THE INDECOMPOSABLE LIFTABLE MODULES IN CYCLIC BLOCKS

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Abstract. Let G be a finite group and let k be an algebraically closed field of characteristic p. We classify the indecomposable liftable kG-modules in blocks with cyclic defect groups. The indecomposable, non-projective, non-simple modules in such a block are constructed from certain paths in the Brauer tree of the block (see [Jan69]). We determine those paths that give rise to liftable modules. We also find the characters of the lifts of these modules and obtain information about their Green correspondents.

1. Introduction

We consider a finite group G and an algebraically closed field k of characteristic p. The aim of this note is to classify the indecomposable liftable modules of kG which lie in a kG-block with a cyclic defect group. This will be called a cyclic block in the following.

The structure of a cyclic block is encoded in its Brauer tree. This is a finite tree, together with a planar embedding and, possibly, a multiplicity assigned to one of its vertices. In [Jan69], Janusz has given a construction of all non-projective, non-simple indecomposable modules in a cyclic block in terms of certain paths on its Brauer tree.

Here, we determine those paths that give rise to the indecomposable liftable modules, and describe the characters of their lifts. We also obtain information about the Green correspondents of the indecomposable liftable modules.

Indecomposable direct summands of permutation kG-modules are trivial source modules and hence are liftable. One of the motivations for our work was the attempt to understand the trivial source modules in cyclic blocks.

Let us now describe our results in more detail. For this purpose we introduce a p-modular system $(K, \mathcal{R}, k)$. A kG-module X is liftable, if there exists an $\mathcal{R}G$-lattice M such that $X \cong k \otimes_{\mathcal{R}} M$. Now let $\mathbf{B}$ denote a kG-block with a cyclic defect group. The embedded Brauer tree of $\mathbf{B}$ is denoted by σ. The embedding is determined by specifying, for each vertex of σ, a cyclic ordering of the edges incident to this vertex. Consider two edges E and F of σ incident to the vertex χ. We say that F is a successor of E aorund χ, if F comes next to E in the cyclic ordering of the edges around χ. We use

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the convention that in a drawing of σ in a plane, the successor of an edge is the counter-clockwise neighbour of this edge.
The vertices of the Brauer tree are labelled by irreducible K-characters of $\mathbf{B}$. (We label the exceptional vertex of σ, if one exists, by any one of the exceptional characters.) The edges of σ are labelled by the simple kG-modules in $\mathbf{B}$. By a leaf of σ we either mean a vertex of valency 1 or the edge incident to it. Thus the leaves of σ correspond to the simple liftable $\mathbf{B}$-modules.
There are two types of non-projective, non-simple indecomposable modules in $\mathbf{B}$, corresponding to the two types of paths displayed in Figures 4 and 5. The modules of Type I, constructed from paths of Type I, and of Type II, constructed from paths of Type II, are distinguished by the fact that the socles and the heads of the Type I modules do not have any common constituent.
The description of the indecomposable liftable $\mathbf{B}$-modules of Type I is rather restrictive.

Theorem 1.1. *Let τ be a path of Type I on the Brauer tree of $\mathbf{B}$. Then the indecomposable module X constructed from τ is liftable if and only if τ is as in Figure 1 and one of the following cases occurs:*

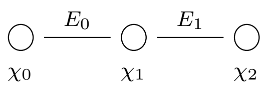


FIGURE 1. Path of the liftable Type I modules

(a) χ_1 is not the exceptional vertex. Then the character of a lift of X equals χ_1.
(b) χ_1 is the exceptional vertex, which has multiplicity m. Each of E_0 and E_1 occurs t times as composition factor of X for some $1 \leq t \leq m$. The character of any lift of X is a sum of t distinct exceptional characters.
In particular, X is uniserial in both cases, and the head of X is either isomorphic to E_0 or E_1. If the head of X is isomorphic to E_0, the successor of E_1 around χ_1 equals E_0, and if the head of X is isomorphic to E_1, the successor of E_0 around χ_1 equals E_1.

