FINITE GROUPS OF LIE TYPE AND THEIR REPRESENTATIONS

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This article is a slightly expanded account of the series of four lectures I gave at the conference. It is intended as a (non-comprehensive) survey covering some important aspects of the representation theory of finite groups of Lie type, where the emphasis is put on the problem of labelling the irreducible representations and of finding their degrees. All three cases are covered, representations in characteristic zero, in defining as well as in non-defining characteristics.

The first section introduces various ways of defining groups of Lie type and some classes of important subgroups of them. The next three sections are devoted to the representation theory of these groups, each section covering one of the three cases.

The lectures were addressed at a broad audience. Thus on the one hand, I have tried to introduce even the most fundamental notions, but on the other hand, I have also tried to get right to the edge of todays knowledge in the topics discussed. As a consequence, the lectures were of a somewhat inhomogeneous level of difficulty. In this article I have omitted the most introductory material. The reader may find all background material needed from representation theory in the textbook [51] by Isaacs.

For this survey I have included a few more examples, as well as most of the references to the results presented in my talks. The sections in this article correspond to the four lectures I have given, the subsections to the sections inside the lectures, and the subsubsections to the individual slides.

1 The finite groups of Lie type

In this first section we give various examples and constructions for finite groups of Lie type, we introduce the concepts of finite reductive groups and groups with BN-pairs. All of this material can be found in the books by Carter [9, 10] and Steinberg [77, 78].

1.1 Various constructions for finite groups of Lie type

One of the motivations to study finite groups of Lie type stems from the fact that this class of groups constitutes a large portion of the class of all finite simple groups.

1.1.1 The classification of the finite simple groups

"Most" finite simple groups are closely related to finite groups of Lie type. This is a consequence of the classification theorem of the finite simple groups.

Theorem 1.1 (Classification of the finite simple groups) *Every finite simple group is*

(1) *one of* 26 *sporadic simple groups; or*
(2) *a cyclic group of prime order; or*
(3) *an alternating group* A_n *with* $n \geq 5$*; or*
(4) *closely related to a finite group of Lie type.*

So what are finite groups of Lie type? A first answer could be: Finite analogues of Lie groups.

1.1.2 The finite classical groups

Examples for finite analogues of Lie groups are the finite classical groups, i.e. full linear groups or linear groups preserving a form of degree 2, defined over finite fields. Let us list a few examples of classical groups.

Example 1.2 $\mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$, $\mathrm{Sp}_{2m}(q)$, $\mathrm{SO}_{2m+1}(q)$ $\ldots$ (q a prime power) are classical groups. To be more specific, we may define

$$\mathrm{SO}_{2m+1}(q) = \{g \in SL_{2m+1}(q) \mid g^{tr} J g = J\},$$

with

$$J = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} \in \mathbb{F}_q^{2m+1 \times 2m+1}.$$

Related groups, e.g. $\mathrm{SL}_n(q)$, $\mathrm{PSL}_n(q)$, $\mathrm{CSp}_{2m}(q)$, the conformal symplectic group, etc. are also classical groups.

Not all classical groups are simple, but closely related to simple groups. For example, the projective special linear group $\mathrm{PSL}_n(q) = \mathrm{SL}_n(q)/Z(SL_n(q))$ is simple (unless $(n, q) = (2, 2), (2, 3)$), but $\mathrm{SL}_n(q)$ is not simple in general.

1.1.3 Exceptional groups

There are groups of Lie type which are not classical, namely, the *exceptional groups* $G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$ (q a prime power), the *twisted groups* ${}^2E_6(q)$, ${}^3D_4(q)$ (q a prime power), the *Suzuki groups* ${}^2B_2(2^{2m+1})$ ($m \geq 0$), and the *Ree groups* ${}^2G_2(3^{2m+1})$ and ${}^2F_4(2^{2m+1})$ ($m \geq 0$). The names of these groups, e.g. $G_2(q)$ or $E_8(q)$ refer to simple complex Lie algebras or rather their root systems.

Some of the questions we are going to discuss in this section are: How are groups of Lie type constructed? What are their properties, subgroups, orders, etc?

1.1.4 The orders of some finite groups of Lie type

The orders of groups of Lie type are given by nice formulae.

Example 1.3 Here are these order formulae for some finite groups of Lie type.
$|\mathrm{GL}_n(q)| = q^{n(n-1)/2}(q-1)(q^2-1)(q^3-1)\cdots(q^n-1)$.
$|\mathrm{GU}_n(q)| = q^{n(n-1)/2}(q+1)(q^2-1)(q^3+1)\cdots(q^n-(-1)^n)$.
$|\mathrm{SO}_{2m+1}(q)| = q^{m^2}(q^2-1)(q^4-1)\cdots(q^{2m}-1)$.
$|F_4(q)| = q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$.
$|{}^2F_4(q)| = q^{12}(q-1)(q^3+1)(q^4-1)(q^6+1)$ $(q = 2^{2m+1})$.

Is there a systematic way to derive these formulae?

1.1.5 Root systems

We take a little detour to discuss root systems and related structures. Let V be a finite-dimensional real vector space endowed with an inner product $(-,-)$.

Definition 1.4 A root system in V is a finite subset $\Phi \subset V$ satisfying:
(1) Φ spans V as a vector space and $0 \notin \Phi$.
(2) If $\alpha \in \Phi$, then $r\alpha \in \Phi$ for $r \in \mathbb{R}$, if and only if $r \in \{\pm 1\}$.
(3) For $\alpha \in \Phi$ let s_α denote the reflection on the hyperspace orthogonal to α:

$$s_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha, \quad v \in V.$$

Then $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.
(4) $2(\beta,\alpha)/(\alpha,\alpha) \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

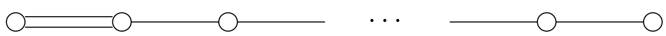
1.1.6 Weyl group and Dynkin diagram

Let Φ be a root system in the inner product space V. The group

$$W := W(\Phi) := \langle s_\alpha \mid \alpha \in \Phi \rangle \leq O(V)$$

is called the *Weyl group* of Φ. Another important notion is that of a *base* of Φ. This is a subset $\Pi \subset \Phi$ such that
(1) Π is a basis of V.
(2) Every $\alpha \in \Phi$ is an integer linear combination of Π with either only non-negative or only non-positive coefficients.

The Weyl group acts regularly on the set of bases of Φ. The *Dynkin diagram* of Φ is defined with respect to one such base. It is the graph with nodes $\alpha \in \Pi$, and $4(\alpha,\beta)^2/(\alpha,\alpha)(\beta,\beta)$ edges between the nodes α and β. For example, the Dynkin diagram of a root system of type B_r looks as follows.

B_r: 

1.1.7 Chevalley groups

Chevalley groups are subgroups of automorphism groups of finite classical Lie algebras. A *Classical Lie algebra* is a Lie algebra corresponding to a finite-dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$.

These have been classified by Killing and Cartan in the 1890s in terms of root systems. Let Φ be the root system of $\mathfrak{g}$, and let Π be a base of Φ. It was shown by Chevalley, that $\mathfrak{g}$ has a particular basis, now called *Chevalley basis*, $\mathcal{C} = \{e_r \mid r \in \Phi, h_r, r \in \Pi\}$, such that all structure constants with respect to $\mathcal{C}$ are integers.

Let $\mathfrak{g}_\mathbb{Z}$ denote the $\mathbb{Z}$-form of $\mathfrak{g}$ constructed from $\mathcal{C}$, i.e. the set of $\mathbb{Z}$-linear combinations of $\mathcal{C}$ inside $\mathfrak{g}$. Then $\mathfrak{g}_\mathbb{Z}$ is a Lie algebra over the integers, free and of finite rank as an abelian group. If k is any field, then $\mathfrak{g}_k := k \otimes_\mathbb{Z} \mathfrak{g}_\mathbb{Z}$ is the *classical Lie algebra corresponding to* $\mathfrak{g}$.

1.1.8 Chevalley's construction (1955, [11])

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ with Chevalley basis $\mathcal{C}$. For $r \in \Phi$, $\zeta \in \mathbb{C}$, there is $x_r(\zeta) \in \mathrm{Aut}(\mathfrak{g})$ defined by

$$x_r(\zeta) := \exp(\zeta \cdot \mathrm{ad}\ e_r).$$

Here, ad e_r denotes the endomorphism $x \mapsto [x, e_r]$ of $\mathfrak{g}$. The matrices of $x_r(\zeta)$ with respect to $\mathcal{C}$ have entries in $\mathbb{Z}[\zeta]$. This allows to define $x_r(t) \in \mathrm{Aut}(\mathfrak{g}_k)$ by replacing ζ by $t \in k$. Then

$$G := \langle x_r(t) \mid r \in \Phi, t \in k \rangle \leq \mathrm{Aut}(\mathfrak{g}_k)$$

is the *Chevalley group* corresponding to $\mathfrak{g}$ over k.

Names such as $A_r(q)$, $B_r(q)$, $G_2(q)$, $E_6(q)$, etc. refer to the type of the root system Φ of $\mathfrak{g}$.

1.1.9 Twisted groups (Tits, Steinberg, Ree, 1957 – 61)

Chevalley's construction gives many of the finite groups of Lie type, but not all. For example, the unitary group $\mathrm{GU}_n(q)$ is not a Chevalley group in this sense. However, $\mathrm{GU}_n(q)$ is obtained from the Chevalley group $\mathrm{GL}_n(q^2)$ by *twisting*:

Let σ denote the automorphism $(a_{ij}) \mapsto (a_{ij}^q)^{-tr}$ of $\mathrm{GL}_n(q^2)$. Then

$$\mathrm{GU}_n(q) = \mathrm{GL}_n(q^2)^\sigma := \{g \in \mathrm{GL}_n(q^2) \mid \sigma(g) = g\}.$$

Similar constructions give the twisted groups ${}^2E_6(q)$, ${}^3D_4(q)$, and the Suzuki and Ree groups ${}^2B_2(2^{2m+1})$, ${}^2G_2(3^{2m+1})$, ${}^2F_4(2^{2m+1})$. These constructions were found by Tits, Steinberg and Ree between 1957 and 1961 (see [80, 75, 70, 71]), although ${}^2B_2(2^{2m+1})$ was discovered in 1960 by Suzuki [79] by a different method.

1.2 Finite reductive groups

The construction discussed in this subsection introduces a decisive class of finite groups of Lie type.

1.2.1 Linear algebraic groups

Let $\bar{\mathbb{F}}_p$ denote the algebraic closure of the finite field $\mathbb{F}_p$. For the purpose of this survey, a *(linear) algebraic group* $\mathbf{G}$ over $\bar{\mathbb{F}}_p$ is a closed subgroup of $\mathrm{GL}_n(\bar{\mathbb{F}}_p)$ for some n. Here, and in the following, topological notions such as closedness refer to the *Zariski topology* of $\mathrm{GL}_n(\bar{\mathbb{F}}_p)$. The closed sets in the Zariski topology are the zero sets of systems of polynomial equations.

Example 1.5 (1) $\mathrm{SL}_n(\bar{\mathbb{F}}_p) = \{g \in \mathrm{GL}_n(\bar{\mathbb{F}}_p) \mid \det(g) = 1\}$.
(2) $\mathrm{SO}_n(\bar{\mathbb{F}}_p) = \{g \in \mathrm{SL}_n(\bar{\mathbb{F}}_p) \mid g^{tr}Jg = J\}$ ($n = 2m+1$ odd).

The algebraic group $\mathbf{G}$ is *semisimple*, if it has no closed connected soluble normal subgroup $\neq 1$. It is *reductive*, if it has no closed connected unipotent normal subgroup $\neq 1$. In particular, semisimple algebraic groups are reductive. For a thorough treatment of linear algebraic group see the textbook by Humphreys [49].

1.2.2 Frobenius maps

Let $\mathbf{G} \leq \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ be a connected reductive algebraic group. A *standard Frobenius map* of $\mathbf{G}$ is a homomorphism

$$F := F_q : \mathbf{G} \to \mathbf{G}$$

of the form $F_q((a_{ij})) = (a_{ij}^q)$ for some power q of p. (This implicitly assumes that $(a_{ij}^q) \in \mathbf{G}$ for all $(a_{ij}) \in \mathbf{G}$.)

Example 1.6 $\mathrm{SL}_n(\bar{\mathbb{F}}_p)$ and $\mathrm{SO}_{2m+1}(\bar{\mathbb{F}}_p)$ admit standard Frobenius maps F_q for all powers q of p.

A *Frobenius map* $F : \mathbf{G} \to \mathbf{G}$ is a homomorphism such that F^m is a standard Frobenius map for some $m \in \mathbb{N}$. If F is a Frobenius map, let $q \in \mathbb{R}$, $q \geq 0$ such that q^m is a power of p with $F^m = F_{q^m}$.

1.2.3 Finite reductive groups

Let $\mathbf{G}$ be a connected reductive algebraic group over $\bar{\mathbb{F}}_p$ and let F be a Frobenius map of $\mathbf{G}$. Then

$$\mathbf{G}^F := \{g \in \mathbf{G} \mid F(g) = g\}$$

is a finite group. The pair $(\mathbf{G}, F)$ or the finite group $G := \mathbf{G}^F$ is called *finite reductive group* or *finite group of Lie type*, though the latter terminology is also used in a broader sense.

Example 1.7 Let q be a power of p and let $F = F_q$ be the corresponding standard Frobenius map of $\mathrm{GL}_n(\bar{\mathbb{F}}_p)$, $(a_{ij}) \mapsto (a_{ij}^q)$. Then $\mathrm{GL}_n(\bar{\mathbb{F}}_p)^F = \mathrm{GL}_n(q)$, $\mathrm{SL}_n(\bar{\mathbb{F}}_p)^F = \mathrm{SL}_n(q)$, $\mathrm{SO}_{2m+1}(\bar{\mathbb{F}}_p)^F = \mathrm{SO}_{2m+1}(q)$.

