delta(M_1)$. 
Consider the mixed system, annihilating Legendre polynomials

$$\mathcal{D} = K\langle n, s_n \mid s_n n = ns_n + s_n \rangle \otimes_K K\langle x, \partial_x \mid \partial_x x = x\partial_x + 1 \rangle.$$ 

$$M = \mathcal{D}/P,$$

$$P = \langle (x^2 - 1)\partial_x^2 + 2x\partial_x - n(1 + n), (n + 2)s_n^2 - (2n + 3)x s_n + n + 1, (n + 1)(s_n \partial_x - x\partial_x + n + 1) \rangle.$$ 

$$\text{GKdim } \mathcal{D} = 4, \quad \text{GKdim } M = 2, \quad t(M) = M = \mathcal{D}/P.$$
The purity filtration of $M = t(M)$ is $0 \subsetneq M_3 \subsetneq M_2 = M$, 

$$M_3 \cong \mathcal{O}/\langle n + 1, s_n, \partial_x \rangle \text{ with } \text{GKdim } M_3 = 1.$$ 

**What are the solutions $g(n, x)$ of this system?**

Since $\partial_x (g) = 0$, one has $g(n, x) = g(n)$. 

However, $g(n)$ should not be identically zero: 

in case $n \in \{-1, 0, 1, \ldots\}$, one can select $g(-1) \in K$ arbitrary 

(step of the jump function). 

**Localization**

The ideal $\langle n + 1, s_n \rangle$ is two-sided and maximal. Hence the 

submodule $M_3$ vanishes under any Ore localization in $K\langle n, s_n \ldots \rangle$, 

for instance when $n$ invertible or $s_n$ invertible (then $s_n^{-1}$ is present and therefore should $n \in \mathbb{Z}$ hold).
The purity filtration of $M = t(M)$ is $0 \subsetneq M_3 \subsetneq M_2 = M$. The pure part of GK dimension 2 is $t(M)/M_3 \cong \mathcal{O}/\langle (x^2 - 1)\partial_x^2 + 2x\partial_x - n(1 + n), \ (n + 2)S_n^2 - (2n + 3)xS_n + n + 1, (1 - x^2)\partial_x + (n + 1)S_n - (n + 1)x \rangle$.

For further investigations of $M$ over localizations w.r.t. $n$ or $S_n$ one should then take the simplified equations.

**Elimination leads to new identities**

The elimination property guarantees, that 1 arbitrary variable can be eliminated; so one gets for instance

$$(n + 1)(n + 2) \cdot ((S_n^2 - 1)\partial_x - (2n + 3)S_n) \cdot P_n(x) = 0,$$

$$(1 - x^2) \cdot ((S_n^2 - 2xS_n + 1)\partial_x - S_n)) \cdot P_n(x) = 0.$$
The hypergeometric series is defined for $|z| < 1$ and $-c \notin \mathbb{N}_0$ as follows:

$$2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

We derive two annihilating ideals from $2F_1(a, b, c; z)$:

- $J_a$ which does not contain $a$,
- $J_c$ which does not contain $c$,

and analyze corresponding modules for purity.
Case $J_a$

The ideal in $\mathcal{O} = K[b, c, z]\langle s_b, s_c, \partial_z \mid \ldots \rangle$ is generated by:

\[\begin{align*}
bcSb - czDz - bc \\
bSbSc - bSc + cSc - c \\
bSb^2 - zSbDz - bSb + Sb^2 - Sb \\
b^2Sb - bzDz - b^2 + bSb - zDz - b \\
& \quad bzSbDz - z^2Dz^2 - bzDz - bSbDz + zDz^2 - bSb + bDz + b + Dz
\end{align*}\]

Let $M = M_a = \mathcal{O}/J_a$. Then $\text{GKdim } \mathcal{O} = 6$, $\text{GKdim } M = 4$.

The purity filtration of $M = t(M)$

\[
0 \subsetneq M_5 \subsetneq M_4 \subsetneq M_3 = M_2 = M_1 = M, \text{ where}
\]

\[
M/M_5 \cong \mathcal{O}/\langle bSb - zDz - b, zDzSc + cSc - c \rangle, \text{ GKdim } M/M_5 = 4
\]
The purity filtration of \( M = t(M) \)

\[
\cdots \text{ and } \quad M_5 \cong \mathcal{O}/\langle c, Sb, b + 1, zDz - Dz - 1 \rangle, \quad \text{GKdim } M_5 = 2.
\]

The solutions can be read off:

\[
\delta_{c,0} \cdot \delta_{b,-1} \cdot (\ln(z - 1) + k_0), \quad k_0 \in K
\]
Case $J_c$

The ideal in $\mathcal{O} = K[a, b, z]\langle s_a, s_b, \partial_z \mid \ldots \rangle$ is generated by:

$$a Sa - b Sb - a + b$$
$$b Sb^2 - SbzDz - b Sb + Sb^2 - Sb$$
$$b^2 Sb - b zDz - b^2 + b Sb - zDz - b$$
$$ab Sb - a zDz - ab + b Sb - zDz - b$$
$$b SbzDz - z^2 Dz^2 - b SbDz - b zDz + zDz^2 - b Sb + b Dz + b + Dz$$

Let $M = M_c = \mathcal{O}/J_c$. Then $\text{GKdim } \mathcal{O} = 6$, $\text{GKdim } M = 4$.

The purity filtration of $M = t(M)$

$$0 \subsetneq M_6 = M_5 = M_4 \subsetneq M_3 = M_2 = M_1 = M,$$ where

$$M/M_6 \cong \mathcal{O}/\langle b Sb - zDz - b, a Sa - zDz - a \rangle,$$ $\text{GKdim } M/M_6 = 4.$
The purity filtration of $M = t(M)$

\[ M_6 \cong \mathcal{O} / \langle Sb, b + 1, Sa, a + 1, zDz - Dz - 1 \rangle, \text{ GKdim } M_6 = 2. \]

The solutions:

\[ \delta_{a,-1} \cdot \delta_{b,-1} \cdot (\ln(z - 1) + k_0), \ k_0 \in K \]
Software

\[ D \]-modules and algebraic analysis:

- **KAN/SM1** by N. Takayama et al.
- **D-modules package in Macaulay2** by A. Leykin and H. Tsai
- **Risa/Asir** by M. Noro et al.
- **OreModules** package suite for Maple by D. Robertz, A. Quadrat et al.
- **Singular:Plural** with a \( D \)-module suite; by V. L. et al.

Holonomic and \( D \)-finite functions:

- **Groebner, Ore algebra, Mgfun, ...** by F. Chyzak
- **HolonomicFunctions** by C. Koutschan
- **Singular:Locapal** (under development) by V. L. et al.
Thanks for your attention!

http://www.singular.uni-kl.de/