SCOPES REDUCTION AND MORITA EQUIVALENCE CLASSES OF BLOCKS IN FINITE CLASSICAL GROUPS II

GERHARD HISS AND RADHA KESSAR

ABSTRACT. We extend our work of [23] on unipotent blocks of classical groups to blocks of orthogonal groups containing characters labelled by degenerate symbols. We also show that, for classical groups in the linear prime case, there are only finitely many Morita types of unipotent blocks with a given defect group.

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1. INTRODUCTION

In [23] we began the investigation of the Morita types of unipotent blocks of classical groups (other than the general linear groups). Here we extend this work to to blocks of orthogonal groups containing characters labelled by degenerate symbols (and correct some errors of [23]). In view of Donovan's conjecture (see [1, Conjecture M]), the ultimate goal is to show that there are only finitely many Morita equivalence classes of unipotent blocks of classical groups with a given defect group.

In [17], Olaf Düvel reduces the proof of Donovan's conjecture for elementary abelian defect groups of rank two to the corresponding problem for quasi-simple groups. He also obtains a reduction to quasi-simple groups of the weaker problem of bounding the maximum entry of the Cartan matrix of a block in terms of its defect group. This weak form of Donovan's conjecture is of course motivated by the question of Brauer posed in [5, Problem 22] and his remark in [6].

The following theorem summarizes our results of [23] and of this paper.

Theorem. *Let $\mathcal{F}$ be an algebraically closed field of characteristic $l > 0$, and let D be a finite l-group. Then there is a finite set $\mathcal{B}$ of $\mathcal{F}$-algebras and a positive integer N, each of $\mathcal{B}$ and N depending only on D, such that the following holds:*

Let B be a unipotent l-block with defect group isomorphic to D of a group algebra $\mathcal{F}G$ where G is one of the classical groups $SL_n(q)$, $SU_n(q)$, $SO_{2m+1}(q)$, $Sp_{2m}(q)$, or $SO^{\pm}_{2m}(q)$ (n, m positive integers, q a prime power). Then one of the following cases occurs.

1. *B is Morita equivalent to an algebra in $\mathcal{B}$.*
2. *There is a classical group $\bar{G}$ of the above list with natural dimension $\bar{n} \leq N$ such that B is Morita equivalent to a unipotent l-block $\bar{B}$ of $\bar{G}$ with defect group D.*
3. *$G = SO^{+}_{2m}(q)$ or $SO^{-}_{2m}(q)$ for some positive integer m and some prime power q, and $l \mid q+1$.*

This theorem is proved at the end of Section 8.

As in [23], the proof of the main finiteness result has two parts. The first one is purely combinatorial, unfortunately rather more complicated than in [23]. The second one, where the Morita equivalence between certain blocks is established, uses a new criterion (Theorem 2.5 below). Our new criterion in fact gives an equivalence of the source algebras of the blocks involved, not only up to an automorphism of the defect group, as in [23, Theorem 8.1], but up to isomorphism of interior algebras. This new criterion, which can of course also be applied to simplify some of the arguments of previous results of the second author in [26, 27], replaces Broué's criterion [7, Théorème 2.4] and the criterion of Scott and Puig [34, Remark 7.5].

We plan a sequel to this paper, where we intend to treat non-unipotent blocks of classical groups. The idea is to use the result of Bonnafé and Rouquier (cf. Theorem 8.2 below) to reduce to the case of "isolated" blocks. It seems plausible that the isolated blocks can be dealt with in a way similar to the unipotent blocks (even though much more complicated).

We give a brief summary of the contents of the individual chapters of our paper. Chapter 2 contains the new criterion for two blocks to have equivalent source algebras. Chapter 3 recalls some basic material on unipotent blocks and the theorem of Cabanes and Enguehard to descend to commutator subgroups. In Chapter 4 we treat the case of linear primes. These results are not an immediate consequence of [22], since there the bounds on the decomposition numbers are given in terms of the natural dimension of the classical groups. In general, there are classical groups of arbitrarily large dimension containing a unipotent block with a given defect group. Chapter 5 sets up the combinatorial results used in the reduction theorem for orthogonal groups of even dimension, contained in Chapter 6. In Chapter 7 we summarize the consequences of the reduction theorem to the unitary prime case of these groups. In Chapter 8 we treat the general linear and unitary groups for all blocks, not just the unipotent ones. Finally, Chapter 9 collects some remarks and clarifications on [23], the first part of this work. The ordering of the chapters follows their decreasing generality.

2. A criterion for Puig equivalence

We give a sufficient criterion for two blocks to have equivalent source algebras.

2.1. **Notation.** Let l a prime and let $(\mathcal{K}, \mathcal{O}, \mathcal{F})$ be an l-modular system. By this we understand a complete discrete valuation ring $\mathcal{O}$ with field of fractions $\mathcal{K}$ of characteristic 0 and residue class field $\mathcal{F}$ of characteristic l. We assume in addition that $\mathcal{F}$ is algebraically closed.

Let G and $\bar{G}$ be finite groups and let B and $\bar{B}$ be blocks of $\mathcal{O}G$ and $\mathcal{O}\bar{G}$, respectively. Following Broué [8, **6.C**], we say that B and $\bar{B}$ are *Puig equivalent*, if there are compatible isomorphisms between the defect groups and the source algebras of the two blocks.

From now on, till the end of this section, we assume that $(\mathcal{K}, \mathcal{O}, \mathcal{F})$ is a splitting l-modular system for G, i.e., that $\mathcal{K}$ is a splitting field for G and all of its subgroups. We denote by $\mathrm{Irr}(G)$ the set of irreducible $\mathcal{K}$-characters of G. If b is a central idempotent of $\mathcal{O}G$, we write $\mathrm{Irr}(G, b)$ for the subset of $\mathrm{Irr}(G)$ arising from the constituents of $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{O}Gb$. Similarly, we write $\mathrm{IBr}(G)$ for the set of irreducible Brauer characters of G with respect to $(\mathcal{K}, \mathcal{O}, \mathcal{F})$, and $\mathrm{IBr}(G, b)$ for those elements of $\mathrm{IBr}(G)$ arising from representations of $\mathcal{F}Gb$.

2.2. **Algebras and idempotent decompositions.** Let A denote either a unitary $\mathcal{O}$-algebra which is $\mathcal{O}$-free of finite rank or a finite dimensional unitary $\mathcal{F}$-algebra.

By an *idempotent decomposition* of A we will mean a set of primitive idempotents of A that are pairwise orthogonal and whose sum equals the identity of A. We denote by $m(A)$ the cardinality of an idempotent decomposition of A.

Suppose now that A is a G-algebra. Then A^G will denote the set of elements of A fixed by G. The algebra A is called an *interior G-algebra* if there is a group homomorphism from G into the group of invertible elements of A. If A is an interior G-algebra, then A is a G-algebra via the conjugation action (by the elements of the image of G under the homomorphism defining the interior G-algebra structure on A). If H is a subgroup of G, then the inclusion of H into G gives $\mathcal{O}G$ the structure of an interior H-algebra. In what follows we will assume this structure without comment.

2.3. **The Brauer homomorphism.** Suppose A as above is an H-algebra for a finite group H. Let D be a subgroup of H. The Brauer homomorphism on A with respect to D, denoted by Br_D^A, is the canonical surjection

$$\mathrm{Br}_D^A : A^D \to A^D/(\sum_Q \mathrm{Tr}_Q^D(A^Q) + J(\mathcal{O})A^D),$$

where Q runs over all proper subgroups of D, Tr_Q^D is the relative trace map and $J(\mathcal{O})$ is the maximal ideal of $\mathcal{O}$ (see [36, §11]). We denote the image $A^D/(\sum_Q \mathrm{Tr}_Q^D(A^Q)+J(\mathcal{O})A^D)$ of Br_D^A by $\overline{A}(D)$. Then $\overline{A}(D)$ is non-zero only if D is an l-group. Since $\sum_Q \mathrm{Tr}_Q^D(A^Q)+J(\mathcal{O})A^D$ is an $N_G(D)$-invariant ideal of A^D, $\overline{A}(D)$ inherits naturally a structure of an $N_H(D)$-algebra, and Br_D^A respects this structure.

Let u be an idempotent of A^H. For any subgroup Q of D, $uA^Qu = (uAu)^Q$ and $\ker(\mathrm{Br}_D^{uAu}) = \ker(\mathrm{Br}_D^A) \cap (uAu)^D$. Thus, the map

$$\iota : \overline{uAu}(D) \to \overline{A}(D)$$

given by $\iota(\mathrm{Br}_D^{uAu}(a)) = \mathrm{Br}_D^A(a)$ is a well defined and injective homomorphism of $N_H(D)$-algebras. Also, $\iota(\overline{uAu}(D)) = \mathrm{Br}_D^A(u)\overline{A}(D)\mathrm{Br}_D^A(u)$.

By the *D-multiplicity of H on A* we will mean the number of elements i in an idempotent decomposition of A^H for which $\mathrm{Br}_D^A(i)$ is non-zero. Note that this makes sense since $A^H \subset A^D$ and that this number is independent of the choice of the idempotent decomposition. We will denote the D-multiplicity of H on A by $m_{H,D}(A)$.

2.4. **Group algebras and the Brauer homomorphism.** Let D be an l-subgroup of G. By [36, Proposition 37.5], there is an isomorphism of $N_G(D)$-algebras $\alpha : \overline{\mathcal{O}G}(D) \to \mathcal{F}C_G(D)$ such that the composition $\alpha \circ \mathrm{Br}_D^{\mathcal{O}G}$ is the surjective map from $(\mathcal{O}G)^D$ to $\mathcal{F}C_G(D)$ which sends an element of $\mathcal{O}C_G(D)$ to its canonical image in $\mathcal{F}C_G(D)$ and maps to zero any D-conjugacy class sum involving elements of G outside $C_G(D)$. Henceforth we shall identify $\mathrm{Br}_D^{\mathcal{O}G}$ with $\alpha \circ \mathrm{Br}_D^{\mathcal{O}G}$ and put $\alpha \circ \mathrm{Br}_D^{\mathcal{O}G} =: \mathrm{Br}_D$.

Let H be a subgroup of G containing D and u an idempotent of $(\mathcal{O}G)^H$. Letting $A = \mathcal{O}G$, and considering the map ι as in Subsection 2.3, we get that for any $a \in (u\mathcal{O}Gu)^D$, $\mathrm{Br}_D(a) = \alpha \circ \iota \circ \mathrm{Br}_D^{u\mathcal{O}Gu}(a)$. Since α is an isomorphism and ι is injective it follows that for any element a of $(u\mathcal{O}Gu)^D$, $\mathrm{Br}_D(a) = 0$ if and only if $\mathrm{Br}_D^{u\mathcal{O}Gu}(a) = 0$. Thus, in particular, we have that $m_{H,D}(u\mathcal{O}Gu)$ is the number of elements i in a primitive idempotent decomposition of $(u\mathcal{O}Gu)^H$ for which $\mathrm{Br}_D(i)$ is non-zero.

From now on till the end of the section, we fix a subgroup H of G. If u is an idempotent of $(\mathcal{O}G)^H$, τ is an irreducible $\mathcal{K}$-character of H and χ is an irreducible $\mathcal{K}$-character of G, we let $r(\tau,\chi,u)$ be the number of composition factors of the $\mathcal{K}G$-module $\mathcal{K}Gu\otimes_{\mathcal{K}H}W_\tau$ isomorphic to V_χ, where V_χ is a $\mathcal{K}G$-module affording χ and W_τ is a $\mathcal{K}H$-module affording τ.

Let us now fix block idempotents b of $\mathcal{O}G$, and c of $\mathcal{O}H$, respectively. We will assume that b and c have a common defect group $D\subset H$. For what follows, note that the algebra $c\mathcal{O}Gbc$ is stable under conjugation by H.

Theorem 2.5. *Let u be an idempotent of $(c\mathcal{O}Gbc)^H$. Then for any $\chi\in\mathrm{Irr}(G,b)$,*

$$\sum_{\tau\in\mathrm{Irr}(H,c)} r(\tau,\chi,u)\geq m_{H,D}(u\mathcal{O}Gu),$$

and for any $\tau\in\mathrm{Irr}(H,c)$,

$$\sum_{\chi\in\mathrm{Irr}(G,b)} r(\tau,\chi,u)\geq m_{H,D}(u\mathcal{O}Gu).$$

Further, if $|\mathrm{Irr}(H,c)|=|\mathrm{Irr}(G,b)|$ and if

$$\sum_{\chi\in\mathrm{Irr}(G,b),\tau\in\mathrm{Irr}(H,c)} r(\tau,\chi,u)\leq m_{H,D}(u\mathcal{O}Gu)|\mathrm{Irr}(H,c)|,$$

then for any primitive idempotent i of $(u\mathcal{O}Gu)^H$, $\mathrm{Br}_D^{u\mathcal{O}Gu}(i)$ is non-zero and the map $\mathcal{O}Hc\to i\mathcal{O}Gi$ given by $x\mapsto xi$ is an isomorphism of interior H-algebras. In particular, if the above holds, then $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are Puig equivalent.

Proof. For $\tau\in\mathrm{Irr}(H,c)$, let f_τ denote the central idempotent of $\mathcal{K}H$ corresponding to the character τ. Similarly for χ in $\mathrm{Irr}(G,b)$ let e_χ denote the central idempotent of $\mathcal{K}G$ corresponding to the character χ. Then, if $\tau\in\mathrm{Irr}(H,c)$, $\chi\in\mathrm{Irr}(G,b)$ and i is an idempotent of $(c\mathcal{O}Gbc)^H$, we have

$$\begin{aligned}\mathcal{K}Ge_\chi if_\tau &\cong \mathcal{K}Ge_\chi i\otimes_{\mathcal{K}H}\mathcal{K}Hf_\tau\\ &\cong \mathcal{K}Ge_\chi i\otimes_{\mathcal{K}H}\tau(1)W_\tau,\end{aligned}\tag{1}$$

as left $\mathcal{K}G$ modules. Here W_τ denotes a $\mathcal{K}H$-module affording τ. Thus, the character of the left $\mathcal{K}G$-module $\mathcal{K}Ge_\chi if_\tau$ is $r(\tau,\chi,i)\tau(1)\chi$. In particular, $r(\tau,\chi,i)$ is non-zero if and only if $e_\chi if_\tau$ is non-zero.

If i and j are orthogonal idempotents of $(c\mathcal{O}Gbc)^H$, then $\mathcal{K}Ge_\chi(i+j)f_\tau=\mathcal{K}Ge_\chi if_\tau\oplus\mathcal{K}Ge_\chi jf_\tau$ and it follows from the above that

$$r(\tau,\chi,i+j)=r(\tau,\chi,i)+r(\tau,\chi,j).\tag{2}$$

Now let i be an idempotent of $(uc\mathcal{O}Gbcu)^H=(u\mathcal{O}Gu)^H$ such that $\mathrm{Br}_D^{u\mathcal{O}Gu}(i)$ is non-zero. There is a primitive idempotent t in $(u\mathcal{O}Gu)^D$ such that $it=t$ and $\mathrm{Br}_D^{u\mathcal{O}Gu}(t)$ is non-zero, whence, as explained in Subsection 2.4, $\mathrm{Br}_D(t)$ is non-zero. Also, t is primitive in $(\mathcal{O}Gb)^D$ (if $t=t'+t''$ with t' and t'' orthogonal idempotents in $(\mathcal{O}Gb)^D$, then $ut'u=u(tt't)u=t'$ and similarly, $ut''u=t''$). In other words, t is a source idempotent of the block b, so that in particular the algebras $t\mathcal{O}Gt$ and $\mathcal{O}Gb$ are Morita equivalent (see [36, Proposition 38.2]). Thus, by [36, Theorem 9.9], if $\mathcal{O}Gbt\mathcal{O}Gb$ is the ideal generated by t in $\mathcal{O}Gb$, then $\mathcal{O}Gbt\mathcal{O}Gb=\mathcal{O}Gb$. But since $it=i$, this clearly means that $\mathcal{O}Gbi\mathcal{O}Gb=\mathcal{O}Gb$. Hence $\mathcal{K}Gbi\mathcal{K}Gb=\mathcal{K}Gb$, so that by [36, Theorem 9.9], $i\mathcal{K}Gi$ and $\mathcal{K}Gb$ are Morita equivalent. In particular, $i\mathcal{K}Gi$

and $\mathcal{K}Gb$ have the same number of isomorphism classes of irreducible modules. On the other hand,

$$\mathcal{K}Gb \cong \prod_{\chi \in \mathrm{Irr}(G,b)} \mathcal{K}Ge_\chi,$$

whence

$$i\mathcal{K}Gi \cong \prod_{\chi \in \mathrm{Irr}(G,b)} i\mathcal{K}Ge_\chi i.$$

So, for all $\chi \in \mathrm{Irr}(G,b)$, $e_\chi i \neq 0$. Since $ic = i$, and $c = \sum_{\tau \in \mathrm{Irr}(H,c)} f_\tau$, this means that for each $\chi \in \mathrm{Irr}(G,b)$, there exists some $\tau \in \mathrm{Irr}(H,c)$ such that $e_\chi i f_\tau$ is non-zero, which by the remark following (1) is equivalent to saying that for some $\tau \in \mathrm{Irr}(H,c)$, $r(\tau,\chi,i)$ is non-zero. Hence for each $\chi \in \mathrm{Irr}(G,b)$, we have

$$\sum_{\tau \in \mathrm{Irr}(H,c)} r(\tau,\chi,i) \geq 1. \tag{3}$$

The first assertion of the theorem now follows from (2) and (3).

Again, let i be an idempotent of $(uc\mathcal{O}Gbcu)^H = (u\mathcal{O}Gu)^H$ such that $\mathrm{Br}_D^{u\mathcal{O}Gu}(i) \neq 0$. Then, as is explained in Subsection 2.4, $\mathrm{Br}_D(i) \neq 0$. Since i is fixed by H, the map from $\mathcal{O}Hc$ to $i\mathcal{O}Gi$ sending an element x of $\mathcal{O}Hc$ to the element xi of $i\mathcal{O}Gi$ is an H-algebra homomorphism and this homomorphism is unitary since $ci = c(ui) = c((bc)u)i = bcui = i$. Note that D stabilizes an $\mathcal{O}$-basis of $i\mathcal{O}Gi$, since $i\mathcal{O}Gi$ is a D-invariant direct summand of $\mathcal{O}G$. Also, $Z(D)$ commutes with $\mathrm{Br}_D(i)$, hence left multiplication by elements of $Z(D)$ gives $\mathrm{Br}_D(i(\mathcal{O}G)^D i) = \mathrm{Br}_D(i)\mathcal{F}C_G(D)\mathrm{Br}_D(i)$ the structure of a left $\mathcal{F}Z(D)$-module. Further, $\mathrm{Br}_D(i)\mathcal{F}C_G(D)\mathrm{Br}_D(i)$ is clearly a direct summand of $\mathcal{F}C_G(D)$ as left $\mathcal{F}Z(D)$-modules. In particular, $\mathrm{Br}_D(i(\mathcal{O}G)^D i)$ is projective as left $\mathcal{F}Z(D)$-module. Hence, by [33, Proposition 3.8] the above map $\mathcal{O}Hc \to i\mathcal{O}Gi$ is a split injection of $\mathcal{O}$-modules.

Let $\tau \in \mathrm{Irr}(H,c)$. Then $f_\tau i$ is non-zero (since the induced map from $\mathcal{K}Hc$ to $i\mathcal{K}Gi$ is injective). On the other hand, $bi = i$, and $b = \sum_{\chi \in \mathrm{Irr}(G,b)} e_\chi$. Thus $e_\chi i f_\tau$ is non-zero for some $\chi \in \mathrm{Irr}(G,b)$. By the remark following (1), this means that for any $\tau \in \mathrm{Irr}(H,c)$, there exists some $\chi \in \mathrm{Irr}(G,b)$ such that $r(\tau,\chi,i)$ is non-zero. In particular,

$$\sum_{\chi \in \mathrm{Irr}(G,b)} r(\tau,\chi,i) \geq 1 \tag{4}$$

for all $\tau \in \mathrm{Irr}(H,c)$. The second statement of the theorem now follows from (2) and (4).

Suppose from now till the end of the proof that $|\mathrm{Irr}(G,b)| = |\mathrm{Irr}(H,c)|$ and that $\sum_{\chi,\tau} r(\tau,\chi,u) \leq m_{H,D}(u\mathcal{O}Gu)|\mathrm{Irr}(H,c)|$.

Let I be a primitive idempotent decomposition of $(u\mathcal{O}Gu)^H$, and let S be the subset of I consisting of those elements which are not in the kernel of $\mathrm{Br}_D^{u\mathcal{O}Gu}$. Then by (2) and (4), letting χ range over $\mathrm{Irr}(G,b)$ and letting τ range over $\mathrm{Irr}(H,c)$,

$$\begin{aligned}\sum_{\chi,\tau} r(\tau,\chi,u) &= \sum_{j\in S}\sum_{\tau,\chi} r(\tau,\chi,j) + \sum_{j\in I\setminus S}\sum_{\tau,\chi} r(\tau,\chi,j)\\ &= \sum_{j\in S}\sum_{\tau}\sum_{\chi} r(\tau,\chi,j) + \sum_{j\in I\setminus S}\sum_{\tau,\chi} r(\tau,\chi,j)\\ &\geq m_{H,D}(u\mathcal{O}Gu)|\mathrm{Irr}(H,c)| + \sum_{j\in I\setminus S}\sum_{\tau,\chi} r(\tau,\chi,j). \qquad (5)\end{aligned}$$

The above implies that $r(\tau,\chi,j) = 0$ for any $j \in I \setminus S$, $\chi \in \mathrm{Irr}(G,b)$, and $\tau \in \mathrm{Irr}(H,c)$. By the remark following (1), this means that $e_\chi j f_\tau = 0$ for any $j \in I \setminus S$, $\chi \in \mathrm{Irr}(G,b)$, and $\tau \in \mathrm{Irr}(H,c)$. But

$$j = bjc = \Big(\sum_{\chi\in\mathrm{Irr}(G,b)} e_\chi\Big) j \Big(\sum_{\tau\in\mathrm{Irr}(H,c)} f_\tau\Big),$$

hence $j = 0$ and $I = S$ as required.