The description of the liftable Type II modules of $\mathbf{B}$ depends on which sorts of paths the Brauer tree admits. The possible structures of the liftable Type II modules are not as restricted as in the case of Type I.

Theorem 1.2. *Let τ be a path of Type II on the Brauer tree. Then the indecomposable module X constructed from τ is liftable if and only if τ is one of the paths given in Figure 2, and the following restrictions apply in the respective cases:*
Case 1: The vertex χ_0 is a leaf of the Brauer tree.

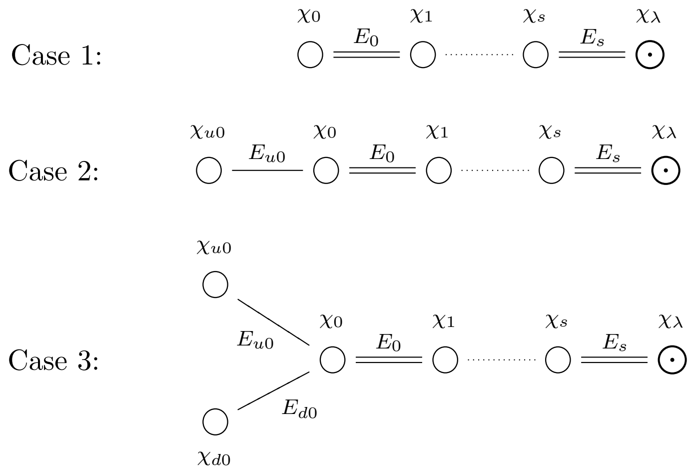


FIGURE 2. Paths of the liftable Type II modules

Case 2: Either E_{u0} is in the head or in the socle of X. If E_{u0} is in the head of X, the successor of E_0 around χ_0 is E_{u0}. If E_{u0} is in the socle of X, the successor of E_{u0} around χ_0 is E_0.
Case 3: The successor of E_{u0} around χ_0 is E_0 and E_{u0} is in the socle of X.
In Cases 1 and 3, there are $m-1$ indecomposable modules, and in Case 2 there are $2(m-1)$ indecomposable modules for the given τ. These are distinguished by a parameter t with $1 \leq t \leq m-1$, indicating that the multiplicity of the composition factor E_s of X equals $t+1$.
The character of any lift of X equals $\chi_0 + \chi_1 + \cdots \chi_s + \mu$, where μ is a sum of t distinct exceptional characters.

In Section 2 we set up our notation and present background material. In Section 3 we restrict the possible paths from which liftable modules can be constructed. The proof of Theorem 1.1 is completed in Subsection 3.1. In Section 4 we determine the number of indecomposable liftable modules of **B** using Green correspondence and show that this agrees with the number of possible candidates found in Section 3. This yields a proof of Theorem 1.2. Finally, in Section 5 we determine the Green correspondents of the indecomposable liftable modules in terms of the Heller operator.

2. BACKGROUND AND PRELIMINARIES

Let us begin fixing some notation and assumptions, which we refer to as Hypothesis 2.1 later on.

Hypothesis 2.1. Fix a finite group G. Let $(K, \mathcal{R}, k)$ be a splitting p-modular system for G such that k is an algebraically closed field of characteristic p. Let **B** be a kG-block with cyclic defect group D of order p^d. We denote its unique subgroup of order p by D_1 and the normaliser of D_1 in G

