A note on the Chevalley property of finite group algebras

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A Hopf algebra H is said to have the *Chevalley property*, if the tensor product of any two simple H-modules is semisimple. This notion was introduced by Andruskiewitsch, Etingof, and Gelaki in [1], and is motivated by a famous theorem of Chevalley which states that a group algebra kG does have this property if k is a field of characteristic 0.

Suppose that G is a finite group and that k is a field of characteristic $p > 0$. It was shown by R. K. Molnar in [7], that under these assumptions kG has the Chevalley property if and only if G has a normal Sylow p-subgroup.

In this note we prove that if the tensor product of every simple kG-module with its dual is semisimple, then the group algebra kG has the Chevalley property. The proof uses the classification of the finite simple groups in case p is odd. For $p = 2$, a result of Okuyama can be applied which does not require the classification.

We also give an example of a group algebra not having the Chevalley property, for which the tensor square of every simple module is semisimple.

The investigations were motivated by a question of Külshammer.

Proposition. *Let G be a finite group and let k be an algebraically closed field of characteristic p. Then the following holds:*

(1) *If $V \otimes_k V^*$ is semisimple for every simple kG-module V, then G has a normal Sylow p-subgroup (and thus kG has the Chevalley property).*

(2) *If $p = 2$ and $V \otimes_k V$ is semisimple for every simple kG-module V, then G has a normal Sylow 2-subgroup (and thus kG has the Chevalley property).*

Proof. Suppose that G does not have a normal Sylow p-subgroup. If $p = 2$, a result of Okuyama (see [8, Theorem 2.33]) shows that there is a non-trivial, self-dual simple kG-module V. By Fong's Lemma (see [5, Theorem VII.8.13]), V has even dimension.

Now let p be odd. By a result of Michler [6, Theorem 2.4], which uses the classification of the finite simple groups, there is a simple kG-module V with $p \mid \dim_k(V)$.

It is well known that $V \otimes_k V^*$ is not semisimple if V is absolutely simple and of dimension divisible by p (see, e.g., [2, Theorem 3.1.9]). This implies both parts of the theorem.

The following example was found with the help of GAP (see [4]). It shows that Part (2) of the above theorem does not hold for odd p.

Example. Let G be the non-abelian group of order 21, and let k be an algebraically closed field of characteristic 3. Then kG has three simple modules (up to isomorphism). Apart from the trivial kG-module, there is a pair S, S^* of dual simple kG-modules of dimension 3. The Brauer character φ of S has the following three values:

$$3, \quad \frac{-1+\sqrt{-7}}{2}, \quad \frac{-1-\sqrt{-7}}{2}.$$

The Brauer character of S^* equals $\bar{\varphi}$, the complex conjugate of φ. We have $\varphi \cdot \varphi = \varphi + 2\bar{\varphi}$. Since S and S^* are projective, this implies $S \otimes_k S \cong S \oplus S^* \oplus S^*$. Dually, $S^* \otimes_k S^* \cong S \oplus S \oplus S^*$. Hence $V \otimes_k V$ is semisimple for every simple kG-module V. However, G does not have a normal Sylow 3-subgroup.

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References

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