In fact, it follows from (4) and (5) that for any $\tau \in \mathrm{Irr}(H,c)$, and $i \in I$,

$$\sum_{\chi\in\mathrm{Irr}(G,b)} r(\tau,\chi,i) = 1.$$

This means that for any $\tau \in \mathrm{Irr}(H,c)$ and any $i \in I$, there is a unique $\chi \in \mathrm{Irr}(G,b)$ for which $r(\tau,\chi,i) = 1$ and $r(\tau,\psi,i) = 0$ for all $\psi \in \mathrm{Irr}(G,b)$ different from χ. Similarly, rewriting (5) by interchanging the order of the summations over χ and τ, and using (3), we get

$$\sum_{\tau\in\mathrm{Irr}(H,c)} r(\tau,\chi,i) = 1$$

for any $\chi \in \mathrm{Irr}(G,b)$ and any $i \in I$, which is to say that for any $\chi \in \mathrm{Irr}(G,b)$ and any $i \in I$, there is a unique $\tau \in \mathrm{Irr}(G,b)$ for which $r(\tau,\chi,i) = 1$ and $r(\phi,\chi,i) = 0$ for all $\phi \in \mathrm{Irr}(H,c)$ different from τ.

Now, fix $i \in I$. The above shows that there is a bijection between $\mathrm{Irr}(H,c)$ and $\mathrm{Irr}(G,b)$ which sends an element τ of $\mathrm{Irr}(H,c)$ to the unique element χ of $\mathrm{Irr}(G,b)$ for which $r(\tau,\chi,i)$ is non-zero (and hence $r(\tau,\chi,i) = 1$). So, by (1) it follows that for any $\chi \in \mathrm{Irr}(G,b)$ the multiplicity of χ as an irreducible character of the $\mathcal{K}G$ module $\mathcal{K}Gi$ is $\tau(1)$, where τ is the element of $\mathrm{Irr}(H,c)$ corresponding to χ via the bijection described above.

In particular,

$$\begin{aligned}\mathrm{rank}_{\mathcal{O}}(i\mathcal{O}Gi) &= \dim_{\mathcal{K}}(i\mathcal{K}Gi)\\ &= \dim_{\mathcal{K}}\mathrm{End}_{\mathcal{K}G}(\mathcal{K}Gi)\\ &= \sum_{\tau\in\mathrm{Irr}(H,c)} \tau^2(1)\\ &= \dim_{\mathcal{K}}(\mathcal{K}Hc)\\ &= \mathrm{rank}_{\mathcal{O}}(\mathcal{O}Hc).\end{aligned}$$

Since we have already shown that the map $x \to xi$ from $\mathcal{O}Hc$ to $i\mathcal{O}Gi$ is a split injection of $\mathcal{O}$-modules, the above proves that in fact the map is an isomorphism as required.

Finally, let g be a source idempotent of $\mathcal{O}Hc$, that is, g is a primitive idempotent of $(\mathcal{O}Hc)^D$ such that $\mathrm{Br}_D^{\mathcal{O}Hc}(g)$ is non-zero. Since the isomorphism between $\mathcal{O}Hc$

and $i\mathcal{O}Gi$ displayed above is an interior H-algebra isomorphism, it is in particular an isomorphism of interior D-algebras. So, gi is a primitive idempotent of $(i\mathcal{O}Gi)^D$ such that $\mathrm{Br}_D^{i\mathcal{O}Gi}(gi) \neq 0$. By [36, Proposition 4.12], gi is primitive in $(\mathcal{O}Gb)^D$, and, as explained in Subsection 2.4 $\mathrm{Br}_D^{\mathcal{O}Gb}(gi) \neq 0$. In other words, gi is a source idempotent of b. Now, restricting the above map to $g\mathcal{O}Hg$ provides the required isomorphism of source algebras. □

With D as above, let $\pi : \mathcal{F}C_G(D) \to \mathcal{F}(C_G(D)/Z(D))$ denote the canonical surjection. This is a map of $N_G(D)/D$-algebras. For the sake of notational convenience let us denote the composition $\pi \circ \mathrm{Br}_D$ by Ψ, again a $N_G(D)/D$-algebra homomorphism. Thus if $u \in (\mathcal{O}G)^H$, then $\Psi(u)$ is stable under $N_G(H)$. Finally, let us denote by X and $\bar{X}$ the groups $C_G(D)$ and $C_G(D)/Z(D)$, respectively, and by Y and $\bar{Y}$ the groups $C_H(D)$ and $C_H(D)/Z(D)$, respectively. We then have the following.

Proposition 2.6. *Let G, H, b, c, u and D be as in the previous theorem. The image of $(u\mathcal{O}Gu)^H$ under Ψ equals $[\Psi(u)\mathcal{F}\bar{X}\Psi(u)]^{N_H(D)}$. In particular, $m_{H,D}(u\mathcal{O}Gu) = m([\Psi(u)\mathcal{F}\bar{X}\Psi(u)]^{N_H(D)})$ (recall that the last quantity is the cardinality of an idempotent decomposition of $[\Psi(u)\mathcal{F}\bar{X}\Psi(u)]^{N_H(D)}$).*

Proof. By the Mackey formula (see [36, Corollary 11.10]), Br_D gives a surjection from $\mathrm{Tr}_D^H((u\mathcal{O}Gu)^D)$ to $\mathrm{Tr}_{\{1\}}^{N_H(D)/D}(\mathrm{Br}_D(u)\mathcal{F}X\mathrm{Br}_D(u))$. Clearly the image of $\mathrm{Tr}_{\{1\}}^{N_H(D)/D}(\mathrm{Br}_D(u)\mathcal{F}X\mathrm{Br}_D(u))$ under π equals $\mathrm{Tr}_{\{1\}}^{N_H(D)/D}(\Psi(u)\mathcal{F}\bar{X}\Psi(u))$. On the other hand $c \in \mathrm{Tr}_D^H((\mathcal{O}H)^D)$, since D is a defect group of $\mathcal{O}Hc$. Now the fact that H stabilizes u implies that $u = uc \in \mathrm{Tr}_D^H((\mathcal{O}G)^D)$. Thus, $\mathrm{Tr}_D^H((u\mathcal{O}Gu)^D) = (u\mathcal{O}Gu)^H$. Since Br_D and π are unitary algebra homomorphism it follows that $\Psi((u\mathcal{O}Gu)^D) = \mathrm{Tr}_{\{1\}}^{N_H(D)/D}(\Psi(u)\mathcal{F}\bar{X}\Psi(u))$. The latter is an ideal in $[\Psi(u)\mathcal{F}\bar{X}\Psi(u)]^{N_H(D)}$ containing $\Psi(u)$, giving the first assertion The second assertion follows from the first by the idempotent lifting lemma and the fact that $\ker(\pi) \subset J(\mathcal{F}X)$. □

The above proposition allows us to calculate $m_{H,D}(u\mathcal{O}Gu)$ in certain situations. Recall that by Brauer's first main theorem, b determines a set, $\Lambda(b)$, of ordinary irreducible defect 0 characters of $\mathcal{K}\bar{X}$; the elements of $\Lambda(b)$ are transitively permuted by $N_G(D)$ (see [31, 5.8.13]).The set $\Lambda(b)$ is characterized as follows: For $\theta \in \Lambda(b)$, if b_θ is the block (of defect zero) of $\mathcal{F}\bar{X}$ corresponding to the character θ, then

$$\Psi(b) = \sum_{\theta \in \Lambda(b)} b_\theta.$$

Let $\Lambda(c)$ be defined similarly. For θ in $\mathrm{Irr}(\bar{X})$, $\eta \in \mathrm{Irr}(\bar{Y})$, let $r(\eta, \theta, \Psi(u))$ stand for the multiplicity of θ as a character of the $\mathcal{K}\bar{X}$-module $\mathcal{K}\bar{X}\widehat{\Psi(u)} \otimes_{\mathcal{K}\bar{Y}} W_\eta$, where $\widehat{\Psi(u)}$ is an idempotent of $(\mathcal{O}\bar{X})^{\bar{Y}}$ lifting $\Psi(u)$ and where W_η is a $\mathcal{K}\bar{Y}$-module affording the character η.

Proposition 2.7. *Suppose that $\theta \in \Lambda(b)$ and $\alpha \in N_H(D, \theta) := \{h \in N_H(D) \mid {}^h\theta = \theta\}$ are such that $\langle Y, \alpha \rangle$ and $N_H(D, \theta)$ have the same image in the group of algebra automorphisms of $\mathcal{F}\bar{X}b_\theta$. If α stabilizes every element of $\Lambda(c)$, then*

$$m_{H,D}(u\mathcal{O}Gu) \geq \sum_{\eta \in \Lambda(c)} r(\eta, \theta, \Psi(u)).$$

Proof. Since b_θ is a central idempotent of $\mathcal{F}\bar{X}$ and $N_H(D, \theta) = N_H(D, b_\theta)$, the map from $[\Psi(u)\mathcal{F}\bar{X}\Psi(u)b_\theta]^{N_H(D,\theta)}$ to $[\Psi(u)\mathcal{F}\bar{X}\Psi(u)]^{N_H(D)}$ given by sending an element x to $\mathrm{Tr}_{N_H(D,\theta)}^{N_H(D)}(x)$ is an injective algebra homomorphism (since for $r, s \in N_H(D)$, $x, y \in [\Psi(u)\mathcal{F}\bar{X}\Psi(u)b_\theta]^{N_H(D,\theta)}$, $rxr^{-1}.sys^{-1} = 0$, unless $rs^{-1} \in N_H(D,\theta)$).

It follows from the previous proposition that

$$m_{H,D}(u\mathcal{O}Gu) \geq m([\Psi(u)\mathcal{F}\bar{X}\Psi(u)b_\theta]^{N_H(D,\theta)}). \tag{6}$$

On the other hand, by hypothesis,

$$[\Psi(u)\mathcal{F}\bar{X}\Psi(u)b_\theta]^{N_H(D,\theta)} = [\Psi(u)\mathcal{F}\bar{X}\Psi(u)b_\theta]^{\langle Y,\alpha\rangle}. \tag{7}$$

Note that since $\Psi(u) = \Psi(uc) = \Psi(u)\Psi(c)$ and $\Psi(c)$ is the sum of the blocks c_η of $\mathcal{F}\bar{Y}$ corresponding to elements η of $\Lambda(c)$, there is an inclusion of algebras

$$\prod_{\eta\in\Lambda(c)} c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta \subseteq \Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u).$$

Since $\alpha \in N_H(D,\theta)$ stabilizes $\Psi(u)$, b_θ, and, by hypothesis, c_η for all $\eta \in \Lambda(c)$, we have the inclusion of algebras

$$\prod_{\eta\in\Lambda(c)} [c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta]^{\langle Y,\alpha\rangle} \subseteq [\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)]^{\langle Y,\alpha\rangle}$$

which yields in particular that

$$m([\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)]^{\langle Y,\alpha\rangle}) \geq \sum_{\eta\in\Lambda(c)} m([c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta]^{\langle Y,\alpha\rangle}). \tag{8}$$

Suppose that $\eta \in \Lambda(c)$ is such that $b_\theta\Psi(u)c_\eta$ is non-zero. Then, since $\mathcal{F}\bar{Y}c_\eta$ and $\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)$ (and hence $c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta$) are simple algebras, the map $\mathcal{F}\bar{Y}c_\eta \to c_\eta\Psi(u)b_\theta\mathcal{F}\bar{X}\Psi(u)c_\eta$ given by $x \to xb_\theta\Psi(u)$ is an injective unitary algebra homomorphism (note that elements of $\mathcal{F}\bar{Y}$ stabilize $\Psi(u)$). Identifying $\mathcal{F}\bar{Y}c_\eta$ with its image in $c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta$ under the above homomorphism, by the double centralizer property, there is an algebra isomorphism

$$c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta \cong \mathcal{F}\bar{Y}c_\eta \otimes_{\mathcal{F}} [c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta]^{\bar{Y}}.$$

On the other hand, by the same type of arguments as used in the proof of Theorem 2.5, we can show that the $\mathcal{F}$-dimension of $c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta$ equals $\eta(1)^2r^2(\eta,\theta,\Psi(u))$ (the change from $\mathcal{K}$ to $\mathcal{F}$ is not significant in this situation as all blocks considered are of defect zero). This, along with the above isomorphism gives that $[c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta]^{\bar{Y}}$ is a matrix algebra and that

$$m([c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta]^{\bar{Y}}) = r(\eta,\theta,\Psi(u)). \tag{9}$$

Also, note that since $b_\theta\Psi(u)c_\eta$ is non-zero precisely when $r(\eta,\theta,\Psi(u))$ is a positive integer, the above holds for all $\eta \in \Lambda(c)$.

Since $\bar{Y}$ is normalized by α, α acts on $[c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta]^{\bar{Y}}$. Further, α induces an l'-automorphism of $[c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta]^{\bar{Y}}$ since $N_G(D,\theta)/DX$ is an l'-group ([31, 5.8.13]). Hence, by the Noether-Skolem theorem, there is a semi-simple, invertible element t in the matrix algebra $[c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta]^{\bar{Y}}$ such that

$$[c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta]^{\langle Y,\alpha\rangle} \cong [c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta]^{\langle \bar{Y},t\rangle}.$$

Since t is diagonalizable, $[c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta]^{\langle \bar{Y},t\rangle}$ is clearly isomorphic to a product of matrix algebras, so that in particular

$$m([c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta]^{\langle Y,\alpha\rangle}) = m([c_\eta\Psi(u)\mathcal{F}\bar{X}b_\theta\Psi(u)c_\eta]^{\bar{Y}}). \tag{10}$$

The desired results now follow from the equations (6)–(10). □

3. Unipotent blocks of classical groups

As in the first paper [23] of this series we are concerned with unipotent blocks of classical groups. Let us recall the usual setup of Deligne-Lusztig theory. For a prime power q, let $\bar{\mathbb{F}}_q$ denote an algebraic closure of the finite field $\mathbb{F}_q$, with q elements. Let $\mathbf{G}$ be a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, and let F be a Frobenius morphism of $\mathbf{G}$ (relative to some $\mathbb{F}_q$-structure of $\mathbf{G}$). We then put $G := \mathbf{G}^F$.

Let l be a prime not dividing q, and let $(\mathcal{K}, \mathcal{O}, \mathcal{F})$ be a splitting l-modular system for G. For further notation concerning groups of Lie type and their unipotent l-blocks we refer the reader to [23, Sections 5,6].

3.1. **A theorem of Cabanes and Enguehard.** To make use of the full power of Deligne-Lusztig theory, it is convenient to assume that $\mathbf{G}$ has connected center. Since we are ultimately interested in quasi-simple groups, we have to descend to commutator subgroups. For this purpose the following general result will be useful.

Let $\theta : \mathbf{G}_{\text{sc}} \to [\mathbf{G}, \mathbf{G}]$ denote a simply-connected covering, defined over $\mathbb{F}_q$, of the commutator group $[\mathbf{G}, \mathbf{G}]$. The theorem below, due to Cabanes and Enguehard gives a bijection, preserving defect groups, between the unipotent l-blocks of G and those of $[\mathbf{G}, \mathbf{G}]^F$.

Theorem 3.2. (Cabanes, Enguehard [12]) *Let K be a subgroup of G containing $\theta(\mathbf{G}_{\text{sc}}^F)$. Then K contains the commutator subgroup of G. Let l be a prime which does not divide any of the numbers q, $|G/K|$ or $|Z(\mathbf{G}_{\text{sc}})^F|$. Then restriction of characters from G to K induces an injection from $\mathcal{E}_l(G,(1))$ into $\mathrm{Irr}(K)$.*

In particular, every block B of $\mathcal{O}G$ contained in $\mathcal{E}_l(G,(1))$ covers a unique block b of $\mathcal{O}K$. Moreover, B and b are isomorphic and have a common defect group.

Proof. The first assertion follows from the fact that $\theta(\mathbf{G}_{\text{sc}}^F)$ is a normal subgroup of G with abelian factor group (see, e.g. [28, (7.4.2)]), the second one is a reformulation of [12, Lemma 2(iii)]. Let B be a unipotent l-block of G covering a block b of $\mathcal{O}K$. Then b is uniquely determined by B. If $\rho \in \mathrm{Irr}(K, b)$, then there is an irreducible character of B restricting to ρ. Therefore, restriction of ordinary irreducible characters from B to b is a bijection. The result now follows from [12, Propostion 6]. □

The correspondence of blocks described in the previous theorem is compatible with Harish-Chandra restriction (for subgroups $K = \mathbf{K}^F$, where $\mathbf{K}$ is a closed connected F-stable subgroup of $\mathbf{G}$). This is a general property of Harish-Chandra restriction, a special case of which was stated as Lemma 5.7 in [23]. However, the proof of the reduction theorem [23, Theorem 7.10] uses a more general version of this lemma, valid for Lusztig induction from d-split Levi subgroups (see e.g., the last paragraph of [23, Page 406]). For this reason we now state this more general result, which was already needed in [23] and which will again be used later on.

Lemma 3.3. *Let $\mathbf{K}$ be a closed connected F-stable subgroup of $\mathbf{G}$ containing $[\mathbf{G}, \mathbf{G}]$. Moreover, let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$, which is a Levi complement in the parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Then $\mathbf{P} \cap \mathbf{K}$ is a parabolic subgroup of $\mathbf{K}$, and $\mathbf{L} \cap \mathbf{K}$ is an F-stable Levi complement of $\mathbf{P} \cap \mathbf{K}$ containing $[\mathbf{L}, \mathbf{L}]$. Moreover,*

$$\mathrm{Res}^G_K(R^{\mathbf{G}}_{\mathbf{L} \subset \mathbf{P}}(\zeta)) = R^{\mathbf{K}}_{\mathbf{L} \cap \mathbf{K} \subset \mathbf{P} \cap \mathbf{K}}(\mathrm{Res}^L_{L \cap K}(\zeta))$$

for every $\zeta \in \mathcal{E}_l(L, (1))$.

Proof. The first two statements are clear. For the last, see [2, Corollary 2.1.3] and the subsequent remarks. □

3.4. **Notations for unipotent blocks.** We shall continue with the notation of [23, Section 6]. Thus let $\mathbf{G}$ be a connected reductive algebraic group with connected center such that $G = \mathbf{G}^F = G_n(q)$ is a classical group as in [23, Subsection 4.1]. (The group $\mathbf{G}$ can be taken to be one of the classical groups described in [23, 4.2.3–4.2.5].)

Let d denote the order of q modulo l. If G is a unitary group, we let $e := e(G, l)$ denote the order of $-q$ modulo l, whereas $e := e(G, l)$ is the order of q^2 modulo l in the other cases. We say that l is *unitary* for G, if $l = 2$ or if l is odd and $e = d/2$. Otherwise, l is called *linear* for G (see [23, 6.1]).

Let B be a unipotent l-block of G. By work of Fong and Srinivasan (see [18]), one can associate a non-negative integer $w := w(B)$ to B, called the *weight* of B (see [23, Subsection 6.2]). By work of Cabanes and Enguehard, one can associate a d-cuspidal pair $(\mathbf{K}, \psi)$ to B, uniquely determined by B up to G-conjugacy (see [13, Theorem 4.4]). The partition or symbol γ labelling the unipotent character ψ of K is called the *unipotent label* of B.

The set of unipotent $\mathcal{K}$-characters in B will be denoted by $\mathrm{Irr}_u(G, B)$.

3.5. **Unipotent blocks for linear primes.** Suppose that l is a linear prime for G. In this case the d-cuspidal pair associated to B can be chosen in a particular way which simplifies some of the arguments below. If $G = GU_n(q)$, then e is even (see [23, Subsection 6.1]) an we put $e' = e/2$. In the other cases we let $e' = e$. Put $w = w(B)$. Using the notation of [23, Subsection 4.4], the (1-split) Levi subgroup $\mathbf{M}_{e'w}$ is isomorphic to $GL_{e'w}(\bar{\mathbb{F}}_q) \times \mathbf{G}_{n-2e'w}$, where $\mathbf{G}_{n-2e'w}$ is a group of the same type as $\mathbf{G}$, with natural dimension $n - 2e'w$. The F-stable direct factor $GL_{e'w}(\bar{\mathbb{F}}_q)$ of $\mathbf{M}_{e'w}$ contains a Φ_d-torus $\mathbf{S}$ with $|\mathbf{S}^F| = \phi_d(q)^w$.

Using an argument similar to the one in the last paragraph of the proof of [21, Theorem 8.3], one sees that $C_{\mathbf{G}}(\mathbf{S}) \leq \mathbf{M}_{e'w}$. Moreoever, the image of the direct factor $\mathbf{G}_{n-2e'w}$ of $\mathbf{M}_{e'w}$ in $\mathbf{G}$ (under the embedding of [23, Subsection 4.4]) centralizes $\mathbf{S}$. It follows that $\mathbf{K} := C_{\mathbf{G}}(\mathbf{S}) \cong \mathbf{T} \times \mathbf{G}_{n-2e'w}$, where $\mathbf{T}$ is an F-stable maximal torus of $GL_{e'w}(\bar{\mathbb{F}}_q)$ containing $\mathbf{S}$ with $\mathbf{T}^F \cong (q^e - 1)^w$.