by N. Note that N contains all the normalisers $N_G(D_i)$ of the subgroups $D_i \leq D$ of order p^i for $1 \leq i \leq d$, so that we can consider Green correspondence between kG and kN. If V is an indecomposable non-projective kG-module with vertex contained in D, we denote its Green correspondent in N by $f(V)$. If V belongs to $\mathbf{B}$, then $f(V)$ belongs to the Brauer correspondent $\mathbf{b}$ of $\mathbf{B}$ (compare [Alp86, Corollary 14.4]). Likewise, if Y is an indecomposable kN-module with vertex in D, then $g(Y)$ denotes its Green correspondent in G.
The Brauer tree of $\mathbf{B}$ is denoted by σ. We consider σ as a tree with a planar embedding in the sense of [Alp86, Sec. V.17]. The Brauer tree of $\mathbf{b}$ is a star with the exceptional vertex (if one exists) at the centre. The number of simple kN-modules belonging to $\mathbf{b}$ is denoted by e. This is also the number of simple modules of $\mathbf{B}$. Put $m := (p^d - 1)/e$. If $m > 1$, it is called the exceptional multiplicity of $\mathbf{B}$.
Throughout the whole paper, we allow, for the sake of simplicity, the imprecision of using the same notation for the edges of σ and the simple $\mathbf{B}$-modules, as well as the same notation for the vertices of σ and the characters of irreducible KG-modules belonging to $\mathbf{B}$.
Finally, let X be an indecomposable non-projective $\mathbf{B}$-module.

Let us fix some further notation. Let U be a kG-module and V a kN-module. Then the Heller-translates of U and V are denoted by $\Omega(U)$ and $\Omega(V)$, respectively. In particular, the notation Ω for the Heller operator is used in either group ring, kG and kN. The head and the socle of U are denoted by $\mathrm{hd}(U)$ and $\mathrm{soc}(U)$, respectively.
If χ is a character of KG, then M_χ denotes an $\mathcal{R}G$-lattice in a KG-module with character χ. Moreover, if M is an $\mathcal{R}G$-lattice, let $\overline{M}$ be its reduction modulo p, i.e. $\overline{M} := k \otimes_{\mathcal{R}} M$, and put $KM := K \otimes_{\mathcal{R}} M$. Finally, if χ is a KG-character, we denote by $\overline{\chi}$ the Brauer character such that χ and $\overline{\chi}$ coincide on p'-conjugacy classes.
A kG-module U is liftable, if there exists an $\mathcal{R}G$-lattice M with $\overline{M} \cong U$. By a lift of U we understand any such lattice M or the character of KM. The following theorem of Zassenhaus is useful as it imposes certain restrictions on indecomposable liftable kG-modules.

Theorem 2.2 (Zassenhaus, [La83, La. I.17.3])**.** *Let M be an $\mathcal{R}G$-lattice and let χ be the character of KM. Let $\chi = \chi_1 + \chi_2$ for two KG-characters χ_1, χ_2. Then M contains $\mathcal{R}$-pure submodules N_1, N_2, such that χ_i is the character of KN_i for $i = 1, 2$. In particular, $\overline{M}$ contains submodules isomorphic to $\overline{N}_i$ with Brauer characters $\overline{\chi}_i$, $i = 1, 2$.*

Here is an application of the above theorem which will be used later on.

Lemma 2.3. *Suppose that X is liftable and let χ be the character of some lift of X. Consider the subgraph σ_X of σ consisting of all the vertices of σ which correspond to constituents of χ, and all the edges of σ connecting such vertices. Then σ_X is connected.*

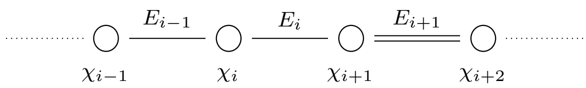
Proof. Suppose the contrary. Then we can write $\chi = \mu + \nu$ with non-zero characters μ, ν of G such that $\overline{\mu}$ and $\overline{\nu}$ have no common irreducible constituent. By Theorem 2.2, there are submodules Y and Z of X with Brauer characters $\overline{\mu}$ and $\overline{\nu}$, respectively. Now $Y \cap Z = 0$, since the two modules have no composition factor in common. Since $\chi = \mu + \nu$, the dimensions of Y and Z add up to the dimension of X. Hence $X = Y \oplus Z$, contradicting the indecomposability of X. $\square$

The following lemma allows us to transfer the analysis of the indecomposable liftable **B**-modules to the respective Green correspondent in **b** (and vice versa).