All groups of Lie type, except the Suzuki and Ree groups can be obtained in this way by a **standard** Frobenius map. The projective special linear group $\mathrm{PSL}_n(q)$ is not a finite reductive group unless n and $q-1$ are coprime (in which case it is equal to $\mathrm{SL}_n(q)$).

For the remainder of this section, $(\mathbf{G}, F)$ denotes a finite reductive group over $\bar{\mathbb{F}}_p$.

1.2.4 The Lang-Steinberg theorem

One of the most important general results for finite reductive group is the following theorem due to Lang and Steinberg.

Theorem 1.8 (Lang-Steinberg, 1956 [60]/1968 [78]) *If* $\mathbf{G}$ *is connected, the map* $\mathbf{G} \to \mathbf{G}$, $g \mapsto g^{-1}F(g)$ *is surjective.*

The assumption that $\mathbf{G}$ is connected is crucial here.

Example 1.9 Let $\mathbf{G} = \mathrm{GL}_2(\bar{\mathbb{F}}_p)$, and $F : (q_{ij}) \mapsto (a_{ij}^q)$, where q is a power of p. Then there exists $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{G}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} a^q & b^q \\ c^q & d^q \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Rewriting this, we obtain the equation

$$\begin{bmatrix} a^q & b^q \\ c^q & d^q \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}.$$

Thus the Lang-Steinberg theorem assert in this case that there is a solution to the system of equations:

$$a^q = b, \quad b^q = a, \quad c^q = d, \quad d^q = c, \quad ad - bc \neq 0.$$

The Lang-Steinberg theorem is used to derive structural properties of $\mathbf{G}^F$.

1.2.5 Maximal tori and the Weyl group

A *torus* of $\mathbf{G}$ is a closed subgroup isomorphic to $\bar{\mathbb{F}}_p^* \times \cdots \times \bar{\mathbb{F}}_p^*$. A torus is *maximal*, if it is not contained in any larger torus of $\mathbf{G}$. It is a crucial fact that any two maximal tori of $\mathbf{G}$ are conjugate. This shows that the following notion is well defined.

Definition 1.10 The Weyl group W of $\mathbf{G}$ is defined by $W := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$, where $\mathbf{T}$ is a maximal torus of $\mathbf{G}$.

Example 1.11 (1) Let $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ and $\mathbf{T}$ the group of diagonal matrices. Then $\mathbf{T}$ is a maximal torus of $\mathbf{G}$, $N_{\mathbf{G}}(\mathbf{T})$ is the group of monomial matrices, and $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ can be identified with the group of permutation matrices, i.e. $W \cong S_n$.

(2) Next let $\mathbf{G} = \mathrm{SO}_{2m+1}(\bar{\mathbb{F}}_p)$ as defined in Example 1.2. Then

$$\mathbf{T} := \{\mathrm{diag}[t_1, \ldots, t_m, 1, t_m^{-1}, \ldots, t_1^{-1} \mid t_i \in \bar{\mathbb{F}}_p^*, 1 \leq i \leq m\}$$

is a maximal torus of $\mathbf{G}$.

For $1 \leq i \leq m-1$ let $\dot{s}_i$ be the permutation matrix corresponding to the double transposition $(i, i+1)(m-i, m-i+1)$. Put

$$\dot{s}_m := \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

where I denotes the identity matrix of degree $m-1$. Then $\dot{s}_1, \ldots, \dot{s}_m$ are elements of $N_{\mathbf{G}}(\mathbf{T})$, and the cosets $s_i := \dot{s}_i\mathbf{T} \in W$, $1 \leq i \leq m$, generate W, which is thus a Coxeter group of type B_m (see below).

1.2.6 Maximal tori of finite reductive groups

A *maximal torus* of $(\mathbf{G}, F)$ is a finite reductive group $(\mathbf{T}, F)$, where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$. A *maximal torus* of $G = \mathbf{G}^F$ is a subgroup T of the form $T = \mathbf{T}^F$ for some maximal torus $(\mathbf{T}, F)$ of $(\mathbf{G}, F)$.

Example 1.12 A *Singer cycle* in $\mathrm{GL}_n(q)$ is an irreducible cyclic subgroup of $\mathrm{GL}_n(q)$ of order $q^n - 1$. We will show below that a Singer cycle is a maximal torus of $\mathrm{GL}_n(q)$.

The maximal tori of $(\mathbf{G}, F)$ are classified (up to conjugation in G) by *F-conjugacy classes* of W. These are the orbits in W under the action $v.w := vwF(v)^{-1}$, $v, w \in W$.

1.2.7 The classification of maximal tori

Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$, $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$.

Let $w \in W$, and $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$ with $w = \dot{w}\mathbf{T}$. By the Lang-Steinberg theorem, there is $g \in \mathbf{G}$ such that $\dot{w} = g^{-1}F(g)$. One checks that ${}^g\mathbf{T}$ is F-stable, and so $({}^g\mathbf{T}, F)$ is a maximal torus of $(\mathbf{G}, F)$. (Indeed, $F({}^g\mathbf{T}) = F(g)F(\mathbf{T})F(g)^{-1} = g(\dot{w}\mathbf{T}\dot{w}^{-1})g^{-1} = {}^g\mathbf{T}$ since $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$.)

The map $w \mapsto ({}^g\mathbf{T}, F)$ induces a bijection between the set of F-conjugacy classes of W and the set of G-conjugacy classes of maximal tori of $(\mathbf{G}, F)$. For more details see [10, Section 3.3].

We say that ${}^g\mathbf{T}$ is obtained from $\mathbf{T}$ by *twisting with w*.

1.2.8 The maximal tori of $\mathrm{GL}_n(q)$

Let $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ and $F = F_q$ a standard Frobenius morphism, where q is a power of p.

Then F acts trivially on $W = S_n$, i.e. the maximal tori of $G = \mathrm{GL}_n(q)$ are parametrised by partitions of n. If $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition of n, we write T_λ for the corresponding maximal torus. We have

$$|T_\lambda| = (q^{\lambda_1} - 1)(q^{\lambda_2} - 1) \cdots (q^{\lambda_l} - 1).$$

Each factor $q^{\lambda_i} - 1$ of $|T_\lambda|$ corresponds to a cyclic direct factor of T_λ of this order. This follows form the considerations in the next subsection.

1.2.9 The structure of the maximal tori

Let $\mathbf{T}'$ be an F-stable maximal torus of $\mathbf{G}$, obtained by twisting the reference torus $\mathbf{T}$ with $w = \dot{w}\mathbf{T} \in W$. This means that there is $g \in \mathbf{G}$ with $g^{-1}F(g) = \dot{w}$ and $\mathbf{T}' = {}^g\mathbf{T}$. Then

$$T' = (\mathbf{T}')^F \cong \mathbf{T}^{wF} := \{t \in \mathbf{T} \mid t = \dot{w}F(t)\dot{w}^{-1}\}.$$

Indeed, for $t \in \mathbf{T}$ we have $gtg^{-1} = F(gtg^{-1})$ $[= F(g)F(t)F(g)^{-1}]$ if and only if $t \in \mathbf{T}^{wF}$.

Example 1.13 Let $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$, and $\mathbf{T}$ the group of diagonal matrices. Let $w = (1, 2, \ldots, n)$ be an n-cycle. Then

$$\mathbf{T}^{wF} = \{\mathrm{diag}[t, t^q, \ldots, t^{q^{n-1}}] \mid t \in \bar{\mathbb{F}}_p, t^{q^n - 1} = 1\},$$

and so $\mathbf{T}^{wF}$ is cyclic of order $q^n - 1$. It also follows that the maximal torus of G corresponding to w acts irreducibly on $\mathbb{F}_q^n$ and thus is a Singer cycle. On the other hand, a maximal torus of G corresponding to an element of W not conjugate to w acts reducibly on V since it lies in a proper Levi subgroup. Since every semisimple element of G, in particular a generator of a Singer cycle, lies in some maximal torus of G, it follows that a a Singer cycle is indeed a maximal torus.

1.3 *BN*-pairs

The following axiom system was introduced by Jacques Tits to allow a uniform treatment of groups of Lie type, not necessarily finite ones.

1.3.1 *BN*-pairs

We begin by defining what it means that a group has a BN-pair.

Definition 1.14 Let G be a group. The subgroups B and N of the group G form a BN-pair, if the following axioms are satisfied:

(1) $G = \langle B, N \rangle$;

(2) $T := B \cap N$ is normal in N;
(3) $W := N/T$ is generated by a set S of involutions;
(4) If $\dot{s} \in N$ maps to $s \in S$ (under $N \to W$), then $\dot{s}B\dot{s} \neq B$;
(5) For each $n \in N$ and $\dot{s}$ as above, $(B\dot{s}B)(BnB) \subseteq B\dot{s}nB \cup BnB$.

The group $W = N/T$ is called the *Weyl group* of the BN-pair of G. It is a Coxeter group with Coxeter generators S.

1.3.2 Coxeter groups

Let $M = (m_{ij})_{1 \leq i,j \leq r}$ be a symmetric matrix with $m_{ij} \in \mathbb{Z} \cup \{\infty\}$ satisfying $m_{ii} = 1$ and $m_{ij} > 1$ for $i \neq j$. The group

$$W := W(M) := \left\langle s_1, \ldots, s_r \mid (s_i s_j)^{m_{ij}} = 1 (i \neq j), s_i^2 = 1 \right\rangle_{\text{group}}$$

(where the relation $(s_i s_j)^{m_{ij}} = 1$ is omitted if $m_{ij} = \infty$), is called the *Coxeter group* of M, the elements $s_1, \ldots, s_r$ are the *Coxeter generators* of W.

The relations $(s_i s_j)^{m_{ij}} = 1$ $(i \neq j)$ are called the *braid relations*. In view of $s_i^2 = 1$, they can be written as

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots \quad m_{ij} \text{ factors on each side.}$$

The matrix M is usually encoded in a *Coxeter diagram*, a graph with nodes corresponding to $1, \ldots, r$, and with number of edges between nodes $i \neq j$ equal to $m_{ij} - 2$.

Example 1.15 The involutions s_i introduced in Example 1.11(2) satisfy the relations $s_i^2 = 1$ for $1 \leq i \leq m$, $(s_i s_{i+1})^3 = 1$ for $1 \leq i \leq m-1$ and $(s_{m-1}s_m)^4 = 1$. All other pairs of the s_i commute. The matrix encoding these relations is called a Coxeter matrix of type B_m. Its Coxeter diagram is as follows.

B_m: 

1.3.3 The BN-pair of $\mathrm{GL}_n(k)$ and of $\mathrm{SO}_n(k)$

Let k be a field and $G = \mathrm{GL}_n(k)$. Then G has a BN-pair with:

- B the group of upper triangular matrices;
- N the group of monomial matrices;
- $T = B \cap N$ the group of diagonal matrices;
- $W = N/T \cong S_n$ the group of permutation matrices.

Let $n = 2m + 1$ be odd and let $\mathrm{SO}_n(k) = \{g \in \mathrm{SL}_n(k) \mid g^{tr} J g = J\} \leq \mathrm{GL}_n(k)$ be the orthogonal group. If B, N are as above for $\mathrm{GL}_n(k)$, then

$$B \cap \mathrm{SO}_n(k), N \cap \mathrm{SO}_n(k)$$

is a BN-pair of $\mathrm{SO}_n(k)$. (This would not have been the case had we defined $\mathrm{SO}_n(k)$ with respect to an orthonormal basis as $\mathrm{SO}_n(k) = \{g \in \mathrm{SL}_n(k) \mid g^{tr} g = I\}$.) Using Examples 1.11 and 1.15 we see that the Weyl group of $\mathrm{SO}_n(k)$ is a Coxeter group of type B_m.

1.3.4 Split BN-pairs of characteristic p

Let G be a group with a BN-pair (B, N). This is said to be a *split BN-pair of characteristic* p, if the following additional hypotheses are satisfied:

(6) $B = UT$ with $U = O_p(B)$, the largest normal p-subgroup of B, and T a complement of U.

(7) $\bigcap_{n \in N} nBn^{-1} = T$. (Recall $T = B \cap N$.)

Example 1.16 (1) A semisimple algebraic group $\mathbf{G}$ over $\bar{\mathbb{F}}_p$ and a finite group of Lie type of characteristic p have split BN-pairs of characteristic p.

In $\mathbf{G}$ one chooses a maximal torus $\mathbf{T}$ and a maximal closed connected soluble subgroup $\mathbf{B}$ of $\mathbf{G}$ containing $\mathbf{T}$. Such a $\mathbf{B}$ is called a *Borel subgroup* of $\mathbf{G}$. Then $\mathbf{B}$ and $N_{\mathbf{G}}(\mathbf{T})$ form a split BN-pair of $\mathbf{G}$ of characteristic p.

(2) If $G = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ or $\mathrm{GL}_n(q)$, q a power of p, then U is the group of upper triangular unipotent matrices. In the latter case, U is a Sylow p-subgroup of G.

1.3.5 Parabolic subgroups and Levi subgroups

Let G be a group with a split BN-pair of characteristic p. Any conjugate of B is called a *Borel subgroup* of G. A *parabolic subgroup* of G is one containing a Borel subgroup.

Let $P \leq G$ be a parabolic subgroup. Then

$$P = U_P L = L U_P \tag{1}$$

such that $U_P = O_p(P)$ is the largest normal p-subgroup of P, and L is a complement to U_P in P. The decomposition (1) is called a *Levi decomposition* of P, and L is a *Levi complement* of P, and a *Levi subgroup* of G.

A Levi subgroup is itself a group with a split BN-pair of characteristic p.