Let $\mathbf{K}$ be as above and let $\mathbf{L}$ be a maximal d-split Levi subgroup of $\mathbf{G}$ containing $\mathbf{K}$. We claim that $\mathbf{L} = C_{\mathbf{G}}(\mathbf{S}_0)$ for some Φ_d-torus contained in $\mathbf{S}$. Indeed, there is some Φ_d-torus $\mathbf{S}'$ of $\mathbf{G}$ with $\mathbf{L} = C_{\mathbf{G}}(\mathbf{S}')$. Since $\mathbf{S} \leq \mathbf{K} \leq \mathbf{L}$, we have $\mathbf{S}' \leq Z(\mathbf{L}) \leq \mathbf{K}$. Hence $\mathbf{S}' \leq Z(\mathbf{K}) = \mathbf{T} \times Z(\mathbf{G}_{n-2e'w})$. Now $Z(\mathbf{G}) \leq Z(\mathbf{K})$ and hence $\mathbf{S}'Z(\mathbf{G}) = \mathbf{S}_0 \times Z(\mathbf{G})$ for some Φ_d-torus $\mathbf{S}_0$ of $GL_{e'w}(\bar{\mathbb{F}}_q)$ contained in $\mathbf{T}$ and hence in $\mathbf{S}$. Since $C_{\mathbf{G}}(\mathbf{S}_0) = C_{\mathbf{G}}(\mathbf{S}'Z(\mathbf{G})) = \mathbf{L}$, this proves the above claim. Since $\mathbf{L}$ is maximal, this easily implies that there is some $1 \leq r \leq w$ such that $\mathbf{L} \cong GL_r(\bar{\mathbb{F}}_q) \times \mathbf{G}_{n-2e'r}$ and $L = \mathbf{L}^F \cong GL_r(q^e) \times G_{n-2e'r}(q)$.

4. Donovan's conjecture for linear primes and unipotent blocks

In this section we prove Donovan's conjecture for unipotent blocks of classical groups in the linear prime case. We begin by showing that the decompositon numbers in a unipotent block are bounded by a function in the defect of the block and the decomposition numbers occuring in unipotent characters. The results of Gruber and Hiss [22] then give bounds for the decompositon numbers in terms of the defect. Finally, we use the fact that the unipotent characters of classical groups are rational valued.

4.1. **Bounds on decomposition numbers.** Here we derive bounds on decomposition numbers in unipotent blocks in terms of decomposition numbers of unipotent characters.

Lemma 4.2. *Let G, l, d, e, B and w be as in Subsection 3.4 and let $(\mathbf{K}, \psi)$ be a d-cuspidal pair in $\mathbf{G}$ giving rise to B.*

Let $\mathbf{L}$ be a d-split Levi subgroup of $\mathbf{G}$ containing $\mathbf{K}$, and let ρ be an irreducible constituent of $R_K^L(\psi)$. Write

$$R_L^G(\rho) = \sum_{\chi \in \mathrm{Irr}_u(G)} a_\chi \chi$$

with $a_\chi \in \mathbb{Z}$. Then $a_\chi = 0$ if χ does not lie in B, and $|a_\chi| \leq (2e)^w w!$ for $\chi \in \mathrm{Irr}_u(B)$.

Proof. The set of irreducible constituents of $R_K^G(\psi)$ coincides with $\mathrm{Irr}_u(B)$ by [13, Theorem 4.4]. The fact that $a_\chi = 0$ if $\chi \notin \mathrm{Irr}_u(B)$ now follows from the transitivity of d-Lusztig induction (see [9, Theorem 3.11]).

Let $W_G(\mathbf{K}, \psi) := N_G(\mathbf{K}, \psi)/K$ denote the relative Weyl group of $(\mathbf{K}, \psi)$. Then $W_G(\mathbf{K}, \psi) \cong C_e \wr S_w$ if G is a unitary group. Otherwise $W_G(\mathbf{K}, \psi)$ is isomorphic to a subgroup of index at most 2 in $C_{2e} \wr S_w$. (For proofs of these facts see [9, pp. 45–52].)

It follows from [9, Fundamental Theorem 3.2] that the multiplicities of the irreducible constituents of $R_L^G(\rho)$ are (up to signs) the same as the multiplicities of the irreducible constituents of $\mathrm{Ind}_{W_L(\mathbf{K},\psi)}^{W_G(\mathbf{K},\psi)}(\vartheta)$, where ϑ is a suitable irreducible character of $W_L(\mathbf{K}, \psi)$. Hence these multiplicities are at most equal to $(2e)^w w!$. $\square$

Lemma 4.3. *Let G, l, d, e, B and w be as in Subsection 3.4. Suppose that l is linear for G and that $w > 0$. Let $(\mathbf{K}, \psi)$ be a d-cuspidal pair in $\mathbf{G}$ giving rise to B.*

Let $\mathbf{L}$ be a maximal d-split Levi subgroup of $\mathbf{G}$ containing $\mathbf{K}$. Then $(\mathbf{K}, \psi)$ is a d-cuspidal pair in $\mathbf{L}$. The structure of $L = \mathbf{L}^F$ is given as $L \cong GL_r(q^e) \times M$, where $M = \mathbf{M}^F$ is a classical group of the same type as G and where $r \leq w$.

Let b be the unipotent block of L arising from $(\mathbf{K}, \psi)$. Then $b \cong b_0 \otimes b_1$, where b_0 is the principal block of $GL_r(q^e)$, and where b_1 is a unipotent block of M of weight $w(b_1) < w$.

Proof. The fact that $(\mathbf{K}, \psi)$ is a d-cuspidal pair in $\mathbf{L}$ follows from the remarks in [13, Subsection 3.1]. The structure of L is derived in Subsection 3.5.

Write $b = b_0 \otimes b_1$ with unipotent blocks of $GL_r(q^e)$ and of M, respectively. Certainly, b_0 is the principal block of $GL_r(q^e)$ since $l \mid q^e - 1$. The d-cuspidal pair of $\mathbf{M}$ giving rise to b_1 equals $(\mathbf{K} \cap \mathbf{M}, \mathrm{Res}_{K \cap M}(\psi))$. If $w' = w(b_1)$ and e' is defined as in Subsection 3.5, then $n - 2e'r - 2e'w' = n - 2e'w$. Since $r \geq 1$, this implies that $w' < w$. $\square$

If G is any finite group, and l is a prime, we write $\gamma_{G,l'}$ for the characteristic function on the l'-elements of G, i.e., $\gamma_{G,l'}(g) = 1$ if g is l-regular, and $\gamma_{G,l'}(g) = 0$, otherwise. Then $\gamma_{G,l'}$ is a class function on G. Moreoever, for a prime $p \neq l$, we have $\gamma_{G,l'}(g) = \gamma_{G,l'}(g_{p'})$ for all $g \in G$ (where $g_{p'}$ denotes the p'-part of g).

We shall view the l-modular Brauer characters as class functions defined on all of G by extending them by 0 to the l-singular elements of G. For $\chi \in \mathrm{Irr}(G, B)$ we may thus write

$$\gamma_{G,l'}\chi = \sum_{\varphi \in \mathrm{IBr}(G,B)} d_{\chi,\varphi}\varphi.$$

For a prime l, an integer e dividing $l-1$ and a non-negative integer w we write $\tilde{\mathcal{B}}_u(l, e, w)$ for the set of pairs (G, B) where G is a classical group as in Subsection 3.4 such that $e(G, l) = e$ and l is linear for G, and B is a unipotent block of G of weight $w(B) \leq w$. Next, we put

$$\tilde{k}_u(e, w) = \max\{|\mathrm{Irr}_u(G, B)| \mid (G, B) \in \tilde{\mathcal{B}}_u(l, e, w)\}$$

and

$$\tilde{m}_u(l, e, w) := \max\{d_{\chi,\varphi} \mid \chi \in \mathrm{Irr}_u(G, B), \varphi \in \mathrm{IBr}(G, B) \text{ for some } (G, B) \in \tilde{\mathcal{B}}_u(l, e, w)\}.$$

Notice that $\tilde{k}_u(e, w)$ only depends on e and not on l, and that $\tilde{k}_u(e, w)$ and $\tilde{m}_u(l, e, w)$ are non-decreasing as functions in w (for fixed l and e).

Proposition 4.4. *Let G, l, d, e, B and w be as in Subsection 3.4 and suppose that l is linear for G. Put $k := \tilde{k}_u(e, w)$ and $M := \tilde{m}_u(l, e, w)$.*

Then the decomposition numbers in B are bounded above by

$$f(l, e, w) := k^{w(k+3)} M^{w(k+1)} (2e)^{w(w+1)/2} \prod_{i=1}^{w} (i!)^4.$$

Proof. The proof is by induction on w. Note that $f(l, e, 0) = 1$, so the assertion is true for $w = 0$. Suppose then that $w > 0$. We choose a d-cuspidal pair $(\mathbf{K}, \psi)$ giving rise to B. Let D be a Sylow l-subgroup of $C^0_{\mathbf{G}}([\mathbf{K}, \mathbf{K}])^F$. Then D is a defect group of B by [13, Theorem 4.4(ii)].

Let ξ be an ordinary irreducible character of B. Then there is an element t of D, a linear character $\hat{t}$ of $C_G(t)$ of l-power order, and a unipotent character λ of $C_G(t)$ such that $\xi = \varepsilon_{\mathbf{G}}\varepsilon_{C_{\mathbf{G}}(t)} R^G_{C_G(t)}(\hat{t}\lambda)$ (see [13, Theorem 4.4] and [23, Subsections 6.2, 6.3, and Lemma 7.7]).

If $t \in Z(G)$, then $\xi = \hat{t}\lambda$ and thus ξ has the same restriction to the l-regular classes as the unipotent character λ of G. Certainly $\tilde{m}_u(l, e, w) = M \leq f(l, e, w)$.

We may thus assume that t does not lie in the center of G. By [13, Propositon 2.5], there is a proper d-split Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ containing $\mathbf{C} := C_{\mathbf{G}}(t)$. Let us take a maximal such Levi subgroup $\mathbf{L}$. By [13, Theorem 4.4(iii)], we have $[\mathbf{K}, \mathbf{K}] \subseteq \mathbf{C} \subseteq \mathbf{L}$. It follows from [13, Proposition 1.7], that there is an element $g \in G$ centralizing $[\mathbf{K}, \mathbf{K}]$ such that $\mathbf{K} \leq g\mathbf{L}g^{-1}$. Replacing t and $\mathbf{L}$ by their g-conjugates, we may assume that $\mathbf{K} \subseteq \mathbf{L}$ and $\mathbf{C} \subseteq \mathbf{L}$.

Again by [13, Theorem 4.4], there is a unipotent l-block b of L arising from the d-cuspidal pair $(\mathbf{K}, \psi)$. We claim that the decomposition numbers in b are bounded above by $w!f(l, e, w-1)$. Indeed, by Lemma 4.3, $L \cong GL_r(q^e) \times G_{n-2e'r}(q)$, and b is the tensor product of the principal block b_0 of $GL_r(q^e)$ and of an l-block b_1 of weight smaller than w of $G_{n-2e'r}(q)$ (here, e' has the same meaning as in Subsection 3.5). Since $r \leq w$, the claim follows from [22, Corollary 8.10] and the induction hypothesis.

Put $\mu := R^L_C(\hat{t}\lambda)$. Then $\pm\mu$ is an ordinary irreducible character in b. Write $\gamma_{L,l'}\mu = \sum_{\vartheta \in \mathrm{IBr}(L,b)} z_\vartheta \vartheta$. Then, by the above,

$$|z_\vartheta| \leq w!f(l, e, w-1), \tag{11}$$

for all $\vartheta \in \mathrm{IBr}(L, b)$.

Let $\vartheta \in \mathrm{IBr}(L, b)$. It follows from the main result of [20] that we may write $\vartheta = \sum_{\rho \in \mathrm{Irr}_u(L,b)} y_{\rho,\vartheta} \gamma_{L,l'} \rho$ for some $y_{\rho,\vartheta} \in \mathbb{Z}$. The decomposition numbers of unipotent characters in b_0 are bounded above by $w!$ and those in b_1 by $\tilde{m}_u(l, e, w-1)$. The decomposition matrix of b_0 is upper triangular. Hence the entries in the inverse of the matrix of unipotent decomposition numbers of b_0 are at most $w!^2$ in absolute value. By crudely applying Hadamard's inequality (see, e.g., [15, Proposition 2.2.4]) to the matrix of unipotent decomposition numbers of b_1 we find that the absolute values of the entries in its inverse matrix are bounded above by $k^k \tilde{m}_u(l, e, w-1)^k$. Hence

$$|y_{\rho,\vartheta}| \leq w!^2 k^k \tilde{m}_u(l, e, w-1)^k, \tag{12}$$

for all $\rho \in \mathrm{Irr}_u(L, b)$.

Let $\rho \in \mathrm{Irr}_u(L, b)$. Then $R_L^G(\rho) = \sum_{\chi \in \mathrm{Irr}_u(G)} a_{\chi,\rho} \chi$, with $a_{\chi,\rho} \in \mathbb{Z}$. By Lemma 4.2 we have

$$|a_{\chi,\rho}| \leq (2e)^w w!. \tag{13}$$

Moreover, $a_{\chi,\rho} = 0$ if $\chi \notin B$. Put $\varepsilon := \varepsilon_{\mathbf{G}} \varepsilon_{C_{\mathbf{G}}(t)}$. By [16, Proposition 12.6], Lusztig induction commutes with the restriction to l'-elements, in the sense that it respects multiplication by $\gamma_{G,l'}$. We thus have

$$\begin{aligned}
\varepsilon \gamma_{G,l'} \xi &= R_L^G(\gamma_{L,l'} \mu) \\
&= \sum_{\vartheta \in \mathrm{IBr}(L,b)} z_\vartheta R_L^G(\vartheta) \\
&= \sum_{\rho \in \mathrm{Irr}_u(L,b)} \sum_{\vartheta \in \mathrm{IBr}(L,b)} z_\vartheta y_{\rho,\vartheta} R_L^G(\gamma_{L,l'} \rho) \\
&= \sum_{\chi \in \mathrm{Irr}_u(G,B)} \left(\sum_{\rho \in \mathrm{Irr}_u(L,b)} \sum_{\vartheta \in \mathrm{IBr}(L,b)} z_\vartheta y_{\rho,\vartheta} a_{\chi,\rho} \right) \gamma_{G,l'} \chi.
\end{aligned}$$

Now write

$$\gamma_{G,l'} \chi = \sum_{\varphi \in \mathrm{IBr}(G,B)} d_{\chi,\varphi} \varphi.$$

Since χ is unipotent, we have

$$d_{\chi,\varphi} \leq \tilde{m}_u(l, e, w). \tag{14}$$

Then the multiplicity $d_{\xi,\varphi}$ of $\varphi \in \mathrm{IBr}(G, B)$ in $\gamma_{G,l'} \xi$ is given by

$$d_{\xi,\varphi} = \left| \sum_\rho \sum_\vartheta \sum_\chi z_\vartheta y_{\rho,\vartheta} a_{\chi,\rho} d_{\chi,\varphi} \right|.$$

Each of the sums above has at most k summands. Thus, using Equations (11), (12), (13), (14), and induction, we find that

$$\begin{aligned} d_{\chi,\xi} &\leq k^3\,[w!f(l,e,w-1)]\cdot\left[w!^2k^k\tilde{m}_u(l,e,w-1)^k\right]\cdot[(2e)^w w!]\cdot[\tilde{m}_u(l,e,w)] \\ &\leq k^3\left[w!k^{(w-1)(k+3)}M^{(w-1)(k+1)}(2e)^{w(w-1)/2}\prod_{i=1}^{w-1}(i!)^4\right]\cdot \\ &\quad\left[w!^2k^kM^k\right]\cdot[(2e)^w w!]\cdot M \\ &= k^{w(k+3)}M^{w(k+1)}(2e)^{w(w+1)/2}\prod_{i=1}^{w}(i!)^4. \end{aligned}$$

□

If the defect groups of B are abelian, we obtain a much better bound, without the assumption that l is linear.

Proposition 4.5. *Let G, l, d, e, B and w be as in Subsection 3.4 and suppose that B has an abelian defect group. Let $m_u(B)$ denote the largest decomposition number occurring in some unipotent character of B.*

Then the decomposition numbers of B are bounded above by

$$(2e)^w w! m_u(B)|\mathrm{Irr}_u(G,B)|.$$

Proof. Let $(\mathbf{K},\psi)$ be a d-cuspidal pair giving rise to B. Since B has an abelian defect group, a Sylow l-subgroup D of $Z^0(\mathbf{K})^F$ is a defect group of B (see [11, (2.3)]). In particular, $\mathbf{K}\subseteq C_{\mathbf{G}}(t)$ for all $t\in D$. By assumption, the prime l is $(Z(\mathbf{K})/Z(\mathbf{G}),F,d)$ adapted (see [11, Définiton 1.14]) and thus $C_{\mathbf{G}}(t)$ is a d-split Levi subgroup of $\mathbf{G}$ for all $t\in D$ (see [11, Proposition 1.15]).

Let ξ be an ordinary irreducible character of B. Then, as in the proof of Proposition 4.4, $\xi=\varepsilon_{\mathbf{G}}\varepsilon_{C_{\mathbf{G}}(t)}R^G_{C_G(t)}(\hat{t}\lambda)$, for some $t\in D$, a linear character $\hat{t}$ of $C_G(t)$ and a unipotent character λ of $C_G(t)$. Moreover, by [13, Theorem 4.4(iii)], there is a d-cuspidal pair $(\mathbf{K}_t,\psi_t)$ in $\mathbf{L}:=C_{\mathbf{G}}(t)$ such that $[\mathbf{K},\mathbf{K}]=[\mathbf{K}_t,\mathbf{K}_t]$, and the restrictions of ψ and ψ_t to $[\mathbf{K},\mathbf{K}]^F$ agree. Since $\mathbf{K}\leq\mathbf{L}$, the pair $(\mathbf{K},\psi)$ is d-cuspidal in $\mathbf{L}$, by the remarks in [13, Subsection 3.1]. This implies, by the statement in [13, Definition 3.4] that the two d-cuspidal pairs $(\mathbf{K},\psi)$ and $(\mathbf{K}_t,\psi_t)$ are conjugate in L. We may thus assume without loss of generality that they are equal. In particular, λ is a constituent of $R^L_K(\psi)$.

Let us write $\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{L}}R^G_L(\lambda)=\sum_{\chi\in\mathrm{Irr}_u(G,B)}a_\chi\chi$. Since Lusztig induction commutes with the restriction to l'-elements, we find

$$\begin{aligned} \gamma_{G,l'}\xi &= \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{L}}\gamma_{G,l'}R^G_L(\hat{t}\lambda)=\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{L}}R^G_L(\gamma_{L,l'}\hat{t}\lambda) \\ &= \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{L}}R^G_L(\gamma_{L,l'}\lambda)=\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{L}}\gamma_{G,l'}R^G_L(\lambda) \\ &= \sum_{\chi\in\mathrm{Irr}_u(G,B)}a_\chi\gamma_{G,l'}\chi \\ &= \sum_{\chi\in\mathrm{Irr}_u(G,B)}a_\chi\left(\sum_{\varphi\in\mathrm{IBr}(G,B)}d_{\chi,\varphi}\varphi\right) \\ &= \sum_{\varphi\in\mathrm{IBr}(G,B)}\left(\sum_{\chi\in\mathrm{Irr}_u(G,B)}a_\chi d_{\chi,\varphi}\right)\varphi. \end{aligned}$$

Since $|a_\chi| \leq (2e)^w w!$ by Lemma 4.2, and since $d_{\chi,\varphi} \leq m_u(B)$ by definition, the result follows. □

4.6. **Linear primes and unipotent blocks.** Here we present the main result of this section. This considerably generalizes [23, Theorem 9.4], where we fixed the dimension of the classical groups considered. In this and the next subsection we let l be a prime and $\mathcal{F}$ an algebraically closed field of characteristic l. Note that, given a finite group G, there is always a splitting l-modular system $(\mathcal{K}, \mathcal{O}, \mathcal{F})$ for G.

Theorem 4.7. *Let D be a finite l-group. Then, up to Morita equivalence, there are only finitely many l-blocks with defect group D occurring as unipotent blocks in the group algebras $\mathcal{F}G$ for groups G in the set*

$$\{G_n(q) \mid G_n(q) \text{ as in Subsection 3.4, } l \nmid q \text{ and } l \text{ linear for } G_n(q)\}. \tag{15}$$

Proof. Let B be a unipotent block of $\mathcal{F}G$, with defect group isomorphic to D, for a group $G = G_n(q)$ as in (15). Let w denote the weight of B and consider a unipotent character χ in B. By Harish-Chandra theory, χ corresponds to an ordered or unordered pair (μ_1, μ_2) of partitions, such that $|\mu_1| + |\mu_2| = n$. Since B has e-weight w, and since l is linear for G, each of μ_1, μ_2, has e-weight at most w. By [22, Theorem 8.11], every modular constituent of χ occurs with multiplicity at most M_w^2, where M_w is the largest decomposition number of a unipotent character lying in a block of weight at most w of some group $GL_{n'}(q')$, where $n' \leq n$ and $q' \in \{q, q^2\}$. Theorem 9.3 of [23] implies that M_w is bounded independently of n' and q', hence $\tilde{m}_u(l, e, w)$ is finite. By Proposition 4.4 the decomposition numbers in B are bounded from above by some number depending only on l and w.

Since $l > 2$, the unipotent characters of B form a basic set by [20]. The unipotent characters in a classical group are rational valued (see the proof of [23, Theorem 9.4]). This implies by [23, Propositions 9.1, 9.2] that there are only finitely many Morita equivalence classes among the unipotent blocks with defect group D occurring in groups algebras of the groups of (15). □

4.8. **Descent to commutator subgroups.** We close this section by an investigation of commutator subgroups. This is necessary for the intended applications to Donovan's conjecture. Since the unitary groups will be dealt with in more detail in the final section, we restrict ourselves to the case of classical groups of types B, C, and D here.

Let $G_n(q)$ denote a classical group as in [23, Subsection 4.1], but exclude Case (1), i.e., assume that $G_n(q) \neq GU_n(q)$. We denote the commutator subgroup of $G_n(q)$ by $K_n(q)$, and call an l-block of $K_n(q)$ unipotent, if it is covered by a unipotent l-block of $G_n(q)$.

Theorem 4.9. *Let D be a finite l-group. Then, up to Morita equivalence, there are only finitely many l-blocks with defect group D occurring as unipotent blocks in the group algebras $\mathcal{F}G$ for groups G in the class*

$$\{K_n(q) \mid K_n(q) \text{ as above, } l \nmid q \text{ and } l \text{ linear for } G_n(q)\}. \tag{16}$$

Proof. There is nothing to prove if D is the trivial group, hence we assume $|D| > 1$ in the following. Let $G = K_n(q)$ for a group $K_n(q)$ as in (16), and put $\tilde{G} := G_n(q)$. Let B be a unipotent block of $\mathcal{F}G$ with defect group isomorphic to D, and let $\tilde{B}$ be a unipotent block of $\tilde{G}$ covering B.