Lemma 2.4. *Let the assumptions be as in Hypothesis 2.1. If Y is an indecomposable liftable non-projective kG-module belonging to* **B***, then $f(Y)$ is an indecomposable liftable non-projective kN-module belonging to* **b***.*
If U is an indecomposable liftable non-projective **b***-module, than $g(U)$ is an indecomposable liftable non-projective kG-module belonging to* **B***.*

Proof. The assertion that $g(U)$ is an indecomposable liftable kG-module, if U is an indecomposable liftable **b**-module, has been proven by Peacock in [Pea77, Thm. 2.8] (with the help of [Tho67, Lemma 1]). Considering [Alp86, Thm. 17.3], the other assertion is proven similarly. $\square$

Let us briefly recall the construction of the indecomposable modules of **B** according to Janusz [Jan69] (see also [Fei82, Section VII.12]). Recall that X is an indecomposable non-projective **B**-module. Assume in addition that X is not simple.
Janusz showed that X can be constructed from a certain path τ on σ, together with a selection of every second edge in τ and, in case m is larger than 1, a multiplicity associated to one of the edges. Two types of paths arise, a Type I path as in Figure 4, and a Type II path as in Figure 5. The Type I path is supposed to start at the edge E_0. Type II paths only arise for $m > 1$. In this case the vertex of the Type II path labelled by χ_λ is the exceptional vertex of the Brauer tree σ of **B**. The path of Figure 5 is understood to start in χ_{u0}, cross χ_0, continue to χ_λ, turn direction, cross $\chi_s, \ldots, \chi_0$ and end in χ_{d0}. An edge is displayed with a double line, as shown in the following figure, if this edge is crossed twice in the path.



The label χ_i on a vertex of the path always denotes an ordinary irreducible character of **B** associated to this vertex. If this is the exceptional vertex, χ_i denotes any one of the exceptional characters of **G**.
Let Δ denote the set of selected edges of τ (or their indices). If $i \in \Delta$, we also say that the edge E_i is marked. The head and the socle constituents of X correspond to the marked edges, and to the unmarked edges respectively. For each $i \in \Delta$ there is a submodule X_i of X with unique head E_i, whose

$$X_i := \begin{tikzcd}[column sep=small] & E_i \arrow[dl, dash, dotted] \arrow[dr, dash, dotted] & \\ E_{i-1} & & E_{i+1} \end{tikzcd}$$

FIGURE 3. Submodule X_i of X

socle consists of the edges of τ adjacent to E_i. In case τ contains the edges E_{i-1} and E_{i+1}, the module X_i may be visualised as in Figure 3. The composition factors of the left leg of X_i, from bottom to top, correspond to the edges of σ adjacent to χ_i encountered on a clock-wise walk around χ_i from E_{i-1} to E_i. The composition factors of the right leg of X_i, from top to bottom, correspond to the edges of σ adjacent to χ_{i+1} encountered on a counter-clockwise walk around χ_{i+1} from E_i to E_{i+1}. A walk around the exceptional vertex of σ may include several full circles. In case E_i is the first or the last edge in τ, the module X_i is uniserial.
We have $X = \sum_{i\in\Delta} X_i$. In fact X is obtained abstractly by amalgamating modules isomorphic to X_i along their common socle constituents. In [Fei82, VII.12], the module X_i is called $V_i^0(n_i)$ which is in fact isomorphic to the one called $V_i(n_i)$. We may visualise X as follows:

$$\begin{tikzcd}[column sep=small] \cdots & E_{i-2} \arrow[dr, dash, dotted] & & E_i \arrow[dl, dash, dotted] \arrow[dr, dash, dotted] & & E_{i+2} \arrow[dl, dash, dotted] \arrow[dr, dash, dotted] & & \\ & & E_{i-1} & & E_{i+1} & & E_{i+3} & \cdots \end{tikzcd}$$

Here, $\{\ldots, E_{i-2}, E_i, E_{i+2}, \ldots\}$ are the head constituents of X, i.e. $\Delta = \{\ldots, i-2, i, i+2, \ldots\}$, whereas $\{\ldots, E_{i-1}, E_{i+1}, E_{i+3}, \ldots\}$ are the socle constituents of X.
The module X is called of Type I or of Type II, if it is constructed from a path of Type I or of Type II, respectively. By construction, X is of Type I if and only if the head and the socle of X have no common irreducible constituent (up to isomorphism).