1.3.6 Examples for parabolic subgroups

In classical groups, parabolic subgroups are the stabilisers of isotropic subspaces. Let $G = \mathrm{GL}_n(q)$, and $(\lambda_1, \ldots, \lambda_l)$ a partition of n. Then

$$P = \left\{ \begin{bmatrix} \mathrm{GL}_{\lambda_1}(q) & \star & \star \\ & \ddots & \star \\ & & \mathrm{GL}_{\lambda_l}(q) \end{bmatrix} \right\}$$

is a typical parabolic subgroup of G. A corresponding Levi subgroup is

$$L = \left\{ \begin{bmatrix} \mathrm{GL}_{\lambda_1}(q) & & \\ & \ddots & \\ & & \mathrm{GL}_{\lambda_l}(q) \end{bmatrix} \right\} \cong GL_{\lambda_1}(q) \times \cdots \times \mathrm{GL}_{\lambda_l}(q).$$

If B denotes, once again, the group of upper triangular matrices in G, then a Levi decomposition of B is given by $B = UT$ with T the diagonal matrices and U the upper triangular unipotent matrices.

1.3.7 The Bruhat decomposition

Let G be a group with a BN-pair. Then

$$G = \dot{\bigcup_{w \in W}} BwB \tag{2}$$

(we write $Bw := B\dot{w}$ if $\dot{w} \in N$ maps to $w \in W$ under $N \to W$). The disjoint union (2) of G into B, B-double cosets, is called the *Bruhat decomposition* of G. (The Bruhat decomposition for $\mathrm{GL}_n(k)$ follows from the Gaussian algorithm.)

Now suppose that the BN-pair is split, $B = UT = TU$. Let $w \in W$. Then $\dot{w}T = T\dot{w}$ since $T \lhd N$, and so $BwB = BwU$. Moreover, there is a subgroup $U_w \in U$ such that $BwU = BwU_w$, with "uniqueness of expression". This means that every element $g \in BwU_w$ can be written in a unique way as $g = b\dot{w}u$ with $b \in B$ and $u \in U_w$. If furthermore, G is finite, this implies

$$|G| = |B| \sum_{w \in W} |U_w|.$$

1.3.8 The orders of the finite groups of Lie type

Let G be a finite group of Lie type of characteristic p. Then G has a split BN-pair of characteristic p. Thus

$$|G| = |B| \sum_{w \in W} |U_w|.$$

Assume for simplicity that $G = \mathbf{G}^F$ for a standard Frobenius map $F = F_q$. Then $|U_w| = q^{\ell(w)}$, where $\ell(w)$ is the *length* of $w \in W$, i.e. the length of the shortest word in the Coxeter generators S of W expressing w.

By theorems of Solomon (1966, [72]) and Steinberg (1968, [78]),

$$\sum_{w \in W} q^{\ell(w)} = \prod_{i=1}^{r} \frac{q^{d_i} - 1}{q - 1},$$

where $d_1, \ldots, d_r$ are the degrees of the basic polynomial invariants of W. This gives the formulae for $|G|$ displayed in Example 1.3. An analogous, but slightly more complicated argument yields the order formulae for the twisted groups. For details see [9, Chapter 14].

2 Representations in defining characteristic

In this section we introduce the fundamental problems in the representation theory of finite groups of Lie type in the defining characteristic case. A comprehensive account of the knowledge in this area is given in Jantzen's monograph [57]. See also [50].

2.1 Classification of representations

2.1.1 A fundamental problem in representation theory

Let G be a finite group and k a field. It is a fundamental fact that there are only finitely many irreducible k-representations of G up to equivalence. This suggests the problem of classifying all irreducible representations of G over k. More ambitious is the following fundamental task:

> Classify all irreducible representations of all finite simple groups over all fields.

As already mentioned, "most" finite simple groups are groups of Lie type, and as a first step towards a classification of their irreducible representations one needs to find labels for these, their degrees, etc. It is useful to begin with the case of algebraically closed fields k. Instead of talking of representations we also use the equivalent language of kG-modules.

2.1.2 Three Cases

In the following, let $G = \mathbf{G}^F$ be a finite reductive group. Recall that $\mathbf{G}$ is a connected reductive algebraic group over $\bar{\mathbb{F}}_p$ and that F is a Frobenius morphism of $\mathbf{G}$. Let k be algebraically closed with $\mathrm{char}(k) = \ell \geq 0$. It is natural to distinguish three cases:

1. $\ell = p$ (usually $k = \bar{\mathbb{F}}_p$); *defining characteristic*
2. $\ell = 0$; *ordinary representations*
3. $\ell > 0$, $\ell \neq p$; *non-defining characteristic*

In this section we consider Case 1, and the remaining two sections are devoted to Cases 2 and 3.

2.2 Representations of (finite) reductive groups

2.2.1 A rough survey

Let us begin with a rough survey. Let $k = \bar{\mathbb{F}}_p$ and let $(\mathbf{G}, F)$ be a finite reductive group over k.

By a k-representation of $\mathbf{G}$ we understand an algebraic homomorphism, i.e. a homomorphism of groups that is also a morphism of algebraic varieties. We list some fundamental facts about the classification of irreducible k-representations of $\mathbf{G}$ and of $G = \mathbf{G}^F$.

1. An irreducible k-representation of $\mathbf{G}$ has finite degree.
2. The irreducible k-representations of $\mathbf{G}$ are classified by *dominant weights*, i.e. we have labels for these irreducible k-representations.
3. Under a natural condition on $\mathbf{G}$, every irreducible k-representation of $G = \mathbf{G}^F$ is the restriction of an irreducible k-representation of $\mathbf{G}$ to G.

We thus have to discuss the following questions. What are dominant weights? Which irreducible representations of $\mathbf{G}$ restrict to irreducible representations of G?

2.2.2 Character group and cocharacter group

For the remainder of this lecture, let $\mathbf{G}$ be a connected reductive algebraic group over $k = \bar{\mathbb{F}}_p$ and let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$. (All of these are conjugate.) Recall that $\mathbf{T} \cong k^* \times k^* \times \cdots \times k^*$. The number r of factors is an invariant of $\mathbf{G}$, the rank of $\mathbf{G}$.

Put $X := X(\mathbf{T}) := \mathrm{Hom}(\mathbf{T}, k^*)$. Again, Hom refers to algebraic homomorphisms of algebraic groups. Then X is an abelian group which we write additively. Thus $X \cong \bigoplus_1^r \mathrm{Hom}(k^*, k^*)$. Now $\mathrm{Hom}(k^*, k^*) \cong \mathbb{Z}$, so X is a free abelian group of rank r. (Indeed, every $\chi \in \mathrm{Hom}(k^*, k^*)$ is of the form $\chi(t) = t^z$ for some $z \in \mathbb{Z}$.) Similarly, $Y := Y(\mathbf{T}) := \mathrm{Hom}(k^*, \mathbf{T})$ is free abelian of rank r.

The groups X and Y are called the *character group* and *cocharacter group*, respectively. There is a natural duality $X \times Y \to \mathbb{Z}$, $(\chi, \gamma) \mapsto \langle \chi, \gamma \rangle$, defined by $\chi \circ \gamma \in \mathrm{Hom}(k^*, k^*) \cong \mathbb{Z}$.

2.2.3 The character groups and cocharacter groups of $\mathrm{GL}_n(k)$ and $\mathrm{SL}_n(k)$

Let $\mathbf{G} = \mathrm{GL}_n(k)$. Take

$$\mathbf{T} := \{\mathrm{diag}[t_1, t_2, \ldots, t_n] \mid t_1, \ldots, t_n \in k^*\},$$

the maximal torus of diagonal matrices. (Thus $\mathrm{GL}_n(k)$ has rank n.) The character group X has basis $\varepsilon_1, \ldots, \varepsilon_n$ with

$$\varepsilon_i(\mathrm{diag}[t_1, t_2, \ldots, t_n]) = t_i.$$

The cocharacter group Y has basis $\varepsilon_1', \ldots \varepsilon_n'$ with

$$\varepsilon_i'(t) = \mathrm{diag}[1, \ldots 1, t, 1, \cdots, 1],$$

where the t is on position i. Clearly, $\{\varepsilon_i\}$ and $\{\varepsilon_i'\}$ are dual with respect to the pairing $\langle -, - \rangle$.

Now let $\mathbf{G} = \mathrm{SL}_n(k)$ with $k = \bar{\mathbb{F}}_p$. Take

$$\mathbf{T} := \{\mathrm{diag}[t_1, t_2, \ldots, t_n] \mid t_1, \ldots, t_n \in k^*, t_1 t_2 \cdots t_n = 1\},$$

the maximal torus of diagonal matrices. (Thus $\mathrm{SL}_n(k)$ has rank $n-1$.) The character group X has basis $\varepsilon_1, \ldots, \varepsilon_{n-1}$ with

$$\varepsilon_i(\mathrm{diag}[t_1, t_2, \ldots, t_n]) = t_i.$$

Y has basis $\varepsilon_1', \ldots \varepsilon_{n-1}'$ with

$$\varepsilon_i''(t) = \mathrm{diag}[1, \ldots 1, t, 1, \cdots, 1, t^{-1}],$$

where the t is on position i. Clearly, $\{\varepsilon_i\}$ and $\{\varepsilon_i''\}$ are dual with respect to the pairing $\langle -, - \rangle$.

2.2.4 Roots and coroots

Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$. Then $\mathbf{B} = \mathbf{U}\mathbf{T}$ with $\mathbf{U} \lhd \mathbf{B}$ and $\mathbf{U} \cap \mathbf{T} = \{1\}$. (Recall that $\mathbf{G}$ has a split BN-pair of characteristic p.)

The minimal subgroups of $\mathbf{U}$ normalised by $\mathbf{T}$ are called *root subgroups*. A root subgroup is isomorphic to $\mathbf{G}_a := (k, +)$. The action of $\mathbf{T}$ on a root subgroup gives rise to a homomorphism $\mathbf{T} \to \mathrm{Aut}(\mathbf{G}_a)$. Since $\mathrm{Aut}(\mathbf{G}_a) \cong k^*$, we obtain an element of X. The characters obtained this way are the *positive roots* of $\mathbf{G}$ with respect to $\mathbf{T}$ and $\mathbf{B}$. The set of positive roots is denoted by Φ^+, and the set $\Phi := \Phi^+ \cup (-\Phi^+) \subset X$ is the *root system* of $\mathbf{G}$.

One can also define a set $\Phi^\vee \subset Y$ of *coroots* of $\mathbf{G}$ with respect to $\mathbf{T}$ and $\mathbf{B}$. Indeed, let $\alpha \in \Phi^+$ and let $\mathbf{U}_\alpha \leq \mathbf{U}$ be the corresponding root subgroup. Then there is a homomorphism $\varphi : \mathrm{SL}_2(k) \to \mathbf{G}$ with

$$\varphi\left\{\left[\begin{array}{cc} 1 & a \\ 0 & 1 \end{array}\right] \mid a \in k\right\} = \mathbf{U}_\alpha,$$

and

$$\varphi\left\{\left[\begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array}\right] \mid a \in k^*\right\} \leq \mathbf{T}.$$

Now define $\alpha^\vee \in Y = \mathrm{Hom}(k^*, \mathbf{T})$ by $\alpha^\vee(a) := \varphi\left\{\left[\begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array}\right]\right\}$ for $a \in k^*$.

2.2.5 The roots and the coroots of $\mathrm{GL}_n(k)$ and of $\mathrm{SL}_n(k)$ and the roots of $\mathrm{SO}_{2m+1}(k)$

Let $\mathbf{G} = \mathrm{GL}_n(k)$ and $\mathbf{T}$ the maximal torus of diagonal matrices. We choose $\mathbf{B}$ as group of upper triangular matrices. Then $\mathbf{U}$ is the subgroup of upper triangular unipotent matrices.

The root subgroups are the groups

$$\mathbf{U}_{ij} := \{I_n + aI_{ij} \mid a \in k\}, \quad 1 \leq i < j \leq n,$$

where I_{ij} denotes the elementary matrix with 1 on position (i, j) and 0 elsewhere. The positive root α_{ij} determined by $\mathbf{U}_{ij}$ equals $\varepsilon_i - \varepsilon_j$.

Indeed, if $\mathbf{t} = \mathrm{diag}[t_1, \ldots, t_n]$, then $\mathbf{t}(I_n + aI_{ij})\mathbf{t}^{-1} = I_n + t_i t_j^{-1} aI_{ij}$. On the other hand, $(\varepsilon_i - \varepsilon_j)(\mathbf{t}) = t_i t_j^{-1}$. We have

$$\Phi = \{\alpha_{ij} \mid \alpha_{ij} = \varepsilon_i - \varepsilon_j, 1 \leq i \neq j \leq n\}$$

and

$$\Phi^\vee = \{\alpha_{ij}^\vee \mid \alpha_{ij}^\vee = \varepsilon'_i - \varepsilon'_j, 1 \leq i \neq j \leq n\}.$$

Note that $\mathbb{Z}\Phi$ and $\mathbb{Z}\Phi^\vee$ have rank $n - 1$.

Now let $\mathbf{G} = \mathrm{SL}_n(k)$ and $\mathbf{T}$ be as in 2.2.3 and let $\mathbf{B}$ and $\mathbf{U}$ be as above. The root subgroups are the same as for $\mathrm{GL}_n(q)$. The positive root α_{ij} determined by $\mathbf{U}_{ij}$ equals $\varepsilon_i - \varepsilon_j$ if $j \neq n$, and $\alpha_{in} = \varepsilon_i + \sum_{j=1}^{n-1} \varepsilon_j$. We have $\alpha_{ij}^\vee = \varepsilon''_i - \varepsilon''_j$ for

$i < j < n$ and $\alpha_{in}^\vee = \varepsilon_i''$ for $i < n$. In this example $X/\mathbb{Z}\Phi$ is cyclic of order n, and $Y = \mathbb{Z}\Phi^\vee$.