There is a simply connected covering $\theta : \mathbf{G}_{\mathrm{sc}} \to [\mathbf{G}, \mathbf{G}]$, defined over $\mathbb{F}_q$, of the commutator group $[\mathbf{G}, \mathbf{G}]$ such that $G = K_n(q) = \theta(\mathbf{G}_{\mathrm{sc}}^F)$. We may thus apply Theorem 3.2. If $l \nmid q-1$, then B and $\tilde{B}$ are Morita equivalent and the latter has defect group D. Suppose then that $l \mid q-1$ and let G_1 denote the subgroup of $\tilde{G}$ containing G such that G_1/G is the Sylow l-subgroup of $\tilde{G}/G$. Then, by Theorem 3.2, $\tilde{B}$ covers a unique block B_1 of G_1, and $\tilde{B}$ and B_1 are Morita equivalent. Moreover, B_1 covers B.

Let Z denote the Sylow l-subgroup of $Z(\tilde{G})$. Then $G_1 = Z \times G$, and thus the defect group of B_1 is isomorphic to $Z \times D$. Since D is non-trivial, the weight of $\tilde{B}$ is larger than 1, and so $|Z| \mid |D|$ by the description of the defect groups (see [13, Theorem 4.4(ii)]). In particular, $|Z \times D| \le |D|^2$. By Theorem 4.7 the Morita equivalence class of B_1 lies in a finite set depending only on the order of $Z \times D$. This implies that there is an upper bound, depending only on $|D|$, for the Cartan invariants of B_1.

Every irreducible character of B is the restriction of an irreducible character of B_1. In particular, the Cartan invariants of B are bounded. Also, the irreducible Brauer character of B are rational valued (since the same is true for those of B_1). By [23, Propositons 9.1, 9.2], there are only finitely many possibilities for the Morita type of B. □

5. Combinatorics

Here we collect the combinatorial results on symbols needed later on in the statement and proof of the reduction theorem.

5.1. **Basic assumptions.** Throughout this section let e, w, i, k denote integers with $e > 1$, $w \ge 0$, $1 \le i \le e-1$, and $k > 0$. We will use freely the language and notation for symbols as given in [23, Section 2] except for the following change. Since e is fixed in this section, if Λ is a $2e$-abacus diagram, we will use Λ^{i_g} to denote $\Lambda_e^{i_g}$, and $w(\Lambda)$ to denote $w_e(\Lambda)$. Also, by a diagram we will always mean a $2e$-abacus diagram.

Lemma 5.2. *Suppose that that Λ and Ψ are diagrams such that Ψ can be obtained from Λ by moving t beads on string i_g one position to the left. If $0 < t < |\Lambda^{i_g}| - |\Lambda^{(i-1)_g}|$, then $w(\Psi^{i_g}) + w(\Psi^{(i-1)_g}) \ge |\Lambda^{i_g}| - |\Lambda^{(i-1)_g}| - 1$.*

Proof. Let S be the set of row indices s such that there is a bead on row s and string i_g of Λ but there is no bead on row s and string $(i-1)_g$ of Λ. Then $|S| \ge |\Lambda^{i_g}| - |\Lambda^{(i-1)_g}|$ and there is a subset T of S of cardinality t such that if for each s in T the bead on row s of string i_g of Λ is moved one position to the left, one obtains Ψ.

Let $s_1 = \min(T)$, $s_2 = \max(T)$, $s_3 = \min(S \setminus T)$, and $s_4 = \max(S \setminus T)$. Note that the assumptions on t imply that T and $S \setminus T$ are non-empty and hence that these integers are all defined.

If $s_1 > s_4$, then $s_2 = \max(S)$ and $s_3 = \min(S)$. Since there is no bead on row s_3 and string $(i-1)_g$ of Ψ and there is a bead on row s_2 and string $(i-1)_g$ of Ψ, we get $w(\Psi^{(i-1)_g}) \ge s_2 - s_3 = |S| - 1 \ge |\Lambda^{i_g}| - |\Lambda^{(i-1)_g}| - 1$.

Similarly, if $s_3 > s_2$, then $s_1 = \min(S)$ and $s_4 = \max(S)$. There is no bead on row s_1 and string i_g of Ψ but there is a bead on row s_4 and string i_g of Ψ, hence $w(\Psi^{i_g}) \ge s_4 - s_1 = |S| - 1 \ge |\Lambda^{i_g}| - |\Lambda^{(i-1)_g}| - 1$. So we may assume that both both s_1 and s_3 are less than or equal to each of s_2 and s_4. We have $w(\Psi^{(i-1)_g}) + w(\Psi^{i_g}) \geq (s_2 - s_3) + (s_4 - s_1) \geq |S| - 1 \geq |\Lambda^{i_g}| - |\Lambda^{(i-1)_g}| - 1$. This completes the proof. $\square$

5.3. **Diagram operations.** Let Δ be a diagram. Denote by $c(\Delta)$ the core of Δ. Let Δ' denote the diagram obtained from Δ by swapping all pairs of strings j_g and j_b, $0 \leq j \leq e-1$. Let Δ_1 be the diagram obtained from Δ by swapping strings i_g and $(i-1)_g$ and let Δ^* be the diagram obtained from Δ by swapping the pair of strings i_g and $(i-1)_g$, and the pair of strings i_b and $(i-1)_b$. Note that $\Delta \mapsto \Delta_1$ and $\Delta \mapsto \Delta^*$ are weight preserving bijections of diagrams. Also note that $c(\Delta') = c(\Delta)'$ and $c(\Delta_1) = c(\Delta)_1$.

In addition, when $e = 2$, we define the diagram $\tilde{\Delta}$ as follows. There is a bead on row t and string 1_g of $\tilde{\Delta}$ if and only if there is a bead on row t and string 0_g of Δ; there is a bead on row t and string 1_b of $\tilde{\Delta}$ if and only if there is a bead on row t and string 0_b of Δ; there is a bead on row 0 and string 0_g of $\tilde{\Delta}$ and for all $t > 0$, there is a bead on row t and string 0_g of $\tilde{\Delta}$ if and only if there is a bead on row $t-1$ and string 1_b of Δ. Similarly, there is a bead on row 0 and string 0_b of $\tilde{\Delta}$ and for all $t > 0$, there is a bead on row t and string 0_b of $\tilde{\Delta}$ if and only if there is a bead on row $t-1$ and string 1_g of Δ. (Note that if Δ is a $2e$-unitary diagram of a symbol δ, then $\tilde{\Delta}$ is the $2e$-unitary diagram of the 1-shift of δ.)

5.4. **Symbols.** Symbols can be represented on an abacus with $2e$ strings either through $2e$-linear diagrams or $2e$-unitary diagrams. We fix a choice of either $2e$-linear diagrams or $2e$-unitary diagrams. So, whenever we speak of a diagram for a symbol (or an equivalence class of symbols) we will consistently either mean a $2e$-linear or a $2e$-unitary diagram. Similarly, when we speak of the weight or core of a symbol we will mean its e-linear core and e-linear weight if we are in the linear case and e-unitary core and e-unitary weight if we are in the unitary case. If λ and δ are equivalence classes of symbols, let $\mathcal{S}_{\lambda,\delta}$ denote the set of sequences

$$\lambda = \delta(0), \delta(1), \cdots, \delta(k) = \delta$$

of classes of symbols such that for each j, $1 \leq j \leq k$, a representative of $\delta(j)$ can be obtained from a representative of $\delta(j-1)$ by the removal of a 1-hook. For $S \in \mathcal{S}_{\lambda,\delta}$, we denote by $d(S)$ the number of integers, j, $1 \leq j \leq k-1$ for which the j-th term of S is a degenerate symbol.

5.5. **The symbol γ.** From now on, till the end of this section let γ be an equivalence class of symbols of weight 0. Let $k = \check{s}_e(\gamma)$ if we are in the linear case and $k = \hat{s}_e(\gamma)$ if we are in the unitary case. Assume that $k > 0$. Choose a diagram Γ for γ with the following properties: There are k more beads on string i_g of Γ than there are on string $(i-1)_g$ of Γ and there are enough beads on Γ so that for any equivalence class of symbols λ whose weight is at most w and whose core is γ, there is a diagram Λ for λ with core Γ.

5.6. **The symbol γ_1.** Let γ_1 be the class of symbols corresponding to Γ_1 (see 5.3). Since there are exactly k more beads on string i_g of Γ than on string $(i-1)_g$ and since all the beads of Γ are as high as possible on their strings, we can also obtain Γ_1 from Γ by moving the bottom k beads on string i_g one position to the left. Note that $\mathrm{rank}(\gamma_1) = \mathrm{rank}(\gamma) - k$ and that if either γ or γ_1 is degenerate, the other is not.

5.7. **The symbol $\bar{\gamma}$.** If neither γ nor γ_1 is degenerate, we put $\bar{\gamma} := \gamma_1$. In this case $\bar{\gamma}$ is non-degenerate and $\mathrm{rank}(\bar{\gamma}) = \mathrm{rank}(\gamma) - k$.

If γ is degenerate and does not contain $\{\{1\},\{1\}\}$, define $\bar{\gamma}$ to be the symbol corresponding to the diagram Γ^* (if Γ is a diagram for $\{\{1\},\{1\}\}$, then Γ^* is not the diagram of a symbol). Note that since γ is degenerate, there are also k more beads on string i_b of Γ_1 than on string $(i-1)_b$ and so Γ^* can be obtained from Γ_1 by moving the bottom k beads on string i_b to string $(i-1)_b$. Note that $\bar{\gamma}$ is degenerate and $\mathrm{rank}(\bar{\gamma}) = \mathrm{rank}(\gamma) - 2k$.

If γ_1 is degenerate, $e = 2$ and we are in the situation where symbols are represented by $2e$-unitary diagrams, we define $\bar{\gamma}$ as the class of symbols corresponding to the diagram $(\widetilde{\Gamma_1})_1$. Note that since we are in the case of $2e$-unitary diagrams, $\widetilde{\Gamma_1}$ represents γ_1. Also, $\widetilde{\Gamma_1}$ has $k-1$ more beads on string 1_g than on 0_g and $k-1$ more beads on string 1_b than on string 0_b. In this case $\bar{\gamma}$ is non-degenerate and $\mathrm{rank}(\bar{\gamma}) = \mathrm{rank}(\gamma) - 2k + 1$.

5.8. **The correspondence $\lambda \mapsto \lambda_1$.** For each class of symbols λ of weight at most w and core γ choose once and for all a diagram Λ representing λ such that Γ is the core of Λ. Note that if γ is non-degenerate, or if λ is degenerate, there is a unique choice for Λ, but if γ is degenerate and Λ is non-degenerate, there are two choices for Λ and we fix one.

If λ is a class of symbols of weight w and core γ and if Λ is the chosen diagram for λ, define λ_1 to be the class of symbols corresponding to the diagram Λ_1. Then, $\lambda \mapsto \lambda_1$ is a map from the set of classes of symbols of weight w and core γ to the set of classes of symbols of weight w and core γ_1.

Lemma 5.9. *Let $\tilde{w}$ be an integer with $0 \le \tilde{w} \le w$. Suppose that both γ and γ_1 are non-degenerate and that $k \ge \tilde{w} + 2$. Let λ and θ be classes of symbols of weight $\tilde{w}$ and core γ, and suppose that δ is a class of symbols of weight $\tilde{w}$ and core γ_1.*

Then $|\mathcal{S}_{\lambda,\lambda_1}| = k!$, $|\mathcal{S}_{\lambda,\delta}| = 0$ if $\delta \ne \lambda_1$, and $|\mathcal{S}_{\theta,\lambda_1}| = 0$ if $\theta \ne \lambda$. Further, $d(S) = 0$ for any $S \in \mathcal{S}_{\lambda,\lambda_1}$. The map $\lambda \mapsto \lambda_1$ is a bijection between classes of symbols of weight $\tilde{w}$ and core γ and classes of symbols of weight $\tilde{w}$ and core γ_1.

Proof. Let Λ be the chosen diagram for λ and suppose that $|\mathcal{S}_{\lambda,\delta}|$ is non-zero. Then there is a sequence of abacus diagrams

$$\Lambda = \Delta(0), \Delta(1), \cdots, \Delta(k)$$

such that for each j, $1 \le j \le k$, $\Delta(j)$ is obtained from $\Delta(j-1)$ by moving a bead one position to the left, and such that δ is the class of symbols defined by $\Delta(k)$. Let us consider such a sequence. The core of $\Delta(k)$ represents the class γ_1. On the other hand, the number of beads on $\Delta(k)$ is equal to the number of beads on the diagram Γ_1. Hence, either the core of $\Delta(k)$ is Γ_1 or it is $(\Gamma_1)'$ (recall that $(\Gamma_1)'$ is the swap of Γ_1, see 5.3).

We claim that the core of $\Delta(k)$ is Γ_1. If not, we would have the following equalities.

(a) $|\Delta(k)^{i_g}| = |\Gamma_1^{i_b}| = |\Gamma^{i_b}| = |\Lambda^{i_b}|$.
(b) $|\Delta(k)^{i_b}| = |\Gamma_1^{i_g}| = |\Gamma^{(i-1)_g}| = |\Lambda^{(i-1)_g}|$.
(c) $|\Delta(k)^{(i-1)_g}| = |\Gamma_1^{(i-1)_b}| = |\Gamma^{(i-1)_b}| = |\Lambda^{(i-1)_b}|$.
(d) $|\Delta(k)^{(i-1)_b}| = |\Gamma_1^{(i-1)_g}| = |\Gamma^{i_g}| = |\Lambda^{i_g}|$.
(e) $|\Delta(k)^{j_g}| = |\Gamma_1^{j_b}| = |\Gamma^{j_b}| = |\Lambda^{j_b}|$ for all j different from i and $i-1$.
(f) $|\Delta(k)^{j_b}| = |\Gamma_1^{j_g}| = |\Gamma^{j_g}| = |\Lambda^{j_g}|$ for all j different from i and $i-1$.

Let $l_1 = |\Lambda^{i_g}| - |\Lambda^{i_b}|$ and $l_2 = |\Lambda^{i_b}| - |\Lambda^{(i-1)_g}|$. If $l_1 \geq 0$, then by (a) in order to get from Λ to $\Delta(k)$, at least l_1 beads on string i_g of Λ must be moved one position to the left. In particular, $l_1 \leq k$. Similarly, if $l_2 \geq 0$, then by (b) at least l_2 beads must be moved from string i_b of Λ to get from Λ to $\Delta(k)$, hence $l_2 \leq k$. But $l_1 + l_2 = k$. Thus $0 \leq l_1, l_2 \leq k$, and the only strings from which beads are removed in going from Λ to $\Delta(k)$ are the strings i_g and i_b. In particular, by (a)–(f) we have $|\Gamma^{j_g}| = |\Delta(k)^{j_b}| = |\Lambda^{j_b}| = |\Gamma^{j_b}|$ for all j different from i and $i-1$, and $|\Gamma^{(i-1)_g}| + |\Gamma^{i_g}| = |\Delta(k)^{(i-1)_b}| + |\Delta(k)^{i_b}| = |\Lambda^{(i-1)_b}| + |\Lambda^{i_b}| = |\Gamma^{(i-1)_b}| + |\Gamma^{i_b}|$. Since neither Γ nor Γ_1 is degenerate, we must have $0 < l_1, l_2 < k$. On the other hand, since $\Delta(k)$ is obtained from Λ by moving l_1 beads on string i_g and l_2 beads on string i_b one position to the left, there exists a sequence of diagrams

$$\Lambda = \Pi(0), \Pi(1), \cdots, \Pi(k) = \Delta(k)$$

such that for each j, $1 \leq j \leq l_1$, $\Pi(j)$ is obtained from $\Pi(j-1)$ by moving a bead on string i_g one position to the left, and for $l_1 + 1 \leq j \leq k$, $\Pi(j)$ is obtained from $\Pi(j-1)$ by moving a bead on string i_b one position to the left. By Lemma 5.2 we have $w(\Pi(l_1)^{i_g}) + w(\Pi(l_1)^{(i-1)_g}) \geq k-1$. Since in going from $\Pi(l_1)$ to $\Delta(k)$ the j_g strings are not disturbed, $w(\Delta(k)) \geq w(\Pi(l_1)^{i_g}) + w(\Pi(l_1)^{(i-1)_g})$. Hence $\tilde{w} = w(\Delta(k)) \geq k-1$, a contradiction. This proves the claim.

Thus the only way to get from Λ to $\Delta(k)$ is by successively moving k beads from string i_g of Λ one position to the left. Since $k \geq \tilde{w}$, there is no row index s such that there is a bead on row s of string $(i-1)_g$ and there is no bead on row s and string i_g of Λ. Thus, there are exactly k beads on string i_g of Λ such that there is no bead on the corresponding position of string $(i-1)_g$ of Λ, and it is exactly these k beads that will be moved. In other words, $\Delta(k)$ is Λ_1 and $\delta = \lambda_1$. (The reasoning in this paragraph follows exactly the original argument of Joanna Scopes in the proof of [35, Lemma 2.1].)

By reversing the argument after (a) through (f) and the argument in Lemma 5.2 in the sense of moving beads to the right rather than to the left, we see that if we start with Λ_1 and successively move k beads to the right in such a way that we arrive at a diagram with core representing γ, then this diagram has to be Λ. It follows that if θ is a class of symbols of weight $\tilde{w}$ and core λ, then $|\mathcal{S}_{\theta,\lambda_1}| = 0$ unless $\theta = \lambda$.

The above also shows that $|\mathcal{S}_{\lambda,\lambda_1}|$ is at most equal to the number of sequences of diagrams of the above type linking Λ to Λ_1 and that this latter number is $k!$. Conversely, since any sequence of diagrams linking Λ to Λ_1 only involves moving beads from string i_g to $(i-1)_g$, it is clear that any two such sequences represent distinct sequences of classes of symbols linking λ and λ_1. Thus $|\mathcal{S}_{\lambda,\lambda_1}| = k!$.

Next, we claim that $d(S) = 0$ for any S in $\mathcal{S}_{\lambda,\lambda_1}$. If not, there is a sequence $\Lambda = \Delta(0), \cdots, \Delta(k) = \Lambda_1$ obtained from Λ by successively moving a bead from string i_g to string $(i-1)_g$ such that for some j, $0 \leq j \leq k$, $\Delta(j)$ is a degenerate diagram. Both γ and γ_1 are non-degenerate, hence $0 < j < k$. By Lemma 5.2, we have $w((\Delta(j))^{i_g}) + ((\Delta(j))^{(i-1)_g}) \geq k-1$. On the other hand, the degeneracy of $\Delta(j)$ along with the fact that the b-strings are not disturbed at any stage implies that

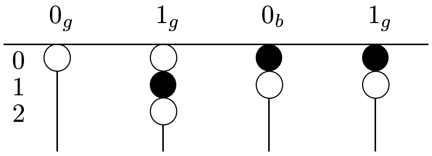
$$\tilde{w} \geq w(\Lambda^{i_b}) + w(\Lambda^{(i-1)_b}) = w((\Delta(j))^{i_g}) + w((\Delta(j))^{(i-1)_g}),$$

a contradiction.

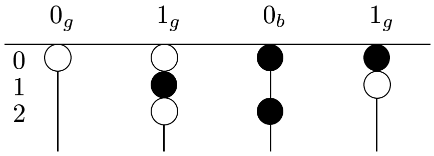
The fact that $\lambda \mapsto \lambda_1$ is a bijection between classes of symbols of weight $\tilde{w}$ and core γ and classes of symbols of weight $\tilde{w}$ and core γ_1 is obvious from the assumption that γ and γ_1 are both non-degenerate. □

5.10. **Examples.** Here we give a couple of examples for the failure of Lemma 5.9 if some of its hypotheses are not satisfied.

(1) Consider the following 4-unitary ($e = 2$) diagrams Γ and Λ.



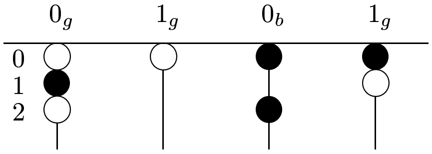
Γ



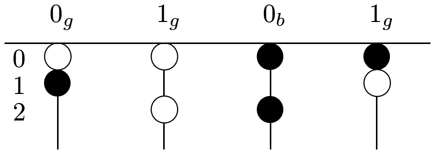
Λ

Thus Γ is a 2-unitary core, $w(\Lambda) = 1$ and $c(\Lambda) = \Gamma$. Also, $k = \hat{s}_e(\gamma) = 2$, where γ is the class of symbols represented by Γ.

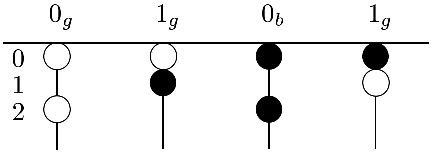
The next four pictures give the diagram Λ_1 and three diagrams $\Delta(i)$, $i = 1, 2, 3$.



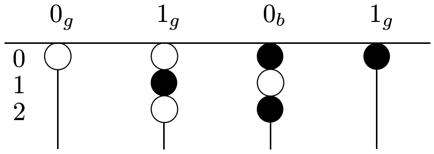
Λ_1



$\Delta(1)$



$\Delta(2)$



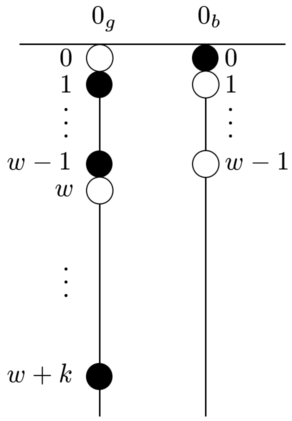
$\Delta(3)$

Denoting by λ_1 the class of symbols represented by Λ_1, and by $\delta(i)$ the classes of symbols represented by $\Delta(i)$, $i = 1, 2, 3$, we find $\mathcal{S}_{\lambda,\lambda_1} = \{(\lambda, \delta(i), \lambda_1) \mid i = 1, 2, 3\}$, the elements of which correspond to the sequences of diagrams $(\Lambda, \Delta(i), \Lambda_1)$, $i = 1, 2$, and $(\Lambda, \Delta(3), {\Lambda_1}')$. It follows that $|\mathcal{S}_{\lambda,\lambda_1}| = 3 > 2 = k!$. Note also that $d(S) = 2$ for the second of the above sequences. This example shows that it does not suffice to assume that $k \geq \tilde{w} + 1$ in Lemma 5.9 (or even only $k \geq \tilde{w}$, as we did in [23, Lemma 7.4]).