Remark 2.5. (a) The liftable simple $\mathbf{B}$-modules are exactly those corresponding to leaves of σ.
(b) Let X be associated to the path τ with first vertex χ_0 and first edge E_0. Then no simple $\mathbf{B}$-module corresponding to an edge of σ adjacent to χ_0 and different from E_0 is a composition factor of X. This follows directly from the construction of X from τ.

Lemma 2.6. *Let X be associated to the path τ of Type I as in Figure 4 with $n \geq 2$. Suppose that χ_1 does not label the exceptional vertex of σ. Suppose further that $1 \in \Delta$, i.e. that E_1 is in the head of X. Then X has no submodule S with Brauer character $\overline{\chi}_1$ and with* $\mathrm{soc}(S) = E_0$.

Proof. Assume that S is such a submodule. Note that S is indecomposable, as its socle is simple. Put

$$W := \sum_{i \in \Delta \setminus \{1\}} X_i,$$

with X_i as in the discussion above, so that $X = X_1 + W$. Then $E_0 \not\leq \operatorname{soc}(W)$, hence $S \cap W = 0$. Thus we have an injective homomorphism

$$S \hookrightarrow X \to (X_1 + W)/W \cong X_1/(X_1 \cap W).$$

There are two cases to be considered:
(i) $S \cong X_1/(X_1 \cap W)$. Then $\dim X = \dim(S + W)$, so that $X = S + W$ is a direct sum, as $S \cap W = 0$. Since $S \neq X_1$, we have $W \neq 0$, contradicting the indecomposability of X.
(ii) S is a proper submodule of $X_1/(X_1 \cap W)$. Then $S \leq \operatorname{rad}(X_1/(X_1 \cap W)) = \operatorname{rad}(X_1)/(X_1 \cap W)$. Now $\operatorname{rad}(X_1)$ is a direct sum of two uniserial modules U_1 and U_2, with socles E_0 and E_2, respectively. It follows that $X_1 \cap W$ is isomorphic to a submodule of U_2, and in turn that S is isomorphic to a submodule of U_1. However, U_1 does not have composition factors isomorphic to E_1, a contradiction. □

Lemma 2.7. *Suppose that $m > 1$. If X is liftable, then the multiplicity of an exceptional character in any lift of X is at most one.*

Proof. Let Q be a projective cover of X. By construction, $\operatorname{hd}(X)$ contains at most one simple module corresponding to an edge of σ connected to the exceptional vertex. Thus Q contains at most one indecomposable summand which is the projective cover of such a simple module. Thus the lift of Q either contains no exceptional character at all, or else the sum of all exceptional characters. By [Gr74, (3.6b)], any lift of X is an epimorphic image of the lift of Q. This proves the assertion. □

3. NECESSARY CONDITIONS FOR LIFTABILITY

Here we derive some necessary conditions on indecomposable liftable modules of **B**. Throughout this section we impose the assumptions and notations of Hypothesis 2.1.

3.1. **Modules of Type I.**

Lemma 3.1. *Suppose that X is of Type I constructed from the path τ as in Figure 4. If X is liftable, the ordinary characters χ_0 and χ_{n+1} (i.e. the left- and rightmost vertices of τ) are not constituents of any lift of X (even if they are leaves of σ).*

Proof. Assume that χ_0 is a constituent of a lift of X. By Remark 2.5(b), χ_0 is a leaf of σ. In this case, E_0 is the only constituent of the reduction modulo p of M_{χ_0} and should therefore occur in the head **and** in the socle of X by Theorem 2.2. But this contradicts the assumption that X is of Type I. The proof for χ_{n+1} is similar. □

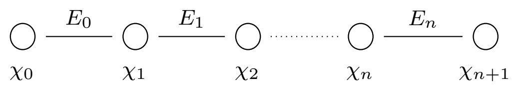


FIGURE 4. Type I path

The following lemma leads to the proof of Theorem 1.1.