Finally, assume that $n = 2m+1$ and p are odd, and let $\mathbf{G} = \mathrm{SO}_n(k)$. Let $\mathbf{T}$, $\mathbf{B}$ and $\mathbf{U}$ denote, respectively, the group of diagonal, upper triangular and upper triangular unipotent matrices contained in $\mathbf{G}$. Then $\mathbf{T}$ is as in Example 1.11. Clearly, a basis of $X = X(\mathbf{T})$ consists of $\varepsilon_1, \ldots, \varepsilon_m$ with $\varepsilon_i(\mathrm{diag}[t_1, \ldots, t_m, 1, t_m^{-1}, \ldots, t_1^{-1}]) = t_i$, $1 \leq i \leq m$. The root subgroups are the groups

$$\mathbf{U}_{ij} := \{I_n + aI_{ij} - aI_{2m-j+2,2m-i+2} \mid a \in k\}, \quad 1 \leq i < j \leq m,$$

together with

$$\mathbf{U}_{ij}' = \{I_n + aI_{i,2m-j+2} - aI_{j,2m-i+2} \mid a \in k\}, \quad 1 \leq i < j \leq m,$$

and

$$\mathbf{U}_i = \{I_n + aI_{i,m+1} - aI_{m+1,2m-i+2} - a^2/2I_{i,2m-i+2}\}, \quad 1 \leq i \leq m.$$

The positive roots α_{ij} and α_{ij}' determined by $\mathbf{U}_{ij}$ and $\mathbf{U}_{ij}'$, respectively, equal $\varepsilon_i - \varepsilon_j$ and $\varepsilon_i + \varepsilon_j$, respectively. Moreover, the positive roots determined by $\mathbf{U}_i$ equals ε_i, $1 \leq i \leq m$. In this case $X = \mathbb{Z}\Phi$.

2.2.6 The root datum

The quadruple $(X, \Phi, Y, \Phi^\vee)$ constructed from $\mathbf{G}$ satisfies:

1. X and Y are free abelian groups of the same rank and there is a duality $X \times Y \to \mathbb{Z}$, $(\chi, \gamma) \mapsto \langle \chi, \gamma \rangle$.
2. Φ and $\Phi^\vee$ are finite subsets of X and of Y, respectively, and there is a bijection $\Phi \to \Phi^\vee$, $\alpha \mapsto \alpha^\vee$.
3. For $\alpha \in \Phi$ we have $\langle \alpha, \alpha^\vee \rangle = 2$. Denote by s_α the "reflection" of X defined by
$$s_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha,$$
and let $s_\alpha^\vee$ be its adjoint ($s_\alpha^\vee(\gamma) = \gamma - \langle \alpha, \gamma \rangle \alpha^\vee$).
Then $s_\alpha(\Phi) = \Phi$ and $s_\alpha^\vee(\Phi^\vee) = \Phi^\vee$.

A quadruple $(X, \Phi, Y, \Phi^\vee)$ as above is called a *root datum*.

The algebraic group $\mathbf{G}$ is determined by its root datum (and p) up to isomorphism. More precisely, we have the following. Suppose that $\mathbf{G}$ and $\mathbf{G}_1$ are connected reductive groups over $\bar{\mathbb{F}}_p$ with Borel subgroups $\mathbf{B} = \mathbf{UT}$ and $\mathbf{B}_1 = \mathbf{U}_1\mathbf{T}_1$, respectively. Let, furthermore, $\Gamma_{\mathbf{G}} := (X, \Phi, Y, \Phi^\vee)$ and $\Gamma_{\mathbf{G}_1} := (X_1, \Phi_1, Y_1, \Phi_1^\vee)$ be the corresponding root data. There is an obvious notion of isomorphism of root data.

Theorem 2.1 (Isomorphism theorem) *Suppose that $f : \Gamma_{\mathbf{G}_1} \to \Gamma_{\mathbf{G}}$ is an isomorphism of root data. Then there exists an isomorphism $\varphi : \mathbf{G} \to \mathbf{G}_1$ of algebraic groups with $\varphi(\mathbf{T}) = \mathbf{T}_1$. Moreover, φ is uniquely determined up to conjugation in $\mathbf{T}$.*

An exact statement and a proof of the isomorphism theorem can be found in Springer's book [73, Thm. 9.6.2].

2.2.7 The Weyl group

The Weyl group $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ acts on X and we have

$$W \cong \langle s_\alpha \mid \alpha \in \Phi \rangle \leq \mathrm{Aut}(X).$$

Suppose that $\mathbf{G}$ is semisimple. Then $\mathrm{rank}\, X = \mathrm{rank}\, \mathbb{Z}\Phi$. In this case Φ is a root system in $V := X \otimes_{\mathbb{Z}} \mathbb{R}$ and W is its Weyl group (where V is equipped with an inner product $(-,-)$ satisfying $\langle \beta, \alpha^\vee \rangle = 2(\beta,\alpha)/(\alpha,\alpha)$ for all $\alpha, \beta \in \Phi$). Moreover, W is a Coxeter group with Coxeter generators $\{s_\alpha \mid \alpha \in \Pi\}$, where $\Pi \subset \Phi^+$ is a base of Φ. Note that Π is uniquely determined by this property. Indeed, Π consists of those elements of Φ^+ which cannot be written as sums of elements of Φ^+.

2.2.8 Weight spaces

Let M be a finite-dimensional algebraic $k\mathbf{G}$-module. This means that the k-representation of $\mathbf{G}$ afforded by M is algebraic. For $\lambda \in X = X(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, k^*)$ put

$$M_\lambda := \{v \in M \mid tv = \lambda(t)v \text{ for all } t \in \mathbf{T}\}.$$

If $M_\lambda \neq \{0\}$, then λ is called a *weight* of M and M_λ is the corresponding *weight space*. (Thus M_λ is a simultaneous eigenspace for all $t \in \mathbf{T}$.) It is a crucial fact, that M is a direct sum of its weight spaces, i.e.

$$M = \bigoplus_{\lambda \in X} M_\lambda.$$

This follows from the fact that the elements of $\mathbf{T}$ act as commuting semisimple linear operators on M.

2.2.9 Dominant weights and simple modules

The elements of the set

$$X^+ := \{\lambda \in X \mid 0 \leq \langle \lambda, \alpha^\vee \rangle \text{ for all } \alpha \in \Phi^+\} \subset X$$

are called the *dominant weights* of $\mathbf{T}$ (with respect to Φ^+).

Example 2.2 Let $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ and let $\mathbf{T}$ be the maximal torus of diagonal matrices. Use the notation for the roots and coroots of $\mathbf{G}$ from subsubsection 2.2.5.

An element $\lambda \in X$ is of the form $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i$ with $\lambda_i \in \mathbb{Z}$ for all $1 \leq i \leq n$. Now $\lambda \in X^+$ if and only if $\langle \lambda, \alpha_{ij}^\vee \rangle \geq 0$ for all $1 \leq i < j \leq n$. Since $\alpha_{ij}^\vee = \varepsilon_i' - \varepsilon_j'$, this is the case if and only if $\lambda_i - \lambda_j \geq 0$ for all $1 \leq i < j \leq n$. It follows that X^+ corresponds to the ordered sequences $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ of integers.

We order X by $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a sum of roots in Π. We then have the following classification of the simple $k\mathbf{G}$-modules.

Theorem 2.3 (Chevalley, late 1950s, cf. [12]) (1) *For each* $\lambda \in X^+$ *there is a simple* $k\mathbf{G}$*-module* $L(\lambda)$*.*

(2) $\dim L(\lambda)_\lambda = 1$*. If* μ *is a weight of* $L(\lambda)$*, then* $\mu \leq \lambda$*. (Consequently,* λ *is called the* highest weight *of* $L(\lambda)$*.)*

(3) *If* M *is a simple* $k\mathbf{G}$*-module, then* $M \cong L(\lambda)$ *for some* $\lambda \in X^+$*.*

The dimensions of the $\dim L(\lambda)$ are not known except for some special cases.

2.2.10 The natural and the adjoint representations of $\mathrm{GL}_n(k)$

Let $\mathbf{G} = \mathrm{GL}_n(k)$. Let $M := k^n$ be the natural module of $k\mathbf{G}$. Then the weights of M are the ε_i, $1 \leq i \leq n$. The highest of these is ε_1 (recall that $\varepsilon_i - \varepsilon_j \in \Phi^+$ for $i < j$). Thus $M = L(\varepsilon_1)$.

Next, let $M := \{x \in k^{n \times n} \mid \mathrm{tr}(x) = 0\}$. Then M is a simple $k\mathbf{G}$-module by conjugation (the *adjoint* module). The weights of M are the roots α_{ij} and 0. The highest one of these is $\alpha_{1n} = \varepsilon_1 - \varepsilon_n$. Thus $M = L(\varepsilon_1 - \varepsilon_n) = L(\alpha_{in})$.

2.2.11 Steinberg's tensor product theorem

For $q = p^m$, put

$$X_q^+ := \{\lambda \in X \mid 0 \leq \langle \lambda, \alpha^\vee \rangle < q \text{ for all } \alpha \in \Pi\} \subset X^+.$$

(Recall that $\Pi \subset \Phi^+$ is a base of Φ.) Let $F = F_p$ denote the standard Frobenius morphism $(a_{ij}) \mapsto (a_{ij}^p)$. If M is a $k\mathbf{G}$-module, we put $M^{[i]} := M$, with *twisted action* $g.v := F^i(g).v$, $g \in G$, $v \in M$.

Theorem 2.4 (Steinberg's tensor product theorem, [76]) *For* $\lambda \in X_q^+$ *write* $\lambda = \sum_{i=0}^{m-1} p^i \lambda_i$ *with* $\lambda_i \in X_p^+$*. (This is called the* p*-adic expansion of* λ*.) Then*

$$L(\lambda) = L(\lambda_0) \otimes_k L(\lambda_1)^{[1]} \otimes_k \cdots \otimes_k L(\lambda_{m-1})^{[m-1]}.$$

Thus it suffices to determine the dimensions of the simple modules L_λ for λ in the finite set X_p^+. The next theorem gives a classification of the simple modules for the finite groups of Lie type.

Theorem 2.5 (Steinberg, [76]) *If* $\lambda \in X_q^+$*, then the restriction of* $L(\lambda)$ *to* $G = \mathbf{G}^{F^m}$ *is simple. If* $\mathbf{G}$ *is simply connected, i.e.* $Y = \mathbb{Z}\Phi^\vee$*, then every simple* kG*-module arises this way.*

2.2.12 The irreducible representations of $\mathrm{SL}_2(k)$

Let $\mathbf{G} = \mathrm{SL}_2(k)$. Then G acts as group of k-algebra automorphisms on the polynomial ring $k[x_1, x_2]$ in two variables, the action being defined by:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}.$$

For $j = 0, 1, \ldots$ let M_j denote the set of homogeneous polynomials in $k[x_1, x_2]$ of degree j. Then M_j is $\mathbf{G}$-invariant, hence a $k\mathbf{G}$-module, and $\dim M_j = j + 1$. Moreover, M_j is a simple $k\mathbf{G}$-module, in fact $M_j = L(j\varepsilon_1)$, if $0 \leq j < p$. In general, write $j = j_0 + pj_1 + \cdots + p^m j_m$ with $0 \leq j_i < p$. Then, by Steinberg's tensor product theorem, $L(j\varepsilon_1) = M_{j_0} \otimes_k M_{j_1}^{[1]} \otimes_k \cdots \otimes_k M_{j_m}^{[m]}$.

Thus $\mathrm{SL}_2(p)$ has exactly the simple modules $M_0, \ldots, M_{p-1}$ of dimensions $1, \ldots, p$. This description of the p-modular irreducible representations of $\mathrm{SL}_2(q)$ (for powers q of p) is due to Brauer and Nesbitt [6].

2.2.13 Weyl modules

From now on assume that $\mathbf{G}$ is simply connected, i.e. $Y = \mathbb{Z}\Phi^\vee$. For each $\lambda \in X^+$, there is a distinguished finite-dimensional $k\mathbf{G}$-module $V(\lambda)$. These $V(\lambda)$ are called *Weyl modules*.

The Weyl modules are constructed via *reduction modulo p*. Let Φ be the root system of $\mathbf{G}$ and let $\mathfrak{g}$ be the semisimple Lie algebra over $\mathbb{C}$ with root system Φ. For $\lambda \in X^+$, let $V(\lambda)_\mathbb{C}$ be a simple $\mathfrak{g}$-module with highest weight λ. This has a suitable $\mathbb{Z}$-form $V(\lambda)_\mathbb{Z}$. Then $V(\lambda) := k \otimes_\mathbb{Z} V(\lambda)_\mathbb{Z}$ can be equipped with the structure of a $k\mathbf{G}$-module.

This construction generalises the construction of the Chevalley groups as groups of automorphisms on a $\mathbb{Z}$-form of their adjoint module.

2.2.14 Formal characters

Let M be a finite-dimensional $k\mathbf{G}$-module. Recall that

$$M = \bigoplus_{\lambda \in X} M_\lambda.$$

Clearly, $\dim M$ can be recovered by the vector $(\dim M_\lambda)_{\lambda \in X}$. It is convenient to view this as an element of $\mathbb{Z}X$, the group ring of X over $\mathbb{Z}$. We introduce a $\mathbb{Z}$-basis e^λ, $\lambda \in X$, of $\mathbb{Z}X$ with $e^\lambda e^\mu = e^{\lambda+\mu}$.