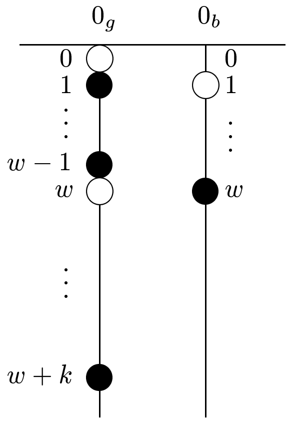
(2) The next example shows why the assumption $e > 1$ in Lemma 5.9 is necessary. Let k, w be integers with $w > 0$ and $k \geq w$ odd. Let $X = \{0, 2, 4, \ldots, k-1\}$ and

$Y = \{1, 3, 5, \dots, k\}$, and let γ denote the equivalence class of symbols containing $\{X, Y\}$. Then γ is its own 1-unitary core, has defect 0 and rank $(k+1)^2/4$.

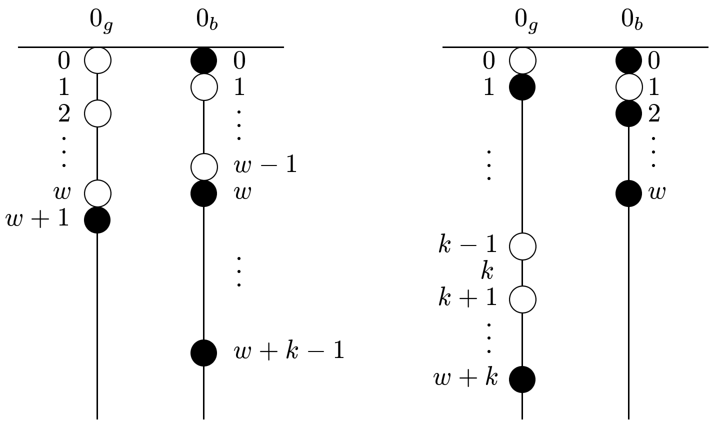
According to our conventions in 5.8, we choose a 2-unitary ($e = 1$) diagram Γ for γ which has at least w beads on each string. For example, if w is even, we can take the following diagram (of the symbol $\{[0, w-1] \cup (X+w), [0, w-1] \cup (Y+w)\}$ equivalent to $\{X, Y\}$):



(a) Let λ be the class of symbols of 1-unitary weight w and 1-unitary core γ represented by the diagram Λ obtained by shifting every bead on the 0_b-string of Γ one position down. Here is a picture of Λ:



Now consider the classes of symbols λ_1 and δ of weight w and core γ_1, represented by the following two diagrams.



It is easy to see that $|\mathcal{S}_{\lambda,\lambda_1}| \neq 0 \neq |\mathcal{S}_{\lambda,\delta}|$, contrary to the assertion in Lemma 5.9. Note that this is true for any odd $k \geq w$.

(b) Now suppose that w is even and let λ be the class of symbols of 1-unitary weight w and 1-unitary core γ obtained by shifting the last w beads on the 0_g-string of Γ one position down. It is easy to see that $\mathcal{S}_{\lambda,\lambda_1}$ does contain sequences S with $d(S) \neq 0$, again contrary to the assertions in Lemma 5.9.

5.11. **The correspondences $\lambda \mapsto {\lambda_1}^*$ and $\lambda \mapsto \lambda^*$.** Suppose that λ is a class of symbols of weight at most w and core γ, and that Λ is the chosen diagram for λ. Define ${\lambda_1}^*$ to be the class of symbols corresponding to the diagram which is obtained from Λ by swapping the strings i_b and $(i-1)_b$ and define λ^* to be the class of symbols corresponding to the diagram which is obtained from Λ_1 by swapping the strings i_b and $(i-1)_b$. Clearly ${\lambda_1}^*$ and λ^* have the same weight as λ and λ_1.

Lemma 5.12. *Let λ be a class of symbols of weight at most w and core γ. If γ is degenerate then ${\lambda_1}^*$ has core γ_1 and $\lambda_1 = {\lambda_1}^*$ if and only if λ is degenerate. If γ_1 is degenerate, then λ^* has core γ, and $\lambda^* = \lambda$ if and only if λ_1 is degenerate.*

Proof. Suppose first that γ is degenerate. Let Θ be the diagram obtained form Λ by swapping strings i_b and $(i-1)_b$. Then Θ is a diagram for ${\lambda_1}^*$. Note that $\Theta' = (\Lambda')_1$. From 5.3 we get that

$$c(\Theta)' = c(\Theta') = c((\Lambda')_1) = c(\Lambda')_1 = (\Gamma')_1.$$

Since γ is degenerate, $\Gamma' = \Gamma$, and hence $c(\Theta) = (\Gamma_1)'$. This proves that γ_1 is the core of ${\lambda_1}^*$.

Now, since

$$|\Theta^{i_g}| = |\Lambda^{i_g}| \neq |\Lambda^{(i-1)_g}| = |\Lambda_1^{i_g}|,$$

Θ is not the same as Λ_1. Hence, ${\lambda_1}^* = \lambda_1$ if and only if $\Theta' = \Lambda_1$. But $\Theta' = (\Lambda')_1$. Hence ${\lambda_1}^* = \lambda_1$ if and only if $(\Lambda')_1 = \Lambda_1$ thus if and only if $\Lambda' = \Lambda$.

This settles the lemma in the situation where γ is degenerate. The statements for the case that γ_1 is degenerate follow in an entirely analogous way. □

Lemma 5.13. *Let $\tilde{w}$ be an integer with $0 \leq \tilde{w} \leq w$. Suppose that γ is degenerate and that $k \geq \tilde{w}$. Let λ and θ be classes of symbols of weight $\tilde{w}$ and core γ, and suppose that δ is a class of symbols of weight $\tilde{w}$ and core γ_1. Then* $|\mathcal{S}_{\lambda,\lambda_1}| =$

$|\mathcal{S}_{\lambda,{\lambda_1}^*}| = k!$, $|\mathcal{S}_{\lambda,\delta}| = 0$ *if* $\delta \notin \{\lambda_1, {\lambda_1}^*\}$, *and* $|\mathcal{S}_{\theta,\lambda_1}| = |\mathcal{S}_{\theta,{\lambda_1}^*}| = 0$ *if* $\theta \neq \lambda$. *Further,* $d(S) = 0$ *for any element* S *of* $\mathcal{S}_{\lambda,\lambda_1}$ *or* $\mathcal{S}_{\lambda,{\lambda_1}^*}$.

Proof. Suppose that $|\mathcal{S}_{\lambda,\delta}|$ is non-zero. Then, as in the proof of Lemma 5.9, there is a sequence of abacus diagrams

$$\Lambda = \Delta(0), \Delta(1), \cdots, \Delta(k)$$

such that for each j, $1 \leq j \leq k$, $\Delta(j)$ is obtained from $\Delta(j-1)$ by moving a bead one position to the left, and such that the core of $\Delta(k)$ is Γ_1 or $(\Gamma_1)'$. By the usual Scopes argument, if the core of $\Delta(k)$ is Γ_1, then $\Delta(k) = \Lambda_1$ and there are $k!$ sequences of diagrams linking Λ to Λ_1; each such sequence of diagrams gives a sequence of classes of symbols linking λ to λ_1 and no two sequences of such diagrams give the same sequence of equivalence classes of symbols.

Similarly, if the core of $\Delta(k)$ is $(\Gamma_1)'$, then $\Delta(k)$ is the diagram obtained from Λ by swapping strings i_b and $(i-1)_b$. (Note that a problem as discussed in the proof of Lemma 5.9, namely that beads are moved from i_g to $(i-1)_g$ as well as from i_b to $(i-1)_b$, in order to get from Λ to $\Delta(k)$, can not happen in this situation since γ is degenerate.) In other words, $\Delta(k) = ((\Lambda')_1)'$ and $\delta = {\lambda_1}^*$. Again, there are $k!$ diagrams linking Λ to $((\Lambda')_1)'$; each such sequence of diagrams gives a sequence of classes of symbols linking λ to ${\lambda_1}^*$ and no two sequences of such diagrams give the same sequence of equivalence classes of symbols. In particular, $|\mathcal{S}_{\lambda,\delta}| = 0$ if $\delta \notin \{\lambda_1, {\lambda_1}^*\}$.

By reversing the above argument, it also follows that $|\mathcal{S}_{\theta,\lambda_1}| = |\mathcal{S}_{\theta,{\lambda_1}^*}| = 0$ if $\theta \neq \lambda$. If λ is non-degenerate, then $\lambda_1 \neq {\lambda_1}^*$ by Lemma 5.12, and it follows that $|\mathcal{S}_{\lambda,\lambda_1}| = |\mathcal{S}_{\lambda,{\lambda_1}^*}| = k!$. If λ (and hence Λ is degenerate), it is easy to see that there is a bijection between the set of sequences of diagrams linking Λ to Λ_1 and the set of sequences of diagrams linking Λ to $((\Lambda')_1)'$ such that a pair of corresponding sequences yields the same sequence of equivalence classes of symbols. Thus again, $|\mathcal{S}_{\lambda,\lambda_1}| = k!$. Since γ is degenerate, and any sequence linking λ to λ_1 or to ${\lambda_1}^*$ comes from a sequence of diagrams (starting with Λ) in which beads are only moved either in the g-strings or only in the b-strings, the last assertion of the lemma follows. $\square$

Lemma 5.14. *Let* $\tilde{w}$ *be an integer with* $0 \leq \tilde{w} \leq w$. *Suppose that* γ_1 *is degenerate and that* $k \geq \tilde{w}$. *Let* λ *and* θ *be classes of symbols of weight* $\tilde{w}$ *and core* γ, *and suppose that* δ *is a class of symbols of weight* $\tilde{w}$ *and core* γ_1. *Then* $|\mathcal{S}_{\lambda,\lambda_1}| = |\mathcal{S}_{\lambda^*,\lambda_1}| = k!$, $|\mathcal{S}_{\lambda,\delta}| = |\mathcal{S}_{\lambda^*,\delta}| = 0$ *if* $\delta \neq \lambda_1$, *and* $|\mathcal{S}_{\theta,\lambda_1}| = 0$ *if* $\theta \notin \{\lambda, \lambda^*\}$. *Further,* $d(S) = 0$ *for any element* S *of* $\mathcal{S}_{\lambda,\lambda_1}$ *or* $\mathcal{S}_{\lambda^*,\lambda_1}$.

Proof. This is analogous to that of Lemma 5.13. $\square$

5.15. **The correspondence $\lambda \mapsto \bar{\lambda}$.** Let λ be a class of symbols of weight at most w and core γ. If neither γ nor γ_1 is degenerate, let $\bar{\lambda}$ denote the class of symbols corresponding to the diagram Λ_1. If γ is degenerate, define $\bar{\lambda}$ to be the class of symbols corresponding to the diagram Λ^*. If γ_1 is degenerate, $e = 2$ and we are in the unitary situation, define $\bar{\lambda}$ to be the class of symbols corresponding to the diagram $(\widetilde{\Lambda_1})_1$ (see 5.3 and 5.7).

Lemma 5.16. *Let* $\tilde{w}$ *be an integer with* $0 \leq \tilde{w} \leq w$. *Suppose that* γ *is degenerate and that* $k \geq \tilde{w}$. *Let* λ *be a class of symbols of weight* $\tilde{w}$ *and core* γ, δ *a class of symbols of weight* $\tilde{w}$ *and core* γ_1, *and* ζ *a class of symbols of weight* $\tilde{w}$ *and core* $\bar{\gamma}$. *Then* $|\mathcal{S}_{\lambda_1,\bar{\lambda}}| = |\mathcal{S}_{{\lambda_1}^*,\bar{\lambda}}| = k!$, $|\mathcal{S}_{\lambda_1,\zeta}| = |\mathcal{S}_{{\lambda_1}^*,\zeta}| = 0$ *if* $\zeta \neq \bar{\lambda}$, *and* $|\mathcal{S}_{\delta,\bar{\lambda}}| = 0$ *if* $\delta \notin \{\lambda_1, {\lambda_1}^*\}$. *Further,* $d(S) = 0$ *for any element* S *of* $\mathcal{S}_{\lambda_1,\bar{\lambda}}$ *or* $\mathcal{S}_{{\lambda_1}^*,\bar{\lambda}}$.

The map $\lambda \mapsto \bar{\lambda}$ is a bijection between classes of symbols of weight $\tilde{w}$ and core γ and classes of symbols of weight $\tilde{w}$ and core $\bar{\gamma}$, preserving degeneracy of symbols.

Proof. Omitted. $\square$

In the following two lemmas, we put $\bar{\Delta} := (\widetilde{\Delta_1})_1$ for a diagram Δ (see 5.7). Also, $\bar{\lambda}^*$ is defined as in 5.11, except that the underlying core is $\bar{\gamma}$ instead of γ.

Lemma 5.17. *Let $\tilde{w}$ be an integer with $0 \leq \tilde{w} \leq w$. Suppose that γ_1 is degenerate, $e = 2$ and that we are in the unitary situation.*

If $k > 1$, then $\lambda \mapsto \bar{\lambda}$ is a bijection between classes of symbols of weight $\tilde{w}$ and core γ and classes of symbols of weight $\tilde{w}$ and core $\bar{\gamma}$. This bijection has the property that $\overline{\lambda^*} = \bar{\lambda}^*$. Further, if λ is a class of symbols of weight $\tilde{w}$ and core γ, $\bar{\lambda}^* = \bar{\lambda}$ if and only if λ_1 is degenerate.

Proof. Let λ be a class of symbols of weight w and core γ. Clearly, $\bar{\lambda}$ has weight $\tilde{w}$ and core $\bar{\gamma}$ whence the above map is well defined. We show first that it is onto. Let ζ be a class of symbols of weight $\tilde{w}$ and core $\bar{\gamma}$ and let Θ be a diagram for ζ with core $\bar{\Gamma}$. Let $\Theta(1)$ be the diagram obtained from Θ by swapping strings 0_g and 1_g, $\Theta(2)$ the diagram such that $\widetilde{\Theta(2)} = \Theta(1)$. So $\Theta(2)$ has weight $\tilde{w}$ and core Γ_1. Let $\Theta(3)$ be the diagram obtained by swapping strings 0_g and 1_g of $\Theta(2)$. Then $\Theta(3)$ has weight $\tilde{w}$ and core Γ. Let λ be the class of symbols represented by $\Theta(3)$. Since γ is non-degenerate, $\Theta(3)$ is the unique diagram for λ with core Γ, from which it is immediate that $\Theta = \bar{\Lambda}$.

Next, suppose that λ and δ are classes of symols of weight $\tilde{w}$ and core γ. Let Λ and Δ be the chosen diagrams for λ and δ respectively. By our hypothesis on k, we have

$$|\bar{\Lambda}^{0_g}| = |\Gamma^{1_g}| \neq |\Gamma^{0_g}| + 1 = |\bar{\Delta}^{0_b}|.$$

So $\bar{\Delta}$ is not the swap of $\bar{\Lambda}$. It follows that $\bar{\lambda} = \bar{\delta}$ if and only if $\bar{\Lambda} = \bar{\Delta}$ or, in other words, if and only if $\lambda = \delta$. Thus $\lambda \mapsto \bar{\lambda}$ is a bijection as claimed.

Let λ be a class of symbols of weight $\tilde{w}$ and core γ and let Λ be the chosen diagram for λ. Now $(\Lambda^*)'$ is a diagram for λ^* and $\Lambda^{*\prime}$ has core Γ (see 5.12). Since γ is non-degenerate, this means that $(\Lambda^*)'$ is the chosen diagram for λ^*. Put $\Delta := (\Lambda^*)'$. It can be checked that $\bar{\Delta}' = \bar{\Lambda}^*$ from which it follows that $\overline{\lambda^*} = \bar{\lambda}^*$. $\square$

Lemma 5.18. *Let the assumptions be as in Lemma 5.17 and suppose that $k \geq \tilde{w}+1$. Let λ be a class of symbols of weight $\tilde{w}$ and core γ, δ a class of symbols of weight $\tilde{w}$ and core γ_1, and ζ a class of symbols of weight $\tilde{w}$ and core $\bar{\gamma}$. Then $|\mathcal{S}_{\lambda_1,\bar{\lambda}}| = |\mathcal{S}_{\lambda_1,\bar{\lambda}^*}| = (k-1)!$. Moreoever, $|\mathcal{S}_{\lambda_1,\zeta}| = 0$ if $\zeta \notin \{\bar{\lambda}, \bar{\lambda}^*\}$ and $|\mathcal{S}_{\delta,\bar{\lambda}}| = |\mathcal{S}_{\delta,\bar{\lambda}^*}| = 0$ if $\delta \neq \lambda_1$. Further, $d(S) = 0$ for any element S of $\mathcal{S}_{\lambda_1,\bar{\lambda}}$ or $\mathcal{S}_{\lambda_1,\bar{\lambda}^*}$.*

Proof. Omitted. $\square$

6. Morita equivalences

In this section we fix an odd prime l and a prime power q not divisible by l.

6.1. **Introductory remarks.** In our previous paper, [23], we showed that the Donovan and Puig conjectures hold for the unipotent l-blocks of the families of finite unitary groups $\{GU_n(q) \mid n \geq 1\}$, the odd dimensional orthogonal groups $\{SO_n(q) \mid n \text{ odd}\}$ and the conformal symplectic groups $\{CSp_n(q) \mid n \text{ even}\}$ when

l is a unitary prime for the family concerned. (For the definiton of unitary prime see Subsection 3.4.)

However, we were unable to settle the case of the even dimensional special orthogonal groups; the difficulty being a degeneracy phenomenon in the combinatorics describing the character theory of these groups (see [23, Theorem 8.3 (iii)]). In the present paper, we circumvent these problems and prove that the Donovan and Puig conjectures hold also for the unipotent l-blocks in the familes of the even dimensional orthogonal groups $\{CSO_n^{\pm}(q) \mid n \text{ even}\}$ (for odd q) and $\{SO_n^{\pm}(q) \mid n \text{ even}\}$ (for even q), when l is unitary and does not divide $q+1$.

Since the publication of [23] we have discovered that the proof of the reduction theorem (Theorem 7.9) therein does not apply to all the situations covered by its hypothesis. Specifically, if we are in the symplectic or orthogonal case, we require that the equivalence class of symbols γ being considered have defect different from 0 and 2, and not merely that γ is an equivalence class of non-degenerate symbols. Since all relevant symbols for the odd dimensional orthogonal groups and the symplectic groups have odd defect, the stronger condition is always satisfied for these groups, and the reduction theorem, as well as all consequences thereof remain valid as stated for these groups. For the even dimensional orthogonal groups however, this extra restriction implies that the phrase "non-degenerate cores" should be replaced by "cores of defect at least 4" in [23, Theorem 8.3(iii)]. Note that this extra hypothesis effectively rules out the case $e = 1$, i.e., $l \mid q+1$. In what follows, we prove an anlogue of the reduction theorem which places no restriction on the defect of cores, but where we assume that $e > 1$.

6.2. **The groups.** For a positive even integer n we let $G_n(q)$ denote one of the following groups.

(1) The conformal special orthogonal group $CSO_n^+(q)$ of an n-dimensional vector space over $\mathbb{F}_q$ with a quadratic form of maximal Witt index, where q is odd.
(2) The conformal special orthogonal group $CSO_n^-(q)$ of an n-dimensional vector space over $\mathbb{F}_q$ with a quadratic form of Witt index $n/2-1$, where q is odd.
(3) The special orthogonal group $SO_n^+(q)$ of a quadratic space as in (1), where q is even.
(4) The special orthogonal group $SO_n^-(q)$ of a quadratic space as in (2), where q is even.

If n is clear from the context, we simply write G for $G_n(q)$. Let us recall some notation introduced in [23].

If Y is any finite-dimensional vector space and t a semisimple invertible linear transformation on Y, we put Y_t for the set of fixed points of t on Y, and we let $[t, Y] = \{ty - y \mid y \in Y\}$. Then $Y = Y_t \oplus [t, Y]$. Moreoever, if Y carries a non-degenerate t-invariant bilinear form, then Y_t is the orthogonal complement of $[t, Y]$.

Recall from [23, Section 4,4.2.5–6] that $G = \mathbf{G}^F$, where $\mathbf{G}$ is a special (conformal) orthogonal group of a non-degenerate quadratic form on an $\bar{\mathbb{F}}_q$-vecor space $\mathbf{V}$ of dimension n, and where F is a Frobenius endomorphism of $\mathbf{G}$. We have $\mathbf{G} = J(\mathbf{V})^\circ$, if q is odd, and $\mathbf{G} = I(\mathbf{V})^\circ$, if q is even. To simplify notation we shall write $J_0(\mathbf{V}) := J(\mathbf{V})^\circ$ and $I_0(\mathbf{V}) := I(\mathbf{V})^\circ$. (Note that the latter notation differs from the one introduced in [23, 4.2.1] in case q is even.) There is a suitable $\mathbb{F}_q$-form V

of $\mathbf{V}$ such that $J_0(V) = J_0(\mathbf{V})^F$ and $I_0(V) = I_0(\mathbf{V})^F$. Here, $I_0(V) = I(V) \cap SL(V)$ if q is odd, and, if q is even, $I_0(V)$ denotes the kernel of the Dickson invariant, i.e., the set of isometries t of V with $\dim_{\mathbb{F}_q}([t, V])$ even.

The form on $\mathbf{V}$ is defined with respect to a particular basis $e_1, \dots, e_m, e'_m, \dots, e'_1$. For an integer k with $0 \leq k \leq m$ we let $\mathbf{V}_{n-2k}$ denote the $\bar{\mathbb{F}}_q$-subspace of $\mathbf{V}$ spanned by $e_{k+1}, \dots, e_m, e'_m, \dots, e'_{k+1}$ (see [23, 4.2.7]). If $\mathbf{Y} = \mathbf{V}_{n-2k}$ for some k, we write Y for the $\mathbb{F}_q$-span of the basis elements $e_{k+1}, \dots, e_m, e'_m, \dots, e'_{k+1}$, so that Y is a particular $\mathbb{F}_q$-form of $\mathbf{Y}$.

6.3. **The reduction theorem.** Let $G = G_n(q)$ be a group as in Subsection 6.2 and let $(\mathcal{K}, \mathcal{O}, \mathcal{F})$ be a splitting l-modular system for G such that $\mathcal{F}$ is algebraically closed. Let d and e denote the orders of q and of q^2 modulo l, respectively. Then l is linear for G if $e = d$, and unitary, if $e = d/2$ (it is easy to see that l is linear for G if and only if d is odd).

We use the notation and assumptions of Section 5. The integer e now has the specific meaning introduced above, and γ and w are the unipotent label and weight of a unipotent l-block of G, respectively. As in 5.5, we let $k = \check{s}_e(\gamma)$ if l is linear for G, and $k = \hat{s}_e(\gamma)$, otherwise. In order to use the results of Section 5, we assume that $e > 1$ for the rest of this section. We also assume that γ is not equivalent to the symbol $\{\{1\}, \{1\}\}$, that $k > 0$, and that $e = 2$ and $k > 1$ whenever γ_1 is degenerate (note that l is a unitary prime for G if $e = 2$).

Let $\tilde{k} = 0$ if both γ and γ_1 are non-degenerate, $\tilde{k} = k$ if γ is degenerate and let $\tilde{k} = k - 1$ if γ_1 is degenerate.

Let G_1 be the group $G_{n-2k}(q)$ and let L_k be the standard Levi subgroup of G isomorphic to $G_{n-2k}(q) \times GL_1(q)^k$ as described in [23, Section 4.4]. Let $\bar{G}$ be $G_{n-2(k+\tilde{k})}(q)$ and $L_{k+\tilde{k}}$ the corresponding Levi subgroup of G. Note that $L_{k+\tilde{k}}$ is naturally a Levi subgroup of L_k. If χ is a character of G_1 (or $\bar{G}$), we will denote the inflation of χ to L_k (or to $L_{k+\tilde{k}}$) also by χ. Note that since $e > 1$, any block of G_1 (or of $\bar{G}$) may be regarded as a block of L_k (or of $L_{k+\tilde{k}}$), and we shall do this without comment. Recall from 5.7 that if γ and γ_1 are non-degenerate, then $\bar{\gamma} = \gamma_1$.

Theorem 6.4. *Suppose that $w > 0$ and that B, B_1 and $\bar{B}$ are unipotent blocks of G, G_1 and $\bar{G}$, respectively, each of weight w, and with unipotent labels γ, γ_1 and $\bar{\gamma}$, respectively, and suppose that* $\operatorname{rank}(\bar{\gamma}) \geq 2$. *Also, suppose that if γ and γ_1 are non-degenerate, then $k \geq w + 2$, if γ is degenerate then $k \geq w$, and if γ_1 is degenerate, then $e = 2$ and $k \geq w + 1$.*

(i) *Suppose that γ and γ_1 are non-degenerate. Then there is a bijection $\chi \mapsto \bar{\chi}$ between* $\operatorname{Irr}(G, B)$ *and* $\operatorname{Irr}(\bar{G}, \bar{B})$ *such that $R^G_{L_k}(\bar{\chi})_B = (k!)\chi$ for all $\bar{\chi} \in \operatorname{Irr}(\bar{G}, \bar{B})$. (Here, the subscript B on a character indicates its restriction to the block B).*

(ii) *Suppose that γ or γ_1 is degenerate. Then there is an involution ι permuting* $\operatorname{Irr}(G, B)$ *and* $\operatorname{Irr}(\bar{G}, \bar{B})$, *and a bijection $\chi \mapsto \bar{\chi}$ between* $\operatorname{Irr}(G, B)$ *and* $\operatorname{Irr}(\bar{G}, \bar{B})$ *commuting with ι, such that the following holds. If χ and $\bar{\chi}$ denote the sums of the characters in corresponding ι-orbits on* $\operatorname{Irr}(G, B)$ *and* $\operatorname{Irr}(\bar{G}, \bar{B})$, *repectively, then* $R^G_{L_k}(R^{L_k}_{L_{k+\tilde{k}}}(\bar{\chi})_{B_1})_B = 2(k!)(\tilde{k}!)\chi$.

Proof. The proof is very similar to that of the reduction theorem [23, Theorem 7.9]. First we construct suitable d-cuspidal pairs for B, B_1 and $\bar{B}$ as follows. Let $\mathbf{V}_1 := \mathbf{V}_{n-2k}$ and $\bar{\mathbf{V}} := \mathbf{V}_{n-2(k+\tilde{k})}$ be the subspaces of $\mathbf{V}$ introduced in Subsection 6.2, so that $\mathbf{G}_1 = J_0(\mathbf{V}_1)$ or $I_0(\mathbf{V}_1)$ and $\bar{\mathbf{G}} = J_0(\bar{\mathbf{V}})$ or $I_0(\bar{\mathbf{V}})$. Since $\bar{B}$ is a block of weight w of $\bar{G}$, there is an orthogonal decomposition $\bar{\mathbf{V}} = \mathbf{V}' \oplus \bar{\mathbf{V}}''$ of $\bar{\mathbf{V}}$, is defined over $\mathbb{F}_q$, such that $\dim_{\bar{\mathbb{F}}_q}(\mathbf{V}')$ equals $2ew$ and $I_0(\mathbf{V}')$ has a Sylow ϕ_d-torus $\mathbf{S}$ with $|\mathbf{S}^F| = \phi_d^w(q)$. Put $\bar{\mathbf{K}} := C_{\bar{\mathbf{G}}}(\mathbf{S})$. Then $\mathbf{V}'$ is the fixed point space of $[\bar{\mathbf{K}}, \bar{\mathbf{K}}]$. There is a unipotent character $\bar{\psi}$ of $\bar{\mathbf{K}}^F$ whose restriction to $I_0(\bar{\mathbf{V}}'')^F$ has label $\bar{\gamma}$ and $(\bar{\mathbf{K}}, \bar{\psi})$ is a unipotent d-cuspidal pair for $\bar{B}$. Let $\mathbf{V}_1''$ and $\mathbf{V}''$ be the orthogonal complements to $\mathbf{V}'$ in $\mathbf{V}_1$ and $\mathbf{V}$, respectively. Put $\mathbf{K}_1 = C_{\mathbf{G}_1}(\mathbf{S})$ and $\mathbf{K} = C_{\mathbf{G}}(\mathbf{S})$. Then clearly, $\mathbf{V}'$ is the fixed point space of $[\mathbf{K}_1, \mathbf{K}_1]$ and of $[\mathbf{K}, \mathbf{K}]$. Let ψ_1 be a character of $\mathbf{K}_1^F$ whose restriction to $I_0(\mathbf{V}_1'')^F$ is a unipotent character with label γ_1 and let ψ be a character of $\mathbf{K}^F$ whose restriction to $I_0(\mathbf{V}'')$ is a unipotent character with label γ. Then $(\mathbf{K}_1, \psi_1)$ is a d-cuspidal pair for B_1 and $(\mathbf{K}, \psi)$ is a d-cuspidal pair for B.

Since $\mathrm{rank}(\bar{\gamma}) \geq 2$, we can argue just as in [23, Section 7.6 and Lemma 7.7] to conclude that there is an integer k' with $k' \geq k + \tilde{k}$ such that $\mathbf{S} \subseteq \mathbf{G}_{n-2k'}$ and such that any Sylow l-subgroup D of $G_{n-2k'}(q)$ is a defect group of B, B_1 and $\bar{B}$.

Let $\mathcal{T}$ denote the set of G^*-conjugacy classes of F-stable l-elements of $\mathbf{G}^*$ for which the corresponding Lusztig series of G contains a character in the block B. Here $\mathbf{G}^*$ denotes the dual group of $\mathbf{G}$. Let $\mathcal{T}_1$ and $\bar{\mathcal{T}}$ have analogous meanings for the pairs (G_1, B_1) and $(\bar{G}, \bar{B})$, respectively. Then, as in the proof of [23, Theorem 7.9], each of $\mathcal{T}$, $\mathcal{T}_1$ and $\bar{\mathcal{T}}$ is in bijection with $\bar{G}$-conjugacy classes of l-elements intersecting D non-trivially. If the class of an element t of D corresponds to the classes $\tilde{t}^*$ of $\tilde{G}^*$, where $\tilde{G}^*$ is one of G^*, G_1^* or $\bar{G}^*$, then $C_{\mathbf{G}}(t)$ is a Levi subgroup of $\mathbf{G}$ in duality with $C_{\tilde{\mathbf{G}}^*}(\tilde{t}^*)$.

Let $t \in D$ correspond to each of $t^* \in G^*$, $t_1^* \in G_1^*$ and $\bar{t}^* \in \bar{G}^*$. Note that $[t, \mathbf{V}] = [t, \mathbf{V}_1] = [t, \bar{\mathbf{V}}] \subseteq \mathbf{V}'$. Put $w(t) := \dim_{\bar{\mathbb{F}}_q}[t, \mathbf{V}]/2e$. Then $w(t) \leq w$. Let $\mathbf{C} = C_{I_0([t,\mathbf{V}])}(t)$ and $C = \mathbf{C}^F$. Then $[C_{\mathbf{G}}(t), C_{\mathbf{G}}(t)] \subseteq I_0(\mathbf{V}_t) \times \mathbf{C} \subset C_{\mathbf{G}}(t)$, and it follows from Lemma 3.3 that a unipotent character of $C_G(t)$ is completely determined by its restriction to $I_0(V_t) \times C$. Using this lemma once more and [13, Theorem 4.4], it follows that a character $\chi_{t^*,\rho}$ of G lies in B if and only if ρ restricts to a unipotent character $\rho'' \times \rho'$ of $I_0(V_t) \times C$ such that the label of ρ'' has e-linear core γ (and hence e-linear weight $w(t)$) if l is a linear prime for G, and has e-unitary core γ (and hence e-unitary weight $w(t)$) if l is a unitary prime for G. Similar characterizations hold for the elements of $\mathcal{E}(G_1, t_1^*)$ in B_1 and the elements of $\mathcal{E}(\bar{G}, \bar{t}^*)$ in $\bar{B}$.

Now let $\chi = \chi_{t^*,\rho}$ be a character of B such that ρ restricts to the character $\rho'' \times \rho'$ of $I_0(V_t) \times C$, and suppose that ρ'' has label λ. Note that λ has weight $w(t)$ and core γ. Let λ_1 and $\bar{\lambda}$ be the symbols defined in Subsections 5.8 and 5.15, respectively. Then λ_1 and $\bar{\lambda}$ have weight $w(t)$ and cores γ_1 and $\bar{\gamma}$, respectively.

Suppose first that both γ and γ_1 are non-degenerate. Then λ and $\lambda_1 = \bar{\lambda}$ are both non-degenerate, hence there is a unique unipotent character $\bar{\rho}''$ of $I_0((V_1)_t)$ with label λ_1. Define $\bar{\chi}$ to be the character $\chi_{t_1^*,\bar{\rho}}$ in B_1 such that $\bar{\rho}$ restricts to the character $\bar{\rho}'' \times \rho'$ of $I_0((V_1)_t) \times C$. By Lemma 5.9, applied with $\tilde{w} = w(t)$, $\chi \mapsto \bar{\chi}$ provides a bijection between the set of characters of $\mathcal{E}(G, t^*)$ in B and the set of characters of $\mathcal{E}(G_1, t_1^*)$ in B_1.

If γ is degenerate, then by Lemma 5.16, λ is degenerate if and only if $\bar{\lambda}$ is degenerate. So, if λ is non-degenerate, there is a unique unipotent character $\bar{\rho}''$ of $I_0(\bar{V}_t)$ with label $\bar{\lambda}$ and we define $\bar{\chi}$ to be the character $\chi_{\bar{t}^*,\bar{\rho}}$ in $\bar{B}$ such that $\bar{\rho}$ restricts to the character $\bar{\rho}'' \times \rho'$ of $I_0(\bar{V}_t) \times C$. Also, set $\chi^\iota := \chi$ and $\bar{\chi}^\iota := \bar{\chi}$ in this case. If λ is degenerate, let $(\rho'')^\iota$ be the other character of $I_0(V_t)$ labelled by

λ, let $\rho^\iota = (\rho'')^\iota \times \rho'$ and set $\chi^\iota := \chi_{t^*,\rho^\iota}$. Also, if λ is degenerate, there are two characters $\bar{\rho}''$ and $(\bar{\rho}'')^\iota$ of $I_0(\bar{V}_t)$ labelled by $\bar{\lambda}$; set $\bar{\rho}$ to be the character of $C_{\bar{G}}(t)$ which restricts to $\bar{\rho}'' \times \rho'$ and $\bar{\rho}^\iota$ to be the character of $C_{\bar{G}}(t)$ which restricts to $(\bar{\rho}'')^\iota \times \rho'$. Set $\bar{\chi} = \chi_{\bar{t}^*,\bar{\rho}}$ and $\bar{\chi}^\iota := \chi_{\bar{t}^*,\bar{\rho}^\iota}$. Note that ι acts as an involution on both $\mathcal{E}(G,t^*)$ and $\mathcal{E}(\bar{G},t^*)$. Moreover, the map $\chi \mapsto \bar{\chi}$ is a bijection between these two sets commuting with ι.

If γ_1 is degenerate, then λ and $\bar{\lambda}$ are both non-degenerate. Let $\bar{\rho}''$ be the unique character of $I_0(\bar{V}_t)$ with label $\bar{\lambda}$, set $\bar{\rho} = \bar{\rho}'' \times \rho'$ and set $\bar{\chi} = \chi_{\bar{t}^*,\bar{\rho}}$. Now Lemma 5.17 provides a bijection between the set of characters of $\mathcal{E}(G,t^*)$ in B and the set of characters of $\mathcal{E}(\bar{G},\bar{t}^*)$ in $\bar{B}$.

Letting t run over the $\bar{G}$-conjugacy classes of l-elements intersecting D, in all three cases we get a bijection $\chi \mapsto \bar{\chi}$ between the irreducible characters of G in B and those of $\bar{G}$ in $\bar{B}$. We now prove the required properties of the bijection $\chi \mapsto \bar{\chi}$. For the rest of the section, let $t \in D$ and let $\chi = \chi_{t^*,\rho}$ be an irreducible character of G in B as above.

Let us first assume that both γ and γ_1 are non-degenerate. Let τ be an irreducible character of G in B such that τ is a constituent of the Harish-Chandra induced character $R^G_{L_k}(\bar{\chi})$. Since Harish-Chandra induction preserves Lusztig series, we may assume that $\tau \in \mathcal{E}(G,t^*)$. Suppose that $\tau = \chi_{t^*,\eta}$ and suppose that η restricts to the character $\eta'' \times \eta'$ of $I_0(V_t) \times C$. Then, arguing as in the proof of [23, Theorem 7.9], we have that $\eta' = \rho'$ and the calculation of the multiplicity of τ as a constituent of $R^G_{L_k}(\bar{\chi})$ reduces to the calculation of the number

$$\left\langle R^{I_0(V_t)}_{I_0(\bar{V}_t)\times GL_1(q)^k}(\bar{\rho}''), \eta'' \right\rangle_{I_0(V_t)}.$$

The Iwahori-Hecke algebras corresponding to the Harish-Chandra series of $\bar{\rho}''$ and of η'' are either both of type B or of type D, and the calculation of the above inner products reduces to a calculation in the corresponding Weyl groups. Let θ be the equivalence class of symbols of weight $w(t)$ and e-core γ labelling η''. From the branching rules (see [23, Section 5.5]), if the Iwahori-Hecke algebras are of type B, then the above number is equal to $|\mathcal{S}_{\theta,\lambda_1}|$ (for the definition of the set $\mathcal{S}_{\theta,\lambda_1}$ see Subsection 5.4). If the Iwahori-Hecke algebras are of type D, then the above number is $\sum_{S\in\mathcal{S}_{\theta,\lambda_1}} 2^{d(S)}$. But now it follows from Lemma 5.9 applied with $\tilde{w} = w(t)$ that in either case, the above number is $k!$ if $\theta = \lambda$ and is zero if $\theta \neq \lambda$. This completes the proof of the reduction theorem in the case that both γ and γ_1 are non-degenerate.

Now suppose that γ is degenerate. We have already defined the permutation $\chi \mapsto \chi^\iota$ and have shown that it commutes with the bijection $\chi \mapsto \bar{\chi}$. Let χ be as above. Let ρ_1'' be the unique character of $I_0((V_1)_t)$ labelled by λ_1, set $\rho_1 = \rho_1'' \times \rho'$ and set $\chi_1 = \chi_{t_1^*,\rho_1}$. Also, let $\rho_1''^*$ be the unique character of $I_0((V_1)_t)$ labelled by $\lambda_1{}^*$, where $\lambda_1{}^*$ is defined in Subsection 5.11, set $\rho_1{}^* = \rho_1''^* \times \rho'$ and set $\chi_1{}^* = \chi_{t_1^*,\rho_1{}^*}$. Then by Lemma 5.12, χ_1 and $\chi_1{}^*$ are characters of G_1 in B_1 and $\chi_1 = \chi_1{}^*$ if and only if λ is degenerate.

Let τ be an irreducible character of G_1 in B_1 such that τ is a constituent of the Harish-Chandra induced character $R^{L_k}_{L_{2k}}(\bar{\chi})$. Then, again, we may assume that $\tau \in \mathcal{E}(G_1, t_1^*)$. Suppose that $\tau = \chi_{t_1^*,\eta}$, and let θ be the class of symbols to which η corresponds as in the case when γ and γ_1 were both non-degenerate. Then, arguing as above, we get that the multiplicity of τ as a constituent of $R^{L_k}_{L_{2k}}(\bar{\chi})$ is either

$|\mathcal{S}_{\theta,\bar\lambda}|$ or $\sum_{S\in\mathcal{S}_{\theta,\bar\lambda}} 2^{d(S)}$. By Lemma 5.16, it follows that θ is either λ_1 or $\lambda_1{}^*$ and thus that $R^{L_k}_{L_{2k}}(\bar\chi)_{B_1} = k!(\chi_1+\chi_1{}^*)$ if λ is non-degenerate, and $R^{L_k}_{L_{2k}}(\bar\chi)_{B_1} = (k!)\chi_1$ if λ is degenerate. Similarly, if λ is degenerate, $R^{L_k}_{L_{2k}}(\overline{\chi^\iota})_{B_1} = (k!)\chi_1$. By invoking Lemma 5.13, one sees in exactly the same way as above that if λ is non-degenerate, then $R^G_{L_k}(\chi_1) = R^G_{L_k}(\chi_1{}^*) = (k!)\chi$ and that if λ is degenerate, then $R^G_{L_k}(\chi_1) = k!(\chi+\chi^\iota)$. This proves statement (ii) of the theorem in case γ is degenerate.

Finally, let us assume that γ_1 is degenerate. Let ρ_1'' be a character of $I_0((V_1)_t)$ labelled by λ_1. Note that if λ_1 is degenerate, there are two choices for ρ_1''. Set $\rho_1 = \rho_1'' \times \rho'$ and set $\chi_1 = \chi_{t_1^*,\rho_1}$. If λ_1 is degenerate, let $(\rho_1'')^\iota$ be the other character of $I_0((V_1)_t)$ labelled by λ_1, set $\rho_1^\iota = (\rho_1'')^\iota \times \rho'$ and $\chi_1^\iota = \chi_{t_1^*,\rho_1^\iota}$. Again, let ρ^* be the character of $I_0(V_t)$ labelled by λ^* where λ^* is defined in Subsection 5.11. Note that by Lemma 5.12, $\chi^\iota := \chi_{t^*,\rho^*}$ is a character of G in B, $\chi = \chi^\iota$ if and only if λ_1 is degenerate, and $\chi \mapsto \chi^\iota$ is a permutation of the set of irreducible characters of G in B. We can define the permutation $\tau \mapsto \tau^\iota$ on the characters of $\bar G$ in $\bar B$ analogously.Note that $\overline{\chi^\iota} = \bar\chi^\iota$ by Lemma 5.17.

It follows from Lemma 5.18 that if λ_1 is degenerate, then $R^{L_k}_{L_{2k-1}}(\bar\chi)_{B_1} = (k-1)!(\chi_1+\chi_1^\iota)$ and if λ_1 is non-degenerate, then $R^{L_k}_{L_{2k-1}}(\bar\chi^\iota)_{B_1} = R^{L_k}_{L_{2k-1}}(\bar\chi)_{B_1} = (k-1)!\chi_1$. Again, by Lemma 5.14, we get that if λ_1 is degenerate then $R^G_{L_k}(\chi_1^\iota)_B = R^G_{L_k}(\chi_1)_B = (k!)\chi$ and if λ_1 is non-degenerate, then $R^G_{L_k}(\chi_1)_B = k!(\chi+\chi^\iota)$. □

6.5. **Source algebra equivalence.** In order to show that the blocks B and $\bar B$ of the reduction theorem have equivalent source algebras, we shall invoke Theorem 2.5. However, it is easier to apply this theorem to the orthogonal groups instead of to the conformal groups. For this reason we introduce the following notation. If G is one of the groups introduced in 6.2, we put $\mathbf{G}' := [\mathbf{G},\mathbf{G}]$ (and $G' := \mathbf{G}'^F$), where $\mathbf{G}$ is the (conformal) orthogonal group over $\bar{\mathbb{F}}_q$ with $G = \mathbf{G}^F$. Thus if $\mathbf{V}$ is the natural module for $\mathbf{G}$, its $\mathbb{F}_q$-form V introduced in Subsection 6.2 is the natural module for G' and we have $G' = I_0(\mathbf{V})^F$. Similarly, we define $L_k' := L_k \cap G'$ etc.