Definition 2.6 The *formal character* of M is the element

$$\operatorname{ch} M := \sum_{\lambda \in X} \dim M_\lambda \, e^\lambda$$

of $\mathbb{Z}X$.

2.2.15 Characters of Weyl modules

The characters of the Weyl modules $V(\lambda)$ can be computed from *Weyl's character formula*, which is not reproduced here. In particular, $\dim V(\lambda)$ is known.

Put $a_{\lambda,\mu} := [V(\lambda) \colon L(\mu)] :=$ multiplicity of $L(\mu)$ as a composition factor of $V(\lambda)$. It is known that $a_{\lambda,\lambda} = 1$, and if $a_{\lambda,\mu} \neq 0$, then $\mu \leq \lambda$. We obviously have

$$\operatorname{ch} V(\lambda) = \operatorname{ch} L(\lambda) + \sum_{\mu < \lambda} a_{\lambda,\mu} \operatorname{ch} L(\mu).$$

Once the $a_{\lambda,\mu}$ are known, $\operatorname{ch} L(\lambda)$ and thus $\dim L(\lambda)$ can be computed recursively from $\operatorname{ch} V(\mu)$ with $\mu \leq \lambda$ (there are only finitely many such μ).

2.3 Lusztig's conjecture

Lusztig's conjecture proposes a formula to compute the multiplicities $a_{\lambda,\mu}$ in certain cases. The conjecture is in terms of Kazhdan-Lusztig polynomials.

2.3.1 The Iwahori-Hecke algebra

Let $M = (m_{ij})_{1 \leq i,j \leq r}$ be a symmetric matrix with $m_{ij} \in \mathbb{Z} \cup \{\infty\}$ satisfying $m_{ii} = 1$ and $m_{ij} > 1$ for $i \neq j$. Recall that

$$W = \left\langle s_1, \ldots, s_r \mid (s_i s_j)^{m_{ij}} = 1 (i \neq j), s_i^2 = 1 \right\rangle_{\text{group}},$$

is the Coxeter group of M. Let A be a commutative ring and $v \in A$. The algebra

$$\mathcal{H}_{A,v}(W) := \left\langle T_{s_1}, \ldots, T_{s_r} \mid T_{s_i}^2 = v1 + (v-1)T_{s_i}, \text{ braid rel's } \right\rangle_{A\text{-alg.}}$$

is called the *Iwahori-Hecke algebra* of W over A with *parameter* v. The braid relations are

$$T_{s_i} T_{s_j} T_{s_i} \cdots = T_{s_j} T_{s_i} T_{s_j} \cdots \quad (m_{ij} \text{ factors on each side}).$$

It is a well known fact that $\mathcal{H}_{A,v}(W)$ is a free A-algebra with A-basis T_w, $w \in W$.

Note that $\mathcal{H}_{A,1}(W) \cong AW$, so that $\mathcal{H}_{A,v}(W)$ is a deformation of AW, the group algebra of W over A.

2.3.2 Kazhdan-Lusztig polynomials

Let W be a Coxeter group as above and let $\leq$ denote the Bruhat order on W. Let v be an indeterminate, put $A := \mathbb{Z}[v, v^{-1}]$ and $u := v^2$.

There is an involution ι on $\mathcal{H}_{A,u}(W)$ determined by $\iota(v) = v^{-1}$ and $\iota(T_w) = (T_{w^{-1}})^{-1}$ for all $w \in W$. (The square root v of u is needed in order for T_w to be invertible in $\mathcal{H}_{A,u}(W)$.)

Theorem 2.7 (Kazhdan-Lusztig, [58]) *There is a unique basis C'_w, $w \in W$ of $\mathcal{H}_{A,u}(W)$ such that*

(1) $\iota(C'_w) = C'_w$ for all $w \in W$;

(2) $C'_w = v^{-\ell(w)} \sum_{y \leq w} P_{y,w} T_w$ with $P_{w,w} = 1$, $P_{y,w} \in \mathbb{Z}[u]$, $\deg P_{y,w} \leq (\ell(w) - \ell(y) - 1)/2$ for all $y < w \in W$.

The $P_{y,w} \in \mathbb{Z}[u]$, $y \leq w \in W$, are called the *Kazhdan-Lusztig polynomials* of W.

2.3.3 The affine Weyl group

Recall that the Weyl group W acts on X as a group of linear transformations. Let $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$, and define the dot-action of W as follows:

$$w.\lambda := w(\lambda+\rho)-\rho, \lambda \in X, w \in W.$$

Define

$$W_p = \langle s_{\alpha,z} \mid \alpha \in \Phi^+, z \in \mathbb{Z}\rangle.$$

Here, $s_{\alpha,z}(\lambda) = s_\alpha.\lambda + zp\alpha$ is an affine reflection of X. Then W_p is a Coxeter group, called the *affine Weyl group.*

Each W_p-orbit on X contains a unique element in $\bar{C} := \{\lambda \in X \mid 0 \leq \langle\lambda + \rho, \alpha^\vee\rangle \leq p \text{ for all } \alpha \in \Phi^+\}$.

2.3.4 Lusztig's conjecture

Let $\lambda_0 \in X$ with $0 < \langle\lambda_0 + \rho, \alpha^\vee\rangle < p$ for all $\alpha \in \Phi^+$. Such a λ_0 only exists if $p \geq h := h(W) := \max\{\langle\rho, \alpha^\vee\rangle \mid \alpha \in \Phi^+\} + 1)$. The following theorem combines special cases of the linkage principle and the translation principle.

Theorem 2.8 (Humhreys, 1971, [48]; Jantzen, 1974, [56]) *For $w \in W_p$ such that $w.\lambda_0 \in X_p^+$ we have $\operatorname{ch} L(w.\lambda_0) = \sum_{w'} b_{w,w'} \operatorname{ch} V(w'.\lambda_0)$, with $w' \in W_p$ such that $w'.\lambda_0 \leq w.\lambda_0$ and $w'.\lambda_0 \in X^+$. The $b_{w,w'}$ are independent of λ_0.*

For $p \geq h$, the computation of $\operatorname{ch} L(\lambda)$ for any $\lambda \in X^+$ can be reduced to one of these cases. In the following formulation of Lusztig's conjecture, let w_0 denote the longest element in $W \leq W_p$.

Conjecture 2.9 (Lusztig's conjecture, 1980, [64]) The numbers $b_{w,w'}$ are given by $b_{w,w'} = (-1)^{\ell(w)+\ell(w')}P_{w_0w',w_0w}(1)$, in particular, the $b_{w,w'}$ are also independent of p.

Theorem 2.10 (Andersen-Jantzen-Soergel, [2]) *Lusztig's conjecture is true provided $p >> 0$.*

3 Representations in characteristic zero

Here we describe, to some extent, the ordinary representation theory of finite groups of Lie type. The material in this section can be found in the textbooks [10, 15] by Carter and Digne-Michel.

3.1 Harish-Chandra theory

In the following, unless otherwise said, let G be a finite reductive group of characteristic p. Also, k denotes an algebraically closed field of characteristic $\ell \geq 0$. In Section 2 we have considered the situation $\ell = p$. In this section we will mainly, but not exclusively, investigate the case $\ell = 0$.

Recall that there is a distinguished class of subgroups of G, the parabolic subgroups. One way to describe them is through the concept of split BN-pairs of characteristic p. A parabolic subgroup P has a Levi decomposition $P = LU$, where $U = O_p(P) \lhd P$ is the unipotent radical of P, and L a Levi complement of U in P, i.e. L is a Levi subgroup of G. Levi subgroups of G resemble G; in particular, they are again groups of Lie type. Inductively, we may use the representations of the Levi subgroups to obtain information about the representations of G. This is the idea behind *Harish-Chandra theory.*

3.1.1 Harish-Chandra induction

Assume from now on that $\ell \neq p$. Let L be a Levi subgroup of G, and M a kL-module. View M as a kP-module via $\pi : P \to L$ ($a.v := \pi(a).v$ for $v \in M$, $a \in P$). Put

$$R_L^G(M) := \{f : G \to M \mid a.f(b) = f(ab) \text{ for all } a \in P, b \in G\}.$$

This construction is analogous to the definition of modular forms in number theory.

Then $R_L^G(M)$ is a kG-module, called *Harish-Chandra induced* module. The action of G is given by right multiplication in the arguments of the functions in $R_L^G(M)$: $g.f(b) := f(bg)$, $g, b \in G$, $f \in R_L^G(M)$.

It is an important fact that $R_L^G(M)$ is independent of the choice of P with $P \to L$. In the case of $\ell > 0$, this result is due to Dipper-Du [18], and, independently, to Howlett-Lehrer [47].

3.1.2 Centraliser algebras

With L and M as before, put

$$\mathcal{H}(L, M) := \mathrm{End}_{kG}(R_L^G(M)).$$

Then $\mathcal{H}(L, M)$ is the *centraliser algebra* (or *Hecke algebra*) of the kG-module $R_L^G(M)$, i.e.

$$\mathcal{H}(L, M) = \left\{\gamma \in \mathrm{End}_k(R_L^G(M)) \mid \gamma(g.f) = g.\gamma(f) \text{ for all } g \in G, f \in R_L^G(M)\right\}.$$

The centraliser algebra $\mathcal{H}(L, M)$ is used to analyse the submodules and quotients of $R_L^G(M)$.

3.1.3 Iwahori's example

The following example is a special case of the results of Iwahori [52]. It marks the first appearance of the Iwahori-Hecke algebra in the representation theory of finite groups. Suppose that $\ell = 0$. Let $G = \mathrm{GL}_n(q)$, $L = T$, the group of diagonal matrices, M the trivial kL-module. Then

$$\mathcal{H}(L, M) = \mathcal{H}_{k,q}(S_n),$$

the Iwahori-Hecke algebra over k with parameter q associated to the Weyl group S_n of G. Recall the k-algebra presentation of $\mathcal{H}_{k,q}(S_n)$:

$$\langle T_1, \ldots, T_{n-1} \mid \text{ braid relations }, T_i^2 = q1_k + (q-1)T_i \rangle_{k\text{-algebra}},$$

with the braid relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad (1 \leq i \leq n-2).$$

3.1.4 Harish-Chandra classification

Let V be a simple kG-module. Then V is called *cuspidal*, if V is not a **submodule** of $R_L^G(M)$ for some **proper** Levi subgroup L of G and some kL-module M. Harish-Chandra theory, i.e. Harish-Chandra induction and the concept of cuspidality yields the following classification.

Theorem 3.1 (Harish-Chandra [40], Lusztig [63, 65], $\ell = 0$; Geck-Hiss-Malle [36], $\ell > 0$) *There is a bijection*

$$\{V \mid V \text{ simple } kG\text{-module}\}/\text{iso.}$$
$$\updownarrow$$
$$\left\{ (L, M, \theta) \mid \begin{array}{c} L \text{ Levi subgroup of } G \\ M \text{ simple, cuspidal } kL\text{-module} \\ \theta \text{ irreducible } k\text{-representation of } \mathcal{H}(L,M) \end{array} \right\} /\text{conj.}$$

This theorem allows to partition the isomorphism classes of the simple kG-modules into *Harish-Chandra series*. Two simple kG-modules lie in the same Harish-Chandra series, if and only if they arise from the same *cuspidal pair* (L, M), where L is a Levi subgroup and M a simple, cuspidal kL-module.

3.1.5 Problems in Harish-Chandra theory

The above theorem leads to the following three fundamental tasks:

(1) Determine the *cuspidal pairs* (L, M).
(2) For each of these, "compute" $\mathcal{H}(L, M)$.
(2) Classify the irreducible k-representations of $\mathcal{H}(L, M)$.

The state of the art in this program in case $\ell = 0$ is mainly due to Lusztig (see [65]):

(1) Lusztig has constructed and classified the simple cuspidal kG-modules. They arise from étale cohomology groups of Deligne-Lusztig varieties.

(2) For each cuspidal pair (L, M) consider the *ramification group*

$$W_G(L, M) := (N_G(L, M) \cap N)L/L$$

(the subgroup $N \leq G$ here is the one from the BN-pair of G). If $G = \mathbf{G}^F$ with $Z(\mathbf{G})$ connected, then it turns out that $W_G(L, M)$ is a Coxeter group. Moreover, the centraliser algebra $\mathcal{H}(L, M)$ is an Iwahori-Hecke algebra corresponding to $W_G(L, M)$.

(3) Furthermore, $\mathcal{H}(L, M) \cong kW_G(L, M)$. This is a consequence of the Tits deformation theorem.

3.1.6 Example: $\mathrm{SL}_2(q)$

Let $G = \mathrm{SL}_2(q)$ and $\ell = 0$. The group T of diagonal matrices is the only proper Levi subgroup; it is a cyclic group of order $q-1$. Put $W_G(T) := (N_G(T) \cap N)/T$ ($= N_G(\mathbf{T})/T$). Then $W_G(T) = \langle T, s\rangle/T$ with $s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and so $|W_G(T)| = 2$.

Let M be a simple kT-module. Then $\dim M = 1$ and M is cuspidal, and $\dim R_T^G(M) = q + 1$ (since $[G : B] = q + 1$).

Case 1: $W_G(T, M) = \{1\}$. Then $\mathcal{H}(T, M) \cong k$ and $R_T^G(M)$ is simple.

Case 2: $W_G(T, M) = W_G(T)$. Then $\mathcal{H}(T, M) \cong kW_G(T)$, and $R_T^G(M)$ is the sum of two simple kG-modules.

3.1.7 Drinfeld's example

The cuspidal simple $k\mathrm{SL}_2(q)$-modules have dimensions $q - 1$ and $(q - 1)/2$ (the latter only occur if p is odd). How can these cuspidal modules be constructed?

Consider the affine curve

$$C = \{(x, y) \in \bar{\mathbb{F}}_p^2 \mid xy^q - x^q y = 1\}.$$

Then $G = \mathrm{SL}_2(q)$ acts on C by linear change of coordinates. Hence G also acts on the étale cohomology group

$$H_c^1(C, \bar{\mathbb{Q}}_r),$$

where r is a prime different from p. It turns out that the simple $\bar{\mathbb{Q}}_r G$-submodules of $H_c^1(C, \bar{\mathbb{Q}}_r)$ are the cuspidal ones (here $k = \bar{\mathbb{Q}}_r$).

3.1.8 Goals and results

Suppose now and for the remainder of this section that $\ell = 0$. We write $\mathrm{Irr}(G)$ for the set of irreducible k-characters of G. Since two irreducible kG-representations are equivalent if and only if their characters are equal, we may reformulate our main goal of the classification of the irreducible representations as follows:

> Describe all ordinary character tables of all finite simple groups and related finite groups.

This aim is almost completed. For the alternating groups and their covering groups it was achieved by Frobenius and Schur. There exists labels for the irreducible characters and the conjugacy classes of these groups, and the character value corresponding to a pair of labels can be computed either explicitly or from a recursive formula.

The work for the groups of Lie type is due to many people: Steinberg, Green, Deligne, Lusztig, Shoji, and many others, where, however, Lusztig played a dominant role. To date, only "a few" character values are missing.

The character tables for the sporadic groups and other "small" groups are contained in the famous Atlas of Finite Groups by Conway, Curtis, Norton, Parker and Wilson [13].

3.2 Deligne-Lusztig theory

We begin this subsection by displaying an example of a generic character table.

3.2.1 The generic character table for $\mathrm{SL}_2(q)$, q even

	C_1	C_2	$C_3(a)$	$C_4(b)$
χ_1	1	1	1	1
χ_2	q	0	1	-1
$\chi_3(m)$	$q+1$	1	$\zeta^{am}+\zeta^{-am}$	0
$\chi_4(n)$	$q-1$	-1	0	$-\xi^{bn}-\xi^{-bn}$

Here, the parameters a, b label conjugacy classes, and m, n label irreducible characters. The range of these parameters is as follows: $a, m = 1, \ldots, (q-2)/2$, $b, n = 1, \ldots, q/2$. Moreover, the entries ζ and ξ are "generic roots of unity", namely

$$\zeta := \exp(\frac{2\pi\sqrt{-1}}{q-1}), \qquad \xi := \exp(\frac{2\pi\sqrt{-1}}{q+1}).$$

The conjugacy classes $C_3(a)$ and $C_4(b)$ have representatives as follows

$$\begin{bmatrix} \mu^a & 0 \\ 0 & \mu^{-a} \end{bmatrix} \in C_3(a) \qquad (\mu \in \mathbb{F}_q \text{ a primitive } (q-1)\text{th root of unity}),$$

$$\begin{bmatrix} \nu^b & 0 \\ 0 & \nu^{-b} \end{bmatrix} \overset{\in}{\sim} C_4(b) \qquad (\nu \in \mathbb{F}_{q^2} \text{ a primitive } (q+1)\text{th root of unity}).$$

(The symbol $\overset{\in}{\sim}$ indicates that an element in class $C_4(b)$ is conjugate in $\mathrm{GL}_2(\bar{\mathbb{F}}_2)$ to the element on the left hand side.) Specialising q to 4, gives the character table of $\mathrm{SL}_2(4) \cong A_5$.

3.2.2 Deligne-Lusztig varieties

Let r be a prime different from p and put $k := \bar{\mathbb{Q}}_r$. Let $(\mathbf{G}, F)$ be a finite reductive group, $G = \mathbf{G}^F$. Deligne and Lusztig [14] construct for each pair $(\mathbf{T}, \theta)$, where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$, and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$, a generalised character $R_{\mathbf{T},\theta}^{\mathbf{G}}$ of G. (A generalised character of G is an element of $\mathbb{Z}[\mathrm{Irr}(G)]$.)

Let $(\mathbf{T}, \theta)$ be a pair as above. Choose a Borel subgroup $\mathbf{B} = \mathbf{TU}$ of $\mathbf{G}$ with Levi subgroup $\mathbf{T}$. (In general $\mathbf{B}$ is not F-stable.) Consider the *Deligne-Lusztig variety* associated to $\mathbf{B}$,

$$X_{\mathbf{B}} = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \mathbf{U}\}.$$

This is an algebraic variety over $\bar{\mathbb{F}}_p$.

3.2.3 Deligne-Lusztig generalised characters

The finite groups $G = \mathbf{G}^F$ and $T = \mathbf{T}^F$ act on $X_{\mathbf{B}}$, and these actions commute. Thus the étale cohomology group $H_c^i(X_{\mathbf{B}}, \bar{\mathbb{Q}}_r)$ is a $\bar{\mathbb{Q}}_r[G \times T]$-module, and so its θ-isotypic component $H_c^i(X_{\mathbf{B}}, \bar{\mathbb{Q}}_r)_\theta$ is a $\bar{\mathbb{Q}}_r G$-module, whose character is denoted by ch $H_c^i(X_{\mathbf{B}}, \bar{\mathbb{Q}}_r)_\theta$.

Only finitely many of the vector spaces $H_c^i(X_{\mathbf{B}}, \bar{\mathbb{Q}}_r)$ are $\neq 0$. Now put

$$R_{\mathbf{T},\theta}^{\mathbf{G}} = \sum_i (-1)^i \text{ch } H_c^i(X_{\mathbf{B}}, \bar{\mathbb{Q}}_r)_\theta.$$

Then $R_{\mathbf{T},\theta}^{\mathbf{G}}$ is independent of the choice of $\mathbf{B}$ containing $\mathbf{T}$.

3.2.4 Properties of Deligne-Lusztig characters

The above construction and the following facts are due to Deligne and Lusztig, [14].

Facts 3.2 Let $(\mathbf{T}, \theta)$ be a pair as above. Then

(4) $R_{\mathbf{T},\theta}^{\mathbf{G}}(1) = \pm[G : T]_{p'}$.

(2) If $\mathbf{T}$ is contained in an F-stable Borel subgroup $\mathbf{B}$, then $R_{\mathbf{T},\theta}^{\mathbf{G}} = R_T^G(\theta)$ is the Harish-Chandra induced character.

(3) If θ is in general position, i.e. $N_G(\mathbf{T}, \theta)/T = \{1\}$, then $\pm R_{\mathbf{T},\theta}^{\mathbf{G}}$ is an irreducible character.

(4) For $\chi \in \text{Irr}(G)$, there is a pair $(\mathbf{T}, \theta)$ such that χ occurs in the (unique) expansion of $R_{\mathbf{T},\theta}^{\mathbf{G}}$ into $\text{Irr}(G)$. (Recall that $\text{Irr}(G)$ is a basis of $\mathbb{Z}[\text{Irr}(G)]$.)

3.2.5 Unipotent characters

Definition 3.3 (Deligne-Lusztig, [14]) A k-character χ of G is called unipotent, if χ is irreducible, and if χ occurs in $R_{\mathbf{T},\mathbf{1}}^{\mathbf{G}}$ for some F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$, where $\mathbf{1}$ denotes the trivial character of $T = \mathbf{T}^F$. We write $\text{Irr}^u(G)$ for the set of unipotent characters of G.

The above definition of unipotent characters uses étale cohomology groups. So far, no elementary description known, except for $\text{GL}_n(q)$; see below. Lusztig classified $\text{Irr}^u(G)$ in all cases, independently of q. Harish-Chandra induction preserves unipotent characters, so it suffices to construct the cuspidal unipotent characters.

3.2.6 The unipotent characters of $\text{GL}_n(q)$

Let $G = \text{GL}_n(q)$. Then $\text{Irr}^u(G) = \{\chi \in \text{Irr}(G) \mid \chi \text{ occurs in } R_T^G(1)\}$. This is the set of constituents in the permutation character with respect to the action on the cosets of the subgroup of upper triangular matrices. Moreover, there is a bijection

$$\mathcal{P}_n \leftrightarrow \text{Irr}^u(G), \lambda \leftrightarrow \chi_\lambda,$$

where $\mathcal{P}_n$ denotes the set of partitions of n.

The degrees of the unipotent characters are "polynomials in q":

$$\chi_\lambda(1) = q^{d(\lambda)} \frac{(q^n-1)(q^{n-1}-1)\cdots(q-1)}{\prod_{h(\lambda)}(q^h-1)},$$

with a certain $d(\lambda) \in \mathbb{N}$, and where $h(\lambda)$ runs through the hook lengths of λ.

3.2.7 The degrees of the unipotent characters of $\mathrm{GL}_5(q)$

λ	$\chi_\lambda(1)$
(5)	1
$(4,1)$	$q(q+1)(q^2+1)$
$(3,2)$	$q^2(q^4+q^3+q^2+q+1)$
$(3,1^2)$	$q^3(q^2+1)(q^2+q+1)$
$(2^2,1)$	$q^4(q^4+q^3+q^2+q+1)$
$(2,1^3)$	$q^6(q+1)(q^2+1)$
(1^5)	q^{10}

3.3 Lusztig's Jordan decomposition of characters

In this subsection we introduce Lusztig's classification of the set of irreducible characters of G, known as *Jordan decomposition of characters*.

3.3.1 Jordan decomposition of elements

An important concept in the classification of elements of a finite reductive group is the Jordan decomposition of elements.

Since $\mathbf{G} \leq \mathrm{GL}_n(\bar{\mathbb{F}}_p)$, every $g \in \mathbf{G}$ has finite order. Hence g has a unique decomposition as

$$g = su = us \qquad (3)$$

with u a p-element and s a p'-element. It follows from Linear algebra that u is *unipotent*, i.e. all eigenvalues of u are equal to 1, and s is *semisimple*, i.e. diagonalisable.

The decomposition (3) is called the *Jordan decomposition* of $g \in \mathbf{G}$. If $g \in G = \mathbf{G}^F$, then so are u and s.

3.3.2 Jordan decomposition of conjugacy classes

This yields a model classification for the classification of the irreducible characters of G in case $\ell = 0$ and, conjecturally, also in case $0 \neq \ell \neq p$.

For $g \in G$ with Jordan decomposition $g = us = su$, we write $C^G_{u,s}$ for the G-conjugacy class containing g. This gives a labelling

$$\begin{gathered}\{\text{conjugacy classes of } G\} \\ \updownarrow \\ \{C^G_{s,u} \mid s \text{ semisimple}, u \in C_G(s) \text{ unipotent}\}.\end{gathered}$$

(In the above, the labels s and u have to be taken modulo conjugacy in G and $C_G(s)$, respectively.) Moreover,

$$|C_{s,u}^G| = |G : C_G(s)||C_{1,u}^{C_G(s)}|.$$

This is the *Jordan decomposition of conjugacy classes.*

3.3.3 Example: The general linear group once more

$G = \mathrm{GL}_n(q)$, $s \in G$ semisimple. Then

$$C_G(s) \cong \mathrm{GL}_{n_1}(q^{d_1}) \times \mathrm{GL}_{n_2}(q^{d_2}) \times \cdots \times \mathrm{GL}_{n_m}(q^{d_m})$$

with $\sum_{i=1}^m n_i d_i = n$. (This gives finitely many class types.) Thus it suffices to classify the set of unipotent conjugacy classes $\mathcal{U}$ of G. By Linear algebra we have

$$\mathcal{U} \longleftrightarrow \mathcal{P}_n = \{\text{partitions of } n\}$$

$$C_{1,u}^G \longleftrightarrow (\text{sizes of Jordan blocks of } u)$$

This classification is generic, i.e., independent of q.

In general, i.e. for other groups, it depends slightly on q. For example, $\mathrm{SL}_2(q)$ has two unipotent conjugacy classes if q is even, and three, otherwise.

3.3.4 Jordan decomposition of characters

Let $(\mathbf{G}, F)$ be a connected reductive group. Let $(\mathbf{G}^*, F)$ denote the dual reductive group. If $\mathbf{G}$ is determined by the root datum $(X, \Phi, Y, \Phi^\vee)$, then $\mathbf{G}^*$ is defined by the root datum $(Y, \Phi^\vee, X, \Phi)$.

Example 3.4 (1) If $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$, then $\mathbf{G}^* = \mathbf{G}$.
(2) If $\mathbf{G} = \mathrm{SO}_{2m+1}(\bar{\mathbb{F}}_p)$, then $\mathbf{G}^* = \mathrm{Sp}_{2m}(\bar{\mathbb{F}}_p)$.

Main Theorem 3.5 (Lusztig, [65]; Jordan decomposition of characters) Suppose that $Z(\mathbf{G})$ is connected. Then there is a bijection

$$\mathrm{Irr}(G) \longleftrightarrow \{\chi_{s,\lambda} \mid s \in G^* \text{ semisimple }, \lambda \in \mathrm{Irr}^u(C_{G^*}(s))\}$$

(s taken modulo conjugacy in G^*). Moreover, $\chi_{s,\lambda}(1) = |G^* \colon C_{G^*}(s)|_{p'}\, \lambda(1)$.

3.3.5 The irreducible characters of $\mathrm{GL}_n(q)$

Let $G = \mathrm{GL}_n(q)$. Then

$$\mathrm{Irr}(G) = \{\chi_{s,\lambda} \mid s \in G \text{ semisimple}, \lambda \in \mathrm{Irr}^u(C_G(s))\}.$$

We have $C_G(s) \cong \mathrm{GL}_{n_1}(q^{d_1}) \times \mathrm{GL}_{n_2}(q^{d_2}) \times \cdots \times \mathrm{GL}_{n_m}(q^{d_m})$ with $\sum_{i=1}^m n_i d_i = n$. Thus $\lambda = \lambda_1 \boxtimes \lambda_2 \boxtimes \cdots \boxtimes \lambda_m$ with $\lambda_i \in \mathrm{Irr}^u(\mathrm{GL}_{n_i}(q^{d_i}))$. Moreover,

$$\chi_{s,\lambda}(1) = \frac{(q^n - 1)\cdots(q-1)}{\prod_{i=1}^m [(q^{d_i n_i} - 1)\cdots(q^{d_i} - 1)]} \prod_{i=1}^m \lambda_i(1).$$

The character table of $\mathrm{GL}_n(q)$ has first been determined by Green (1955, [37]) after preliminary work by Steinberg (1951, [74]).