Theorem 6.6. *Under the hypotheses of Theorem 6.4, the blocks B and $\bar B$ are Puig equivalent.*

Proof. By Theorem 3.2 there is a unique block B' of G' covered by B, and restriction yields an isomorphism between the corresponding block algebras. Similarly, there is a unique block B_1' of G_1' covered by B_1 and a unique block $\bar B'$ of $\bar G'$ covered by $\bar B$. Since

$$L_k' \cong G_1' \times GL_1(q)^k$$

and l does not divide $q-1$, it follows that the block B_1' can naturally be considered as a block of L_k' etc. Then, by Lemma 3.3, it follows that the reduction theorem holds on replacing G by G', L_k by L_k', B by B' etc.

Let b, b_1, $\bar b$ be the central primitive idempotents of $\mathcal{O}G'$, $\mathcal{O}L_k'$ and $\mathcal{O}L'_{k+\tilde k}$, respectively, corresponding to the blocks B', B_1' and $\bar B'$. We let

$$u := \frac{1}{|U|}\sum_{v\in U} v \in \mathcal{O}G',$$

where U is the unipotent radical of Q, the standard parabolic subgroup of G with Levi complement L'_k. Also, we let

$$u_1 := \frac{1}{|U_1|}\sum_{v\in U_1} v \in \mathcal{O}L'_k,$$

where U_1 is the unipotent radical of Q_1, the standard parabolic subgroup of L'_k with Levi complement $L'_{k+\tilde{k}}$.

Note that by Theorem 6.4, we have

(17)
$$\sum r(\tau,\chi,bub_1u_1\bar{b}) = \begin{cases} |\mathrm{Irr}(L'_k,\bar{b})|k!, & \text{if } \gamma \text{ and } \gamma_1 \text{ are non-degenerate,}\\ 2|\mathrm{Irr}(L'_{2(k+\tilde{k})},\bar{b})|(k!)^2(\tilde{k}!)^2, & \text{if } \gamma \text{ or } \gamma_1 \text{ is degenerate,}\end{cases}$$

where χ in the above sum ranges over $\mathrm{Irr}(G',b)$ and τ ranges over $\mathrm{Irr}(L'_{k+\tilde{k}},\bar{b})$. We shall apply Theorem 2.5 with G replaced by G', H replaced by $L'_{k+\tilde{k}}$, c by $\bar{b}$ and u by $bub_1u_1\bar{b}$.

In order to proceed, we choose a particular defect group D of the blocks involved as follows. With the notation introduced at the beginning of the proof of Theorem 6.4, recall that $\mathbf{S}$ is a Sylow ϕ_d-torus of $I_0(\mathbf{V}')$ with $|\mathbf{S}^F| = \phi_d^w(q)$. Put $\mathbf{T} := C_{I_0(\mathbf{V}')}(\mathbf{S})$. Then $\mathbf{T}$ is a maximal F-stable torus of $I_0(\mathbf{V}')$ with $|\mathbf{T}^F| = (q^e \pm 1)^w$. Let D_0 be a Sylow l-subgroup of $T := \mathbf{T}^F$ and let D be Sylow l-subgroup of $I_0(\mathbf{V}')^F \cong SO^-_{2ew}(q)$ containing D_0. Then D is a defect group of the blocks B', B'_1 and $\bar{B}'$. Since $I_0(\mathbf{V}')^F$ is a subgroup of index 2 of $I(\mathbf{V}')^F$, $N_{I_0(\mathbf{V}')^F}(D)$ has index 2 in $N_{I(\mathbf{V}')^F}(D)$. Let us fix an element τ of $N_{I(\mathbf{V}')^F}(D)\setminus N_{I_0(\mathbf{V}')^F}(D)$. Also, let us fix an element ρ of $I(\bar{\mathbf{V}}'')^F \setminus I_0(\bar{\mathbf{V}}'')^F$. (Note that $\dim_{\bar{\mathbb{F}}_q}(\bar{\mathbf{V}}'') \geq 4$ since $\mathrm{rank}(\bar{\gamma}) \geq 2$.)

Lemma 6.7. (i) $C_{\mathbf{G}'}(D) = C_{\mathbf{T}}(D) \times I_0(\mathbf{V}'')$.

(ii) *$N_{\mathbf{G}'}(D)$ is generated by $N_{I_0(\mathbf{V}')}(D) \times I_0(\mathbf{V}'')$ and $\rho\tau$.*

Proof. It is clear that an element of $C_{\mathbf{G}'}(D)$ fixes the orthogonal decomposition $\mathbf{V} = \mathbf{V}' \oplus \mathbf{V}''$. Since l is odd, the centralizers in $\mathbf{G}'$ of l-elements are connected and thus $C_{\mathbf{G}'}(D) = C_{I_0(\mathbf{V}')}(D) \times I_0(\mathbf{V}'')$. Now D_0 certainly contains a regular element of $\mathbf{T}$. Hence $\mathbf{T} = C_{I_0(\mathbf{V}')}(D_0) \geq C_{I_0(\mathbf{V}')}(D)$, and the first result follows. Part (ii) is obvious. ☐

Taking F-fixed points in Lemma 6.7 gives

$$C_{G'}(D) = C_T(D) \times I_0(V''),$$
$$C_{L'_k}(D) = C_T(D) \times I_0(V''_1) \times GL_1(q)^k,$$

and

$$C_{L'_{k+\tilde{k}}}(D) = C_T(D) \times I_0(\bar{V}'') \times GL_1(q)^{k+\tilde{k}}.$$

Moreover, $I_0(V''_1) \times GL_1(q)^k$ is a standard Levi subgroup of $I_0(V'')$ and $I_0(\bar{V}'') \times GL_1(q)^{k+\tilde{k}}$ is a standard Levi subgroup of $I_0(V''_1) \times GL_1(q)^k$. Since D is a Sylow l-subgroup of $I_0(V')$, and T contains $C_{I_0(V')}(D)$, putting $Z = C_T(D)/Z(D)$, we have that Z is a an abelian l'-group and the above gives canonical isomorphisms

$$C_{G'}(D)/Z(D) \cong Z \times I_0(V''),$$
$$C_{L'_k}(D)/Z(D) \cong Z \times I_0(V''_1) \times GL_1(q)^k,$$

and

$$C_{L'_{k+\tilde{k}}}(D)/Z(D) \cong Z \times I_0(\bar{V}'') \times GL_1(q)^{k+\tilde{k}}.$$

Henceforth we will use these isomorphisms without comment. To simplify notation, we denote a chosen unipotent character of $I_0(V'')$ labelled by γ with the same symbol γ. Similarly, γ_1 denotes the character of $I_0(V_1'') \times GL_1(q)^k$ which is the product of the unipotent character of $I_0(V_1'')$ labelled by γ_1 with the trivial character of $GL_1(q)^k$ and $\bar{\gamma}$ the unipotent character of $I_0(\bar{V}'') \times GL_1(q)^{k+\tilde{k}}$ which is the product of the unipotent character of $I_0(\bar{V}'')$ labelled by $\bar{\gamma}$ with the trivial character of $GL_1(q)^{k+\tilde{k}}$. Also, whenever γ is degenerate, let γ^ι denote the other character labelled by γ. We will also use γ etc to denote the inflation of γ to $C_{G'}(D)/Z(D)$; b_γ will be used to denote the defect 0 block of $\mathcal{F}I_0(V'')$ containing γ as well as the defect 0 block of $\mathcal{F}(C_{G'}(D)/Z(D))$ containing γ. The terms ${\gamma_1}^\iota, \bar{\gamma}^\iota, b_{\gamma^\iota}, b_{\gamma_1}$ etc will all have the obvious meanings.

As in Proposition 2.7 we denote by $\Lambda(b)$ the set of irreducible characters of defect zero of $C_{G'}(D)/Z(D)$ connected with b through the Brauer homomorphism. It follows from [13, Lemma 4.5] that if γ and γ_1 are both non-degenerate, then $\Lambda(b) = \{\gamma\}$, and $\Lambda(\bar{b}) = \{\bar{\gamma}\}$ (recall that $\bar{b} = b_1$ and $\bar{\gamma} = \gamma_1$ in this case). If γ is degenerate, $\Lambda(b) = \{\gamma, \gamma^\iota\}$, $\Lambda(b_1) = \{\gamma_1\}$ and $\Lambda(\bar{b}) = \{\bar{\gamma}, \bar{\gamma}^\iota\}$. If γ_1 is degenerate, then $\Lambda(b) = \{\gamma\}$, $\Lambda(b_1) = \{\gamma_1, {\gamma_1}^\iota\}$ and $\Lambda(\bar{b}) = \{\bar{\gamma}\}$. Also, it is clear from the descriptions given above that there is a natural isomorphism

$$\mathcal{F}(C_{G'}(D)/Z(D))b_\gamma \cong \mathcal{F}I_0(V'')b_\gamma.$$

In case both γ and γ_1 are non-degenerate, $N_{G'}(D, \gamma) = N_{G'}(D)$, so by Part (iii) of Lemma 6.7 and the above isomorphism it follows that $N_{I_0(V')}(D)$ acts trivially on $\mathcal{F}(C_{G'}(D)/Z(D))b_\gamma$ and the image of $N_{L'_{k+\tilde{k}}}(D, \gamma)$ in the automorphism group of $\mathcal{F}(C_{G'}(D)/Z(D))b_\gamma$ is generated by $\rho\tau$ and $C_{L'_{k+\tilde{k}}}(D)$. Since $\Lambda(b_1)$ contains a single element, the hypotheses of Proposition 2.7 apply in this case.

In case γ is degenerate, by Part (iii) of Lemma 6.7, we have $N_{G'}(D, \gamma) = N_{I_0(V')}(D) \times I_0(V'')$, from which it is clear that the image of $N_{L'_{k+\tilde{k}}}(D)$ in the automorphism group of $(\mathcal{F}C_{G'}(D)/Z(D))b_\gamma$ is generated by $C_{L'_{k+\tilde{k}}}(D, \gamma)$. So, clearly the hypotheses of Proposition 2.7 apply in this case as well. An argument similar to the first case shows that the hypotheses of Proposition 2.7 apply in case γ_1 is degenerate.

On the other hand, note that by the equations following the proof of Lemma 6.7, the intersection of $C_{G'}(D)$ with U is the unipotent radical of the standard parabolic subgroup of $I_0(V'')$ containing $I_0(V_1'') \times GL_1(q)^k$ as Levi complement. Thus, denoting by Ψ the homomorphism introduced before Proposition 2.6 (with $G = G'$) one sees that $\Psi(u) = v$, where v is the central idempotent corresponding to the trivial representation of this unipotent group. (Here we are considering $I_0(V'')$ as a subgroup of $C_G(D)/Z(D)$ via the isomorphism given above.) Similarly $\Psi(u_1) = v_1$, where v_1 is the central idempotent corresponding to the trivial representation of the unipotent radical of the standard parabolic subgroup of $I_0(V_1'')$ containing $I_0(\bar{V}'') \times GL_1(q)^{k+\tilde{k}}$ as Levi complement.

In case both γ and γ_1 are non-degenerate, $\Psi(b) = b_\gamma$ and $\Psi(b_1) = \Psi(\bar{b}) = b_{\gamma_1}$. Thus, $r(\gamma_1, \gamma, \Psi(bub_1u_1\bar{b}))$ is the multiplicity of γ in

$$R^{I_0(V'')}_{I_0(V_1'') \times GL_1(q)^k}(b_{\gamma_1})_{b_\gamma}.$$

But the latter number is $|\mathcal{S}_{\gamma,\gamma_1}|$ which by Lemma 5.9 equals $k!$. Now (17), Proposition 2.7 and Theorem 2.5 imply that b and $\bar{b}$ are Puig equivalent.

In case γ is degenerate, $\Psi(b) = b_\gamma + b_{\gamma^\iota}$, $\Psi(b_1) = b_{\gamma_1}$ and $\Psi(\bar{b}) = b_{\bar{\gamma}} + b_{\bar{\gamma}^\iota}$. So, $r(\bar{\gamma}, \gamma, \Psi(bub_1u_1\bar{b}))$ is the multiplicity of γ in

$$R^{I_0(V'')}_{I_0(V_1'')\times GL_1(q)^k}(R^{I_0(V_1'')\times GL_1(q)^k}_{I_0(\bar{V}'')\times GL_1(q)^{2k}}(b_{\bar{\gamma}})_{b_{\gamma_1}})_{b_\gamma}$$

and again by Lemma 5.13 and Lemma 5.16, the above number equals $(k!)^2$. Similarly, we get that $r(\bar{\gamma}^\iota, \gamma, \Psi(bub_1u_1\bar{b}))$ is $(k!)^2$. Again, (17), Proposition 2.7 and Theorem 2.5 imply that b and $\bar{b}$ are Puig equivalent.

The proof for the case that γ_1 is degenerate is similar to the above two cases, except that since in this case, $\Psi(b_1) = b_{\gamma_1} + b_{\gamma_1^\iota}$, $r(\bar{\gamma}, \gamma, \Psi(bub_1u_1\bar{b}))$ is equal to the sum of the multiplicities of γ in

$$R^{I_0(V'')}_{I_0(V_1'')\times GL_1(q)^k}(R^{I_0(V_1'')\times GL_1(q)^k}_{I_0(\bar{V}'')\times GL_1(q)^{2k-1}}(b_{\bar{\gamma}})_{b_{\gamma_1}})_{b_\gamma}$$

and in

$$R^{I_0(V'')}_{I_0(V_1'')\times GL_1(q)^k}(R^{I_0(V_1'')\times GL_1(q)^k}_{I_0(\bar{V}'')\times GL_1(q)^{2k-1}}(b_{\bar{\gamma}})_{b_{\gamma_1^\iota}})_{b_\gamma}.$$

□

7. Finiteness results

As a consequence of the results of the previous section, we get the following finiteness theorem. Again, we let l be a prime and choose an l-modular system $(\mathcal{K}, \mathcal{O}, \mathcal{F})$ with $\mathcal{F}$ algebraically closed.

Theorem 7.1. *Let q be a prime power not divisible by l, and let w be a positive integer. Let $\{G_n(q) \mid n = 2, 4, \ldots\}$ be one of the families in 6.2 and suppose that l is a unitary prime for this family. Also, assume that l does not divide $q^2 - 1$. Then the number of Puig (Morita) equivalence classes of unipotent l-blocks of weight w occurring in the group algebras $\mathcal{O}G_n(q)$, $n = 2, 4, \ldots$, is finite. There are at most*

$$\left[\frac{l-1}{2}(w+1)+2\right]\prod_{i=1}^{(l-3)/2}\left[(\frac{l-1}{2}i+i^2)(w+1)^2+(\frac{l-1}{2}+3i)(w+1)+2\right]$$

$$+\left[(\frac{l-3}{2})(w+1)+2\right]\prod_{i=1}^{(l-5)/2}[i(w+1)+1] \tag{18}$$

such blocks. Every unipotent block is Puig (Morita) equivalent to a block of $\mathcal{O}G_n(q)$ for some n where n is less than or equal to

$$\frac{1}{2}(l-1)^2\left[(l-2)(w+1)+1\right]^2+(l-1)(w+2).$$

Proof. Let $G = G_n(q)$ for some n. Choose a finite extension $(\tilde{\mathcal{K}}, \tilde{\mathcal{O}}, \mathcal{F})$ of $(\mathcal{K}, \mathcal{O}, \mathcal{F})$ such that $\tilde{\mathcal{K}}$ is a splitting field for all subgroups of G. Let B be a unipotent block of $\mathcal{F}G$ with label γ. If the hypothesis of Theorem 6.4 is satisfied, we may apply Theorem 6.6 to obtain an equivalence of source algebras of B and $\bar{B}$ over $\tilde{\mathcal{O}}$. Thus by [36, Proposition 38.8], there is an equivalence of the source algebras of B and $\bar{B}$ over $\mathcal{O}$.

Suppose there does not exist a $\bar{\gamma}$ such that the hypotheses of Theorem 6.4 are satisfied.

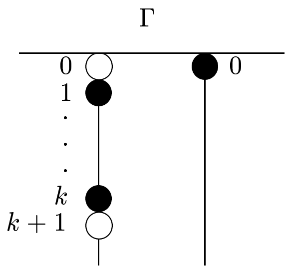
If γ and γ_1 are non-degenerate (so that $\bar{\gamma} = \gamma_1$) and $\mathrm{rank}(\bar{\gamma}) = 1$, then $\bar{\gamma} = \{\{1\}, \{0\}\}$ and $\gamma = \{\{2\}, \{0\}\}$, or $\{\{1, 2\}, \{0, 1\}\}$ or $\bar{\gamma} = \{\{0, 1\}, \{\}\}$ and $\gamma =$

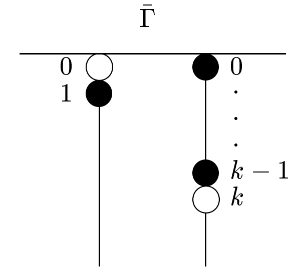
$\{\{0,2\},\{\}\}$ or $\{\{0,1,3\},\{1\}\}$ (the latter only occurs for $e=2$). If γ is degenerate then γ is not equivalent to $\{\{1\},\{1\}\}$ by assumption, and hence $\operatorname{rank}(\bar{\gamma}) \geq 2$. If γ_1 is degenerate, $e=2$ and $\operatorname{rank}(\bar{\gamma})=1$, then $\bar{\gamma}=\{\{1\},\{0\}\}$ and $\gamma=\{\{1,2\},\{0,3\}$. In any of these cases, $\hat{s}_e(\gamma) \leq 2 < w+2$ and $\operatorname{rank}(\gamma) \leq 4$.

Suppose then that either $\hat{s}_e(\gamma)=0$ or that $\hat{s}_e(\gamma)>0$ and there exists a $\bar{\gamma}$ of rank at least 2, but the hypothesis of Theorem 6.4 is not satisfied for any such $\bar{\gamma}$.

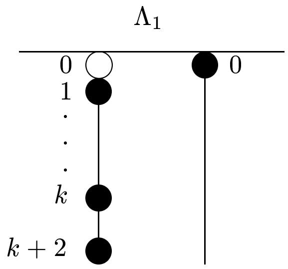
If $e>2$, this can happen only if γ has the property given in [23, Proposition 8.2(iii)] (with w replaced by $w+2$). In other words, if Γ is a $2e$-unitary abacus diagram for some element of γ, then for all i, $1 \leq i \leq e-1$ and for any $a \in \{g,b\}$ such that $\hat{s}_e^{i_a}(\Gamma)>0$, either $\hat{s}_e^{i_a}(\Gamma)<w+2$ or the diagram obtained by swapping strings i_a and $(i-1)_a$ is degenerate. Then the result follows from Theorem 6.6 and [23, Proposition 8.2(iii)]. If $e=2$, then $\hat{s}_e(\gamma)<w+2$ and the result follows from Theorem 6.6 and [23, Proposition 8.2(ii)]. □

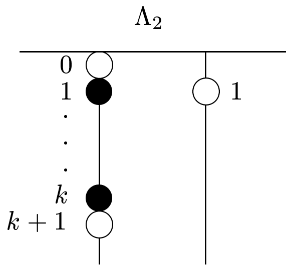
Example 7.2. To illustrate the difficulties in case $e=1$, let us consider the simplest non-trivial example, the case $w=1$. Let k be an odd positive integer and let γ denote the equivalence class of symbols introduced in Example 5.10. We choose a 2-unitary diagram Γ for γ which has exactly one bead on string 0_b. A picture of Γ as well as of $\bar{\Gamma}$ (where this time $\bar{\Gamma}$ is the diagram associated to Γ as in [23, Subsection 7.5, Case 2.2.2]) is given below.



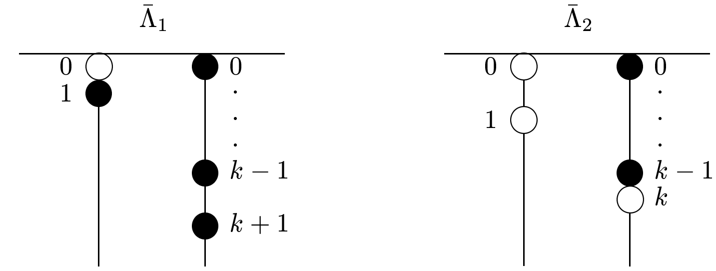


Let l be an odd prime and let q be a prime power with $l \mid q+1$. Then γ determines a unipotent 2-cuspidal character ψ of $H := CSO_{n-2}^{+}(q)$ with $n-2=(k+1)^2/2$. In turn, ψ determines a unipotent l-block B of $G := CSO_n^{-}(q)$ of weight 1 and 1-unitary core γ. The unipotent characters of B are labelled by symbols λ_1 and λ_2. We obtain diagrams Λ_1 and Λ_2 for these two symbols by adding a 1-unitary hook to Γ in two ways.

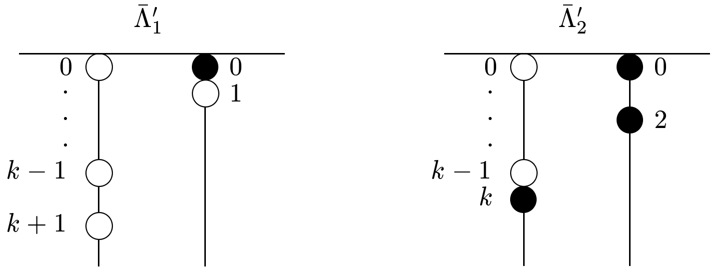




The symbols $\bar{\lambda}_1$ and $\bar{\lambda}_2$ (notation as in the proof of [23, Theorem 7.10]) can be represented by the following diagrams $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$.



Of course, they can also be represented by the swaps $\bar{\Lambda}_1'$ and $\bar{\Lambda}_2'$ (see the diagram operations introduced in 5.3) of the above diagrams.



Let χ_1 and χ_2 denote the two unipotent characters of G with labels λ_1 and λ_2, respectively. Similarely, $\bar{\chi}_1$ and $\bar{\chi}_2$ denote the two unipotent characters of the Levi subgroup L_k of G with labels $\bar{\lambda}_1$ and $\bar{\lambda}_2$, respectively. Then, with the notation of Theorem 6.4 we find in this case that

$$R_{L_k}^G(\bar{\chi}_1)_B = (k!)\chi_1 + f(k)\chi_2,$$

and

$$R_{L_k}^G(\bar{\chi}_2)_B = f(k)\chi_1 + (k!)\chi_2$$

for some function f with $f(k) > 0$. More precisely, $f(k)$ is the number of "paths" of length k, from the diagram $\bar{\Lambda}_2'$ to the diagram Λ_1, where a single step consists in the addition of a 1-hook. This is also equal to the number of such paths from $\bar{\Lambda}_1'$ to Λ_2. For example, $f(3) = 2$ and $f(5) = 16$.

8. Groups of type A

In this section we let l denote a prime and $\mathcal{F}$ an algebraically closed field of characteristic l.

Let $\mathfrak{C}$ be a class of finite groups. We say that *Donovan's conjecture (for l or over $\mathcal{F}$) holds for* $\mathfrak{C}$, if for all finite l-groups D, the number of Morita equivalence classes of blocks with defect groups isomorphic to D occuring in group algebras $\mathcal{F}G$ for groups G in $\mathfrak{C}$, is finite. If the number of Cartan matrices of such blocks is finite, we say that the *weak Donovan conjecture holds for* $\mathfrak{C}$.

8.1. **The theorem of Bonnafé and Rouquier.** Bonnafé and Rouquier have recently proved a conjecture of Broué. In this section we discuss some consequences of this result for groups of type A. For a prime power q and a positive integer n, we shall write $GL_n(-q)$ for the unitary group $GU_n(q)$ in the following.

Theorem 8.2. (Bonnafé and Rouquier) *Let $G = GL_n(\varepsilon q)$ $(\varepsilon = \pm 1)$, such that l does not divide q and let s be a semisimple l'-element of $G^* = G$. Let B be an $\mathcal{F}G$-block with defect group D. Assume that the ordinary characters of B are contained in $\mathcal{E}_l(G, (s))$.*

Then B is Morita equivalent to a unipotent block B' of a group isomorphic to $GL_{n_1}((\varepsilon q)^{m_1}) \times \cdots \times GL_{n_k}((\varepsilon q)^{m_k})$ with $\sum_{i=1}^{k} n_i m_i = n$. Moreoever, B and B' have isomorphic defect groups. Thus $B' \cong B_1 \otimes \cdots \otimes B_k$, where B_i is a unipotent block of $GL_{n_i}((\varepsilon q)^{m_i})$ with defect group D_i such that $D \cong D_1 \times \cdots \times D_k$.

Proof. By the main result of Bonnafé and Rouquier [4, Théorème 11.8], B is Morita equivalent to a unipotent block B' of $C_G(s)$. The centralizer of every semisimple element of G is of the form given in the theorem. The statement about the defect group follows from a result of Cabanes and Enguehard [14, Remark 3.6]. □

This theorem shows that, in order to prove Donovan's conjecture for the general linear or unitary groups, it suffices to consider only their unipotent blocks.

8.3. **Donovan's conjecture for the linear groups.** Let us begin with an example which combines the results of Jost [25], and [23, Theorem 9.3] with the new results of Bonnafé and Rouquier stated above.

In the following, we denote the set of non-trivial powers of prime numbers by $\hat{\mathbb{P}}$.

Theorem 8.4. *Donovan's conjecture holds for $\mathfrak{C} = \{GL_n(q) \mid n \in \mathbb{N}, q \in \hat{\mathbb{P}}\}$.*

Proof. Let D be a finite non-trivial l-group and let B be an l-block with defect group isomorphic to D of the group algebra $\mathcal{F}GL_n(q)$ for some $GL_n(q) \in \mathfrak{C}$. If q is a power of l, then B has maximal defect. This implies that $GL_n(q)$ is uniquely determined by l and D and the result follows in this case.

We may thus assume that q is not divisible by l. Next, by Theorem 8.2, we may assume that B is unipotent. In this case, the result follows from [23, Theorem 9.3]. □

It is natural to ask for the number of Morita equivalence classes occuring in the above theorem, or, more demanding, for the exact invariants of a block determining its Morita equivalence class. Although it is not too diffcult to conjecture the answer, we can only give a weak result in this direction. We show that under certain, very natural necessary conditions on two prime powers q_1 and q_2, the unipotent blocks of $GL_n(q_i)$, $i = 1, 2$, have the same decompositon numbers.

Theorem 8.5. *Let n be a positive integer. Suppose that $q_1, q_2 \in \hat{\mathbb{P}}$ are not divisible by l and have the same residue modulo l. In particular, they have the same order d modulo l. Assume in addition that the l-parts of $q_1^d - 1$ and of $q_2^d - 1$ are the same.*

Then there is a bijection between $\mathcal{E}_l(GL_n(q_1), (1))$ and $\mathcal{E}_l(GL_n(q_2), (1))$ such that corresponding characters have the same l-modular decompositon numbers.

Proof. We match the unipotent characters of $G_i := GL_n(q_i)$, $i = 1, 2$ via their canonical labelling by partitions of n. The result in this case follows from Geck's work on factorizations of decomposition maps. A survey of this is given in Section 3 of [19], and Example 3.3 thereof treats the case of Iwahori-Hecke algebras. This gives, by the way, a proof for a somewhat weaker form of [30, Conjecture 6.38].

The G_i-conjugacy classes of l-elements are classified, via their rational Jordan canonical form, by multisets of irreducible polynomials over $\mathbb{F}_{q_i}$ of degree at most n, whose roots have l-power order. Let f be the degree of such a polynomial. Then

$f \leq n$ and $l \mid (q_i^f - 1)/(q_i^t - 1)$ for all divisors $t \neq f$ of f. Hence $l \mid \Phi_f(q_i)$ (recall that Φ_f denotes the f-th cyclotomic polynomial). It follows from [24, Lemma IX.8.1] that $f = l^a d$ for some non-negative integer a. Let a be the largest integer such that $l^a d \leq n$ and put $f := l^a d$. Then the field $\mathbb{F}_{q_i^f}$ contains the roots of all such polynomials. Our assumption, together with [24, Lemma IX.8.1] now show, that $q_1^f - 1$ and $q_2^f - 1$ are divisible by the same power of l. Hence there is an isomorphism between the Sylow l-subgroup of $\mathbb{F}^*_{q_1^f}$ and that of $\mathbb{F}^*_{q_2^f}$, compatible with the action of the Galois groups of $\mathbb{F}_{q_i^f}$ over $\mathbb{F}_{q_i}$.

This implies that there is a bijection between the G_1-conjugacy classes of l-elements and the G_2-conjugacy classes of l-elements such that if $t_1 \in G_2$ and $t_2 \in G_2$ lie in corresponding classes, then $C_1 := C_{G_1}(t_1)$ and $C_2 := C_{G_2}(t_2)$ arise from the same generic subgroup of $\mathbb{GL}_n$. In particular, there is a natural bijection $\lambda_1 \mapsto \lambda_2$ between their unipotent characters. Moreover, $R_{C_1}^{G_1}(\lambda_1)$ and $R_{C_2}^{G_2}(\lambda_2)$ have the same decompositon into unipotent characters of G_1 and G_2 respectively (if these unipotent characters are matched as above).

There is a linear character $\hat{t}_i$ of $C_{G_i}(t_i)$ such that $\pm R_{C_i}^{G_i}(\hat{t}_i \lambda_i)$ is an irreducible character χ_{t_i,λ_i} of G_i lying in $\mathcal{E}(G_i, (t_i))$, the sign being independent of i. Also, χ_{t_i,λ_i} and $R_{C_i}^{G_i}(\lambda_i)$ have the same restriction to the l-regular conjugacy classes of G_i. This restriction is thus determined by the coefficients of the unipotent characters occuring in $R_{C_i}^{G_i}(\lambda_i)$ and the decomposition numbers of unipotent characters of G_i. Since, under the above bijection of unipotent characters, these coefficients are the same for each i, the result follows. ☐

We finally describe the consequences for the special linear groups of our results for the general linear groups.

Theorem 8.6. (a) *Let D be a finite l-group. Then, up to Morita equivalence, there are only finitely many l-blocks with defect group D occurring as unipotent blocks in the group algebras $\mathcal{F}G$ for groups G in $\mathfrak{C}' := \{SL_n(q) \mid n \in \mathbb{N}, q \in \hat{\mathbb{P}}\}$.*

(b) *The weak Donovan conjecture holds for $\mathfrak{C}'$.*

Proof. As in the proof of Theorem 8.4 we may assume that q is not divisible by l. To prove the result in this case, we introduce the following notation. If $G = SL_n(q)$ for some $n \geq 1$ and q not divisible by l, we write $\tilde{G}$ for the corresponding general linear group containing G as a normal subgroup of index $q - 1$, i.e., $\tilde{G} = GL_n(q)$.

Let B be an l-block with defect group isomorphic to D of the group algebra $\mathcal{F}G$ for $G = SL_n(q)$. Let $\tilde{B}$ be an l-block of $\tilde{G}$ covering B.

Suppose first that B is unipotent. We may choose $\tilde{B}$ to be unipotent as well. If $l \nmid q - 1$, then B and $\tilde{B}$ are isomorphic by Theorem 3.2. Moreoever, they have the same defect group. By Theorem 8.4, there are only finitley many possible Morita equivalence classes for B. On the other hand, if $l \mid q - 1$, then $\tilde{B}$ is the principal block by the results of Fong and Srinivasan. Hence B is the principal block of $\mathcal{F}G$, and there are only finitely many possibilities for G.

Let us now assume that B is not a unipotent block. By a result of Broué and Michel [10], there is a semisimple l'-element $\tilde{s} \in \tilde{G}$ such that $\tilde{B}$ is contained in $\mathcal{E}_l(\tilde{G}, \tilde{s})$. Then B is contained in $\mathcal{E}_l(G, s)$, where s is the image of $\tilde{s}$ under the canonical epimorphism of $\tilde{G} = GL_n(q)$ onto $PGL_n(q)$, the dual group of G (see, e.g., [3, Proposition 2.3.5]).

Let $\chi \in \mathrm{Irr}(B)$, and let $\tilde{\chi} \in \mathrm{Irr}(\tilde{B})$ such that χ occurs in the restriction of $\tilde{\chi}$ to G. By [29, Proposition, p. 162], the multiplicity of χ in the restriction is 1. Moreoever, if $\tilde{\chi}'$ is another irreducible character in $\tilde{B}$ whose restriction to G contains χ, then $\tilde{\chi}' = \lambda\tilde{\chi}$ for some linear character λ of l-power order (see, e.g., [3, (0.7.5) and Corollary 2.3.4]).

Let $\tilde{D}$ be a defect group of $\tilde{B}$ such that $D = \tilde{D} \cap G$. Then $\tilde{D}/D \cong \tilde{D}G/G \leq \tilde{G}/G$. Hence $|\tilde{D}\colon D|$ is bounded above by the l-part l^a of $q-1$. If l does not divide $q-1$, then $\tilde{D} = D$. Suppose that l divides $q-1$. By the description of the blocks of $GL_n(q)$ given by Dipper and James, $\tilde{B} = \mathcal{E}_l(\tilde{G}, (\tilde{s}))$. Moreoever, a Sylow l-subgroup of $C_{\tilde{G}}(\tilde{s})$ is a defect group of $\tilde{B}$. Now $C_{\tilde{G}}(\tilde{s}) \cong \times_{i=1}^{k} GL_{n_i}(q^{m_i})$ with $\sum_{i=1}^{k} n_i m_i = n$. If k, or one of the n_i is greater than 1, then l^{2a} divides $|\tilde{D}|$. In this case $l^a \mid |D|$, and so $|\tilde{D}| \leq |D|^2$. On the other hand, if $k = 1$ and $n_1 = n$, then $C_{\tilde{G}}(\tilde{s}) \cong GL_1(q^n)$ is the Coxeter torus of $\tilde{G}$. Hence $\tilde{D}$ and thus also D are cyclic. Thus if $l \mid q-1$, either D is cyclic or $|\tilde{D}| \leq |D|^2$.

If D is cyclic, there is nothing to prove. So suppose that D is non-cyclic and let M be the largest decomposition number occuring in an l-block with defect group of order at most $|D|^2$ of some group in $\mathfrak{C}$. Let $\tilde{\Phi}$ be the character of a projective indecomposable module of $\tilde{B}$ corresponding to the irreducible Brauer character $\tilde{\varphi}$. Then, for $\chi \in \mathrm{Irr}(G, B)$, we find

$$\begin{aligned}
(\tilde{\Phi}|_G, \chi) &= \sum_{\tilde{\chi} \in \mathrm{Irr}(\tilde{G},\tilde{B}), (\tilde{\chi}|_G,\chi) \neq 0} d_{\tilde{\chi},\tilde{\varphi}} \\
&\leq M \, |\{\tilde{\chi} \in \mathrm{Irr}(\tilde{G}, \tilde{B}) \mid (\tilde{\chi}|_G, \chi) \neq 0\}| \\
&\leq M \, l^a \\
&\leq M \, |D|.
\end{aligned}$$

The last inequality follows from $l^a \leq |D|$. Since every projective indecomposable character of B occurs in the restriction of a $\tilde{\Phi}$ as above, this implies the desired result. □

8.7. **Donovan's conjecture for the unitary groups.**

Theorem 8.8. *Let D be a finite l-group. Then the following holds.*

(a) *Up to Morita equivalence, there are only finitely many l-blocks with defect group D occurring in group algebras $\mathcal{F}G$ for groups G in*

$$\mathfrak{D} := \{GU_n(q) \mid n \in \mathbb{N}, q \in \hat{\mathbb{P}}, l \nmid q^{2s+1} + 1 \text{ for all } s \in \mathbb{N}\}.$$

(b) *Up to Morita equivalence, there are only finitely many l-blocks with defect group D occurring as unipotent blocks in group algebras $\mathcal{F}G$ for groups G in*

$$\mathfrak{D}' := \{SU_n(q) \mid n \in \mathbb{N}, q \in \hat{\mathbb{P}}, l \nmid q^{2s+1} + 1 \text{ for all } s \in \mathbb{N}\}.$$

(c) *The weak Donovan conjecture holds for $\mathfrak{D}'$.*

Proof. The condition on q just means that l is a linear prime for $GU_n(q)$. By Theorem 8.2, B is Morita equivalent to a unipotent block B' of a group isomorphic to $GL_{n_1}((-q)^{m_1}) \times \cdots \times GL_{n_k}((-q)^{m_k})$ with $\sum_{i=1}^{k} n_i m_i = n$. Now $GL_{n_i}((-q)^{m_i})$ equals $GL_{n_i}(q^{m_i})$, if m_i is even, whereas for odd m_i, it is equal to $GU_{n_i}(q^{m_i})$. In the latter case, the order of $-q^{m_i}$ modulo l is even, i.e., l is linear for $GU_{n_i}(q^{m_i})$.

We conclude that $B' \cong B_1 \otimes \cdots \otimes B_k$, where B_i is a unipotent block of $GL_{n_i}(q^{m_i})$ whose defect is contained in D, or of $GU_{n_i}(q^{m_i})$ with defect group isomorphic to a subgroup of D. Moreover, l is linear for $GU_{n_i}(q^{m_i})$. In view of Theorem 8.4, we may thus assume that B is a unipotent block of $G = GU_n(q)$ and that l is linear for G. In this case the result follows from Theorem 4.7.

The proof for the special unitary groups is similar to the one for the special linear groups (Theorem 8.6). Note that by assumption, l does not divide $q + 1$, so that the corresponding part of that proof is not relevant here. □

8.9. **The proof of the theorem in the introduction.** Let l, $\mathcal{F}$ and D be as in the theorem and suppose that D is non-trivial. Let $\mathcal{B}$ be a finite set of basic $\mathcal{F}$-algebras, such that each of the unipotent blocks occurring in Theorem 4.9, in Theorem 8.6(a) and in Theorem 8.8(b) is Morita equivalent to an element of $\mathcal{B}$. Suppose also that $\mathcal{B}$ contains the basic algebras of the principal l-blocks of the finitely many groups occurring in the theorem, which are of characteristic l and have a Sylow l-subgroup isomorphic to D.

Let G be one of the classical groups listed in the theorem and let B be an l-block of $\mathcal{F}G$ with defect group isomorphic to D. Suppose that B is not Morita equivalent to an element of $\mathcal{B}$ and that G and l are not as in Case (3) of the theorem.

Let N be an integer which is larger than any of the bounds (on the dimension n of the classical groups involved) given in [23, Theorem 8.3.(i)(ii)] and Theorem 7.1. (Notice that $l^w \leq |D|$ so that N only depends on D.) Suppose also that N is larger than the maximum n such that $SU_n(q)$ for some q with $l \mid q + 1$ contains a Sylow l-subgroup isomorphic to D. Finally, if $l = 2$, suppose also that N is larger than the natural dimension of any other classical group with a Sylow 2-subgroup isomorphic to D.

If $G = SU_n(q)$ and $l \nmid q+1$, it follows from [23, Theorem 8.3.(i)] and Theorem 3.2 that Case (2) occurs. If $l \mid q+1$, let $\tilde{B}$ be a unipotent l-block of $GU_n(q)$ covering B. Then, by the work of Fong and Srinivasan, $\tilde{B}$ is the principal l-block of $GU_n(q)$ and thus D is isomorphic to a Sylow l-subgroup of $SU_n(q)$. It follows that $n \leq N$.

Finally suppose that G is one of the other classical groups. If $l > 2$, then [23, Theorem 8.3.(ii)], Theorem 7.1 and Theorem 3.2 show that we are in Case (2). If $l = 2$, then B is the principal block of G (this follows from [12, Theorem 13]) and thus D is a Sylow 2-subgroup of G. This implies that the natural dimension of G is at most equal to N.

9. Remarks and corrections

The following page numbers refer to our paper [23].

Conjecture 1.2 on Page 379 is incorrectly credited to [34]. Puig's conjecture is not formulated in this reference, but in the unpublished manuscript [32].

The definition of a symbol $\{X, Y\}$ on Page 382, Lines 19, 20 should include the condition that not both of X and Y are empty.

Replace $\check{w}_e(\lambda)$ by $\hat{w}_e(\lambda)$ on Page 386, Line 21.

Delete "conformal" and replace $CSO_n^-(q)$ by $SO_n^-(q)$ on Page 390, Line 6. Replace "of" by "to" on Page 390, Line 22. Omit "and by $I_0(V)$ the intersection of $I(V)$ with $SL(V)$" on the same page, Lines −12, −11.

Replace $\mathbf{G} := I_0(\mathbf{V})$ by $\mathbf{G} := I_0(\mathbf{V}) := I(\mathbf{V}) \cap SL(\mathbf{V})$ on Page 391, Line 11.

The embedding of $\mathbf{G}_{n-2k}$ into $\mathbf{M}_k$ on Page 394 is of course F-equivariant.

Replace $SCO_n^-(q)$ by $CSO_n^-(q)$ on Page 397, Line 14. It is helpful to note that the unique unipotent character of $(C)SO_2^{\pm}(q)$ is labelled by the class of symbols of rank 1 containing $\{\{1\},\{0\}\}$ (in the '+'-case), respectively $\{\{0,1\},\{\}\}$ (in the '$-$'-case). Moreover, there is no symbol of rank 0 and even defect.

On Page 403, Line -7 we write: "Cases 2.1 and 2.2.1 are proved similarly." This is of course not true under the assumptions of [23, Lemma 7.4], as Example 5.10(1) above shows.

Insert "in" before "Case 1" on Page 404, Line -18. Replace $\bar{\mathbf{K}} := C_{\bar{\mathbf{G}}}(\mathbf{T})$ by $\bar{\mathbf{K}} := C_{\bar{\mathbf{G}}}(\mathbf{S})$ on the last line of Page 404. The resulting $\bar{\mathbf{K}}$ is of course the same, but the new version makes it evident that $\bar{\mathbf{K}}$ is d-split.

Replace $[\mathbf{K},\mathbf{K}] \subseteq C_{\mathbf{G}}(t), C_{\mathbf{G}}(t)]$ by $[\mathbf{K},\mathbf{K}] \subseteq [C_{\mathbf{G}}(t), C_{\mathbf{G}}(t)]$ on Page 405, Line -12.

Replace $k!_\chi$ by $k!\chi$ on Page 406, Line 14. The notation $I_0(\bar{\mathbf{V}})$ on Line -16, Page 406, stands for the identity component $I(\bar{\mathbf{V}})^0$. The reference to Lemma 5.7 on the same page, Line -2, is not sufficient for the purpose of the proof, since it is stated only for Harish-Chandra induction. This reference should be replaced by a reference to Lemma 3.3 of this paper. Replace ψ_γ in Equation (10) of the same page by ψ.

On Page 408, Lines $-3, -2$, we write: "Arguing as in Case 1, one sees immediately that $\Delta = \bar{\Lambda} \ldots$". In Case 2.2.1 it can also happen that $\Delta = \bar{\Lambda}'$ (notation of Subsection 5.3), as Example 5.10(1) shows. For the same reason, our statement "This shows immediately that $\Delta = \bar{\Lambda} \ldots$" on Page 409, Line 8, is wrong, as shown by Examples 5.10(2) and 7.2.

Replace $w.$ by $w,$ on Page 412, Line -12.

Replace $\frac{l-2}{2}$ by $\frac{l-1}{2}$ in Equation (20) on Page 416. Equation (21) on the same page appears to contain various misprints. It is, however, made obsolete by Equation (18) of this paper.

Replace "in" by "is" on Page 417, Line 9.

Replace $\{\{0,1,2\},\{0\}\}$ by $\{\{1,2\},\{0,1\}\}$ and $\mathrm{rank}(\bar{\gamma}) > 2$ by $\mathrm{rank}(\bar{\gamma}) \geq 2$ on Page 418, Line 8. (Notice that the extra case $\gamma = \{\{0,1,3\},\{1\}\}$ in the proof of Theorem 7.1 of this paper only occurs for $e = 2$.)

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G.H.: Lehrstuhl D für Mathematik, RWTH Aachen, Templergraben 64, 52062 Aachen, Germany

R.K.: Department of Mathematics, Ohio State University, 231, W.18th Avenue, Columbus, OH 43210, USA

E-mail address: `G.H.: gerhard.hiss@math.rwth-aachen.de`

E-mail address: `R.K.: kessar@math.ohio-state.edu`