3.3.6 The degrees of the irreducible characters of $\mathrm{GL}_3(q)$

As an example, we give the degrees of all irreducible characters of $\mathrm{GL}_3(q)$.

$C_G(s)$	λ	$\chi_{s,\lambda}(1)$
$\mathrm{GL}_1(q^3)$	(1)	$(q-1)^2(q+1)$
$\mathrm{GL}_1(q^2) \times \mathrm{GL}_1(q)$	$(1) \boxtimes (1)$	$(q-1)(q^2+q+1)$
$\mathrm{GL}_1(q)^3$	$(1) \boxtimes (1) \boxtimes (1)$	$(q+1)(q^2+q+1)$
$\mathrm{GL}_2(q) \times \mathrm{GL}_1(q)$	$(2) \boxtimes (1)$	q^2+q+1
	$(1,1) \boxtimes (1)$	$q(q^2+q+1)$
$\mathrm{GL}_3(q)$	(3)	1
	$(2,1)$	$q(q+1)$
	$(1,1,1)$	q^3

3.3.7 Concluding remarks

There are also results by Lusztig [66] in case $Z(\mathbf{G})$ is not connected, e.g. if $\mathbf{G} = \mathrm{SL}_n(\bar{\mathbb{F}}_p)$ or $\mathbf{G} = \mathrm{Sp}_{2m}(\bar{\mathbb{F}}_p)$ with p odd.

For such groups, $C_{\mathbf{G}^*}(s)$ is not always connected, and the problem then is to define unipotent characters for $C_{\mathbf{G}^*}(s)^F$.

The Jordan decomposition of conjugacy classes and characters allow for the construction of generic character tables in all cases.

Let $\{G(q) \mid q \text{ a prime power}\}$ be a series of finite groups of Lie type, e.g. $\{\mathrm{GU}_n(q)\}$ or $\{\mathrm{SL}_n(q)\}$ (n fixed). Then there exists a finite set $\mathcal{D}$ of polynomials in $\mathbb{Q}[x]$ such that the following holds: If $\chi \in \mathrm{Irr}(G(q))$, then there is $f \in \mathcal{D}$ with $\chi(1) = f(q)$.

4 Representations in non-defining characteristic

In this final section we report on the knowledge in the representation theory of groups of Lie type in the non-defining characteristic case. The reference [22] contains a more detailed survey. The current knowledge in this area is presented in the monograph by [8] by Cabanes and Enguehard.

Throughout this section let G be a finite group and let k be an algebraically closed field of characteristic $\ell \geq 0$. If G is a finite group of Lie type of characteristic p, we also assume that $\ell \neq p$.

4.1 Harish-Chandra theory

We begin with a recollection of Harish-Chandra theory.

4.1.1 Harish-Chandra Classification: Recollection

Let G be a finite group of Lie type of characteristic $p \neq \ell$. Recall that Harish-Chandra theory yields a classification of the simple kG-modules according to Theorem 3.1. This implies three tasks, on whose state of the art we now comment.

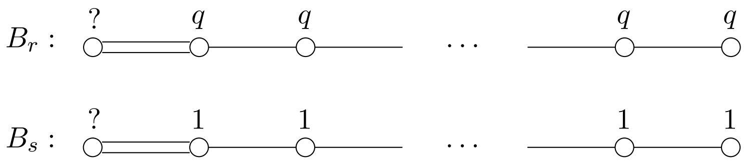
- $\mathcal{H}(L, M)$ is an Iwahori-Hecke algebra corresponding to an "extended" Coxeter group, namely $W_G(L, M)$ (Geck-Hiss-Malle [36], which follows the arguments of Howlett-Lehrer [46]); the parameters of $\mathcal{H}(L, M)$ are not known in general.
- If $G = \mathrm{GL}_n(q)$, everything is known with respect to these tasks (Dipper, [16, 17] and Dipper-James, [19, 20, 21])
- if G is classical group and ℓ is "linear" for G, everything known with respect to these tasks (Gruber-Hiss [39]). (We shall introduce linear primes and discuss these results below.)
- In general, the classification of the cuspidal pairs is open.

4.1.2 Example: $\mathrm{SO}_{2m+1}(q)$

This example is a special case of the results in [36]. Let $G = \mathrm{SO}_{2m+1}(q)$, assume that $\ell > m$, and put $e := \min\{i \mid \ell \text{ divides } q^i - 1\}$, the order of q in $\mathbb{F}_\ell^*$. Any Levi subgroup L of G containing a cuspidal unipotent (see below) module M is of the form

$$L = \mathrm{SO}_{2m'+1}(q) \times \mathrm{GL}_1(q)^r \times \mathrm{GL}_e(q)^s.$$

In this case $W_G(L, M) \cong W(B_r) \times W(B_s)$, where $W(B_j)$ denotes a Weyl group of type B_j. Moreover, $\mathcal{H}(L, M) \cong \mathcal{H}_{k,\mathbf{q}}(B_r) \otimes \mathcal{H}_{k,\mathbf{q}}(B_s)$, with $\mathbf{q}$ as follows:



The question marks indicate the unknown parameters.

4.2 Decomposition numbers

Decomposition numbers allow the passage from characteristic zero representations of a group to representation in positive characteristic. For groups of Lie type they also allow to define the concept of unipotent modules.

From now on assume that $\ell > 0$.

4.2.1 Brauer Characters

Let $\mathfrak{X}$ be a k-representation of G of degree d. The character $\chi_\mathfrak{X}$ of $\mathfrak{X}$ defined as usual by $g \mapsto \mathit{Trace}(\mathbf{X}(g))$ has some deficiencies, e.g. $\chi_\mathfrak{X}(1)$ only gives the degree d of $\mathfrak{X}$ modulo ℓ. Instead one considers the *Brauer character* $\varphi_\mathfrak{X}$ of $\mathfrak{X}$. This is obtained by consistently lifting the eigenvalues of the matrices $\mathfrak{X}(g)$ for $g \in G_{\ell'}$ to characteristic 0. (Here, $G_{\ell'}$ is the set of ℓ-regular elements of G.) Thus $\varphi_\mathfrak{X} : G_{\ell'} \to K$, where K is a suitable field with $\mathrm{char}(K) = 0$, and $\varphi_\mathfrak{X}(g) =$ sum of the eigenvalues of $\mathfrak{X}(g)$ (viewed as elements of K). In particular, $\varphi_\mathfrak{X}(1)$ equals the degree of $\mathfrak{X}$.

We write $\mathrm{IBr}_\ell(G)$ for the set of irreducible Brauer characters of G, $\mathrm{IBr}_\ell(G) = \{\varphi_1, \ldots, \varphi_n\}$. (If $\ell \nmid |G|$, then $\mathrm{IBr}_\ell(G) = \mathrm{Irr}(G)$.) Let $g_1, \ldots, g_n$ be representatives of the conjugacy classes contained in $G_{\ell'}$ (same n as above!). The square matrix

$$[\varphi_i(g_j)]_{1 \leq i,j \leq l}$$

is the *Brauer character table* or *ℓ-modular character table* of G.

4.2.2 Goals and Results

Once more, we reconsider our aim.

> Describe all Brauer character tables of all finite simple groups and related finite groups.

In contrast to the case of ordinary character tables (cf. Section 3), this is wide open:

(1) For alternating groups the knowledge is complete only up to A_{17}.

(2) For groups of Lie type only partial results are known, on which we shall comment below.

(3) For sporadic groups up to McL and other “small” groups (of order $\leq 10^9$), there is an Atlas of Brauer Characters, see [55]. More information is available on the web site of the Modular Atlas Project: (http://www.math.rwth-aachen.de/~MOC/).

4.2.3 The Decomposition Numbers

For $\chi \in \mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_m\}$, write $\hat{\chi}$ for the restriction of χ to $G_{\ell'}$. Then there are integers $d_{ij} \geq 0$, $1 \leq i \leq m$, $1 \leq j \leq n$ such that $\hat{\chi}_i = \sum_{j=1}^{l} d_{ij}\, \varphi_j$. These integers are called the *decomposition numbers* of G modulo ℓ. The matrix $D = [d_{ij}]$ is the decomposition matrix of G.

4.2.4 Properties of Brauer characters

Two irreducible k-representations are equivalent if and only if their Brauer characters are equal. $\mathrm{IBr}_\ell(G)$ is linearly independent (in $\mathrm{Maps}(G_{\ell'}, K)$) and so the decomposition numbers are uniquely determined. The elementary divisors of D are all 1, i.e. the decomposition map defined by $\mathbb{Z}[\mathrm{Irr}(G)] \to \mathbb{Z}[\mathrm{IBr}_\ell(G)], \chi \mapsto \hat{\chi}$ is surjective. Thus:

> Knowing $\mathrm{Irr}(G)$ and D is equivalent to knowing $\mathrm{Irr}(G)$ and $\mathrm{IBr}_\ell(G)$.

If G is ℓ-soluble, $\mathrm{Irr}(G)$ and $\mathrm{IBr}_\ell(G)$ can be sorted such that D has shape

$$D = \left[\frac{I_n}{D'} \right],$$

where I_n is the $(n \times n)$ identity matrix (Fong-Swan theorem).

4.3 Unipotent Brauer characters

The concept of decomposition numbers can be used to define unipotent Brauer characters of a finite reductive group.

4.3.1 Unipotent Brauer characters

Let $(\mathbf{G}, F)$ be a finite reductive group of characteristic p. Recall that $\mathrm{char}(k) = \ell \neq p$. Recall also that

$$\mathrm{Irr}^u(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi \text{ occurs in } R_{\mathbf{T},\mathbf{1}}^{\mathbf{G}} \text{ for some maximal torus } \mathbf{T} \text{ of } \mathbf{G}\}.$$

This yields a definition of $\mathrm{IBr}_\ell^u(G)$.

Definition 4.1 (Unipotent Brauer characters) $\mathrm{IBr}_\ell^u(G) = \{\varphi_j \in \mathrm{IBr}_\ell(G) \mid d_{ij} \neq 0 \text{ for some } \chi_i \in \mathrm{Irr}^u(G)\}$. The elements of $\mathrm{IBr}_\ell^u(G)$ are called the *unipotent Brauer characters* of G.

A simple kG-module is *unipotent*, if its Brauer character is.

4.3.2 Jordan decomposition of Brauer characters

The investigations are guided by the following main conjecture.

Conjecture 4.2 Suppose that $Z(\mathbf{G})$ is connected. Then there is a labelling

$$\mathrm{IBr}_\ell(G) \leftrightarrow \{\varphi_{s,\mu} \mid s \in G^* \text{ semisimple }, \ell \nmid |s|, \mu \in \mathrm{IBr}_\ell^u(C_{G^*}(s))\},$$

such that $\varphi_{s,\mu}(1) = |G^* : C_{G^*}(s)|_{p'}\, \mu(1)$.

Moreover, D can be computed from the decomposition numbers of **unipotent** characters of the various $C_{G^*}(s)$.

This conjecture is known to be true for $\mathrm{GL}_n(q)$ (Dipper-James, [19, 20, 21]) and in many other cases (Bonnafé-Rouquier, [5]). The truth of this conjecture would reduce the computation of decomposition numbers to unipotent characters. Consequently, we will restrict to this case in the following.

4.3.3 The unipotent decomposition matrix

Put $D^u :=$ restriction of D to $\mathrm{Irr}^u(G) \times \mathrm{IBr}_\ell^u(G)$.

Theorem 4.3 (Geck-Hiss, [33]; Geck, [32]) *Under some mild conditions on ℓ (for the exact form of these see [32]), $|\mathrm{Irr}^u(G)| = |\mathrm{IBr}_\ell^u(G)|$ and D^u is invertible over $\mathbb{Z}$.*

Thus under these conditions, the numbers of unipotent ordinary characters and of unipotent ℓ-modular characters are the same. This already indicates a close connection between the two representation theories.

The following conjecture is due to Geck, who has formulated it in a much more precise form, which is published in [34, Conjecture 3.4]

Conjecture 4.4 (Geck) Under some mild conditions on ℓ, the sets $\mathrm{Irr}^u(G)$ and $\mathrm{IBr}^u_\ell(G)$ can be ordered in such a way that D^u has shape

$$\begin{bmatrix} 1 & & & \\ \star & 1 & & \\ \vdots & \vdots & \ddots & \\ \star & \star & \star & 1 \end{bmatrix}.$$

This would give a canonical bijection $\mathrm{Irr}^u(G) \longleftrightarrow \mathrm{IBr}^u_\ell(G)$.

4.3.4 About Geck's Conjecture

Geck's conjecture on D^u is known to hold in the following cases:

- $\mathrm{GL}_n(q)$ (Dipper-James [19, 20])
- $\mathrm{GU}_n(q)$ (Geck [29])
- G classical and ℓ "large" (cyclic defect) (Fong-Srinivasan, [26, 28])
- G a classical group and ℓ "linear" (Gruber-Hiss [39])
- $\mathrm{Sp}_4(q)$ (White [82, 83, 84])
- $\mathrm{Sp}_6(q)$ (White [85]; An-Hiss [1])
- $G_2(q)$ (Hiss-Shamash [43, 44, 45])
- $F_4(q)$ (Köhler [59])
- $E_6(q)$ (Geck-Hiss [34]; Miyachi [67])
- Steinberg triality groups ${}^3D_4(q)$, q odd (Geck [30]; Himstedt [41])
- Suzuki groups (cyclic defect)
- Ree groups (Himstedt-Huang [42])

4.3.5 Linear primes

Let $(\mathbf{G}, F)$ be a finite reductive group, where $F = F_q$ is the standard Frobenius morphism $(a_{ij}) \mapsto (a_{ij}^q)$. Put $e := \min\{i \mid \ell \text{ divides } q^i - 1\}$, the order of q in $\mathbb{F}_\ell^*$. If G is classical ($\neq \mathrm{GL}_n(q)$) and e and ℓ are odd, then ℓ is *linear* for G. This notion is due to Fong and Srinivasan [25, 27].

Example 4.5 $G = \mathrm{SO}_{2m+1}(q)$, $|G| = q^{m^2}(q^2-1)(q^4-1)\cdots(q^{2m}-1)$. If $\ell \,|\, |G|$ and $\ell \nmid q$, then $\ell \mid q^{2d}-1$ for some minimal d. Thus $\ell \mid q^d - 1$ (ℓ linear and $e = d$ odd) or $\ell \mid q^d + 1$ ($e = 2d$).

Now $\mathrm{Irr}^u(G)$ is a union of Harish-Chandra series $\mathcal{E}_1, \ldots, \mathcal{E}_r$. This follows from the fact that Harish-Chandra induction preserves unipotent characters, i.e. the irreducible constituents of a Harish-Chandra induced unipotent character are unipotent.

Theorem 4.6 (Fong-Srinivasan, [25, 27]) *Suppose that $G \neq \mathrm{GL}_n(q)$ is classical and that ℓ is linear. Then $D^u = \mathrm{diag}[\Delta_1, \ldots, \Delta_r]$ with square matrices Δ_i corresponding to $\mathcal{E}_i$.*

In fact it follows from the results of Fong and Srinivasan that if ℓ is linear, unipotent characters of distinct Harish-Chandra series lie in distinct ℓ-blocks.

Let $\Delta := \Delta_i$ be one of the decomposition matrices from above. Assume however, that Δ does not correspond to the principal series of the orthogonal group $\mathrm{SO}_{2m}^{+}(q)$. Then the rows and columns of Δ are labelled by bipartitions of a for some integer a. This is a consequence of Harish-Chandra theory and the fact that the Iwahori-Hecke algebras of the Harish-Chandra induced cuspidal characters are of type B.

Theorem 4.7 (Gruber-Hiss, [39]) *Under the above assumptions,*

$$\Delta = \begin{bmatrix} \Lambda_0 \otimes \Lambda_a & & & & \\ & \ddots & & & \\ & & \Lambda_i \otimes \Lambda_{a-i} & & \\ & & & \ddots & \\ & & & & \Lambda_a \otimes \Lambda_0 \end{bmatrix}.$$

Here $\Lambda_i \otimes \Lambda_{a-i}$ is the Kronecker product of matrices, labelled by those bipartitions whose first component is a partition of i, and Λ_i is the ℓ-modular unipotent decomposition matrix of $\mathrm{GL}_i(q)$.

In the cases where the theorem applies, the decomposition matrices are described by decomposition matrices of general linear groups. This justifies the term "linear" for these primes.

4.4 (q-)Schur algebras

4.4.1 The v-Schur algebra

Let v be an indeterminate an put $A := \mathbb{Z}[v, v^{-1}]$. Dipper and James [21] have defined a remarkable A-algebra $\mathcal{S}_{A,v}(S_n)$, called the *generic v-Schur algebra*, satisfying:

(1) $\mathcal{S}_{A,v}(S_n)$ is free and of finite rank over A.

(2) $\mathcal{S}_{A,v}(S_n)$ is constructed from the generic Iwahori-Hecke algebra $\mathcal{H}_{A,v}(S_n)$, which is contained in $\mathcal{S}_{A,v}(S_n)$ as an embedded subalgebra (a subalgebra with a different unit).

(3) $\mathbb{Q}(v) \otimes_A \mathcal{S}_{A,v}(S_n)$ is a quotient of the quantum group $\mathcal{U}_u(\mathfrak{gl}_n)$ with $v = u^2$. (This is due to Beilinson, Lusztig and MacPherson [4]; see also [23].)

4.4.2 The q-Schur algebra

Let $G = \mathrm{GL}_n(q)$. Then $D^u = (d_{\lambda,\mu})$, with $\lambda, \mu \in \mathcal{P}_n$. Let $\mathcal{S}_{A,v}(S_n)$ be the v-Schur algebra, and let $\mathcal{S} := \mathcal{S}_{k,q}(S_n)$ be the finite-dimensional k-algebra obtained by specialising v to the image of $q \in k$. This is called the *q-Schur algebra*, and satisfies (cf. [21]):

(1) $\mathcal{S}$ has a set of (finite-dimensional) *standard modules* $\mathbf{S}^\lambda$, indexed by $\mathcal{P}_n$.

(2) The simple $\mathcal{S}$-modules $\mathbf{D}^\lambda$ are also labelled by $\mathcal{P}_n$.

(3) If $[\mathbf{S}^\lambda : \mathbf{D}^\mu]$ denotes the multiplicity of $\mathbf{D}^\mu$ as a composition factor in $\mathbf{S}^\lambda$, then $[\mathbf{S}^\lambda : \mathbf{D}^\mu] = d_{\lambda,\mu}$.

As a consequence, the $d_{\lambda,\mu}$ are bounded independently of q and of ℓ.

4.4.3 Connections to defining characteristics

Let $\mathcal{S}_{k,q}(S_n)$ be the q-Schur algebra introduced above. Suppose that $\ell \mid q-1$ so that $q \equiv 1 (\mathrm{mod}\ \ell)$. Then $\mathcal{S}_{k,q}(S_n) \cong \mathcal{S}_k(S_n)$, where $\mathcal{S}_k(S_n)$ is the *Schur algebra* defined by Schur and investigated by J. A. Green [38].

A partition λ of n may be viewed as a dominant weight of $\mathrm{GL}_n(k)$ (identifying $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathcal{P}_n$ with the dominant weight $\lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_m\varepsilon_m$; see Example 2.2). Thus there are corresponding $k\mathrm{GL}_n(k)$-modules $V(\lambda)$ and $L(\lambda)$.

If λ and μ are partitions of n, we have

$$[V(\lambda) : L(\mu)] = [\mathbf{S}^\lambda : \mathbf{D}^\mu] = d_{\lambda,\mu}.$$

The first equality comes from the significance of the Schur algebra, the second from that of the q-Schur algebra.

Thus the ℓ-modular decomposition numbers of $\mathrm{GL}_n(q)$ for prime powers q with $\ell \mid q-1$, determine the composition multiplicities of **certain** simple modules $L(\mu)$ in **certain** Weyl modules $V(\lambda)$ of $\mathrm{GL}_n(k)$, namely if λ and μ are partitions of n.

Facts 4.8 (Schur, Green) Let λ and μ be partitions with at most n parts. Then:

1. $[V(\lambda) : L(\mu)] = 0$, if λ and μ are partitions of different numbers (see [38, 6.6]).
2. If λ and μ are partitions of $r \geq n$, then the composition multiplicity $[V(\lambda) : L(\mu)]$ is the same in $\mathrm{GL}_n(k)$ and $\mathrm{GL}_r(k)$ (see [38, Remark in 6.6]).

The theory of Schur considers only polynomial representations, i.e. homomorphisms which are also morphisms of algebraic varieties. This is a subclass of all algebraic representations. The highest weights of polynomial representations are characterised by the fact that the coefficients λ_i (with respect to the basis $\varepsilon_1, \ldots, \varepsilon_n$; see 2.2.3) are all non-negative. Hence the ℓ-modular decomposition numbers of **all** $\mathrm{GL}_r(q)$, $r \geq 1$, $\ell \mid q-1$ determine the composition multiplicities of **all polynomial** Weyl modules of $\mathrm{GL}_n(k)$.

4.4.4 Connections to symmetric group representations

As for the Schur algebra, there are standard kS_n- modules S^λ, called *Specht modules*, labelled by the partitions λ of n. The simple kS_n-modules D^μ are labelled by the *ℓ-regular* partitions μ of n (no part of μ is repeated ℓ or more times). The ℓ-modular decomposition numbers of S_n are the numbers $[S^\lambda : D^\mu]$. We write λ' for the conjugate of a partition λ.

Theorem 4.9 (James, [53]) $[S^\lambda\!: D^\mu] = [V(\lambda')\!: L(\mu')]$, *if μ is ℓ-regular (notation on the right hand side from* $\mathrm{GL}_n(k)$ *case).*

Karin Erdmann has shown, that any $[V(\lambda): L(\mu)]$ occurs as a decomposition number of a symmetric group, even if μ is not ℓ-regular.

Theorem 4.10 (Erdmann, [24]) *For partitions λ, μ of n, there are ℓ-regular partition $t(\lambda')$, $t(\mu')$ of $\ell n + (\ell - 1)n(n-1)/2$ such that*

$$[V(\lambda): L(\mu)] = [S^{t(\lambda')}: D^{t(\mu')}].$$

4.4.5 Amazing conclusion

Recall that ℓ is a fixed prime and k an algebraically closed field of characteristic ℓ. Each of the following three families of numbers can be determined from any one of the others:

1. $\{[S^\lambda : D^\mu] \mid \lambda, \mu \in \mathcal{P}_n, n \in \mathbb{N}\}$, i.e. the ℓ-modular decomposition numbers of S_n for all n.
2. The ℓ-modular decomposition numbers of the unipotent characters of $\mathrm{GL}_n(q)$ for all primes powers q with $\ell \mid q-1$ and all n.
3. The composition multiplicities of the simple polynomial $k\mathrm{GL}_n(k)$-modules in the polynomial Weyl modules of $\mathrm{GL}_n(k)$ for all n.

Thus all these problems are really hard.

4.4.6 James' conjecture

Let $G = \mathrm{GL}_n(q)$. Recall that $e = \min\{i \mid \ell \text{ divides } q^i - 1\}$. James [54] has computed all matrices D^u for $n \leq 10$.

Conjecture 4.11 (James, [54]) If $e\ell > n$, then D^u only depends on e (neither on ℓ nor q).

Theorem 4.12 (1) *The conjecture is true for $n \leq 10$ (James, [54]).*
(2) *If $\ell >> 0$, D^u only depends on e (Geck, [31]).*

In fact, Geck proved $D^u = D_e D_\ell$ for two square matrices D_e and D_ℓ, and that $D_\ell = I$ for $\ell >> 0$. This result has later been extended by Geck and Rouquier [35].

Theorem 4.13 (Lascoux-Leclerc-Thibon [61, 62]; Ariki [3]; Varagnolo-Vasserot [81]) *The matrix D_e can be computed from the canonical basis of a certain highest weight module of the quantum group $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$.*

In order to compute all unipotent decomposition matrices D^u for $\mathrm{GL}_n(q)$, one needs to determine the matrices D_ℓ. Once James' Conjecture 4.11 is proved, it suffices to consider the primes $\ell < e/n$. Notice that $e = 1$ if $\ell \mid q - 1$. If, in addition, $\ell > n$, then D_e is the identity matrix, since in this case the Schur algebra $\mathcal{S}_k(S_n)$ is semisimple. Thus the result of Theorem 4.13 can not be used to compute any decomposition number of a symmetric group along the lines indicated in 4.4.5.

4.4.7 A unipotent decomposition matrix for $\mathrm{GL}_5(q)$

Let $G = \mathrm{GL}_5(q)$, $e = 2$ (i.e., $\ell \mid q+1$ but $\ell \nmid q-1$), and assume $\ell > 2$. Then D^u equals

(5)	1						
$(4,1)$		1					
$(3,2)$			1				
$(3,1^2)$	1		1	1			
$(2^2,1)$			1	1	1		
$(2,1^3)$		1				1	
(1^5)	1			1			1

The triangular shape defines φ_λ, $\lambda \in \mathcal{P}_5$.

4.4.8 On the degree polynomials

The degrees of the φ_λ are "polynomials in q".

λ	$\varphi_\lambda(1)$
(5)	1
$(4,1)$	$q(q+1)(q^2+1)$
$(3,2)$	$q^2(q^4+q^3+q^2+q+1)$
$(3,1^2)$	$(q^2+1)(q^5-1)$
$(2^2,1)$	$(q^3-1)(q^5-1)$
$(2,1^3)$	$q(q+1)(q^2+1)(q^5-1)$
(1^5)	$q^2(q^3-1)(q^5-1)$

Theorem 4.14 (Brundan-Dipper-Kleshchev, [7]) *The degrees of $\chi_\lambda(1)$ and of $\varphi_\lambda(1)$ as polynomials in q are the same.*

4.4.9 Genericity

Let $\{G(q) \mid q$ a prime power with $\ell \nmid q\}$ be a series of finite groups of Lie type, e.g. $\{\mathrm{GU}_n(q)\}$ or $\{\mathrm{SO}_{2m+1}(q)\}$ (n, respectively m fixed).

Question 4.15 Is an analogue of James' conjecture true for $\{G(q)\}$?

If **yes**, there are only finitely many matrices D^u to compute (there are only finitely many e's and finitely many "small" ℓ's). The following is a weaker form.

Conjecture 4.16 The entries of D^u are bounded independently of q and ℓ.

This conjecture is known to be true for $\mathrm{GL}_n(q)$ (Dipper-James [21]), G classical and ℓ linear (Gruber-Hiss, [39]), and for $\mathrm{GU}_3(q)$ and $\mathrm{Sp}_4(q)$ (Okuyama-Waki, [69, 68]).

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