Lemma 3.2. *Let X be a liftable module of Type I associated to the path τ as in Figure 4. Then the following assertions hold.*
(a) Only $n = 1$ is possible, so that X is uniserial with one of the following structures:

$$(i)\ \begin{array}{c} E_1 \\ \vdots \\ E_0 \end{array} \qquad (ii)\ \begin{array}{c} E_0 \\ \vdots \\ E_1 \end{array} .$$

(b) If X is as in (b)(i), the successor of E_0 around the vertex χ_1 is E_1. If X is as in (b)(ii), the successor of E_1 around the vertex χ_1 is E_0.
(c) If χ_1 does not label the exceptional vertex of σ, then any lift of X has character χ_1.
(d) Let χ_1 label the exceptional vertex of σ and let t be the multiplicity of E_0 as a composition factor of X. Then any lift of X is the sum of t distinct exceptional characters.

Proof. (a) Assume that $n > 1$. By relabelling, if necessary, we may assume that χ_1 does not label the exceptional vertex of σ. Suppose first that E_0 is in the socle of X. By Lemma 3.1, any lift of X must contain χ_1 as a constituent. By Theorem 2.2, there is a submodule S of X with Brauer character $\overline{\chi}_1$. Since E_0 is the only constituent of S in $\mathrm{soc}(X)$, we must have $\mathrm{soc}(S) = E_0$. Since $n > 2$, this contradicts Lemma 2.6. If E_0 is in the head of X, we consider X^*, the contragredient dual of X, a module of the dual block $\mathbf{B}^*$. The path of the Brauer tree of $\mathbf{B}^*$ giving rise to X^* is the same as the path for X, with the complementary choice of Δ, i.e. E_0^* is in the socle of X^*. As above we obtain $n = 1$.
(b) This follows from the liftability and the construction of X.
(c) This follows from (a).
(d) From (a) we get that any lift of X is a sum of t exceptional characters. By Lemma 2.7, these are all distinct. □

The proof of Theorem 1.1 is now an immediate consequence of Lemma 3.2. Note that every irreducible KG-module belonging to a cyclic block and not being associated to a leaf of the Brauer tree, gives rise to a module of Type I as described in Theorem 1.1(a). Note also that these modules occur in Green's walk around the Brauer tree (see [Gr74]).

3.2. **Modules of Type II.** Indecomposable modules of Type II belonging to $\mathbf{B}$ are constructed from a path τ of Type II as shown in Figure 5. It is understood that if $i < 0$ or $j < 0$, the corresponding branches of τ are not present.

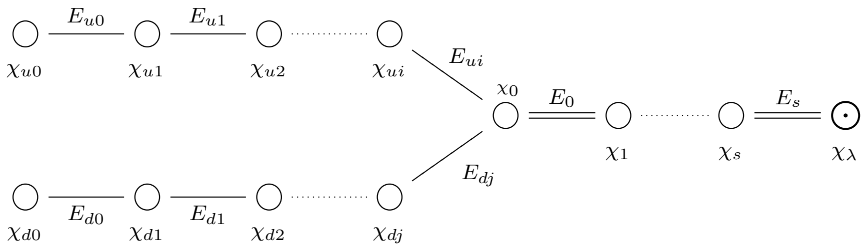


FIGURE 5. Type II path

Lemma 3.3. *Assume that X is liftable of Type II, associated to a path τ as in Figure 5. If $i \geq 0$ or $j \geq 0$, then χ_{u0}, respectively χ_{d0} are not constituents of any lift of X.*

Proof. This follows immediately from Theorem 2.2, Remark 2.5(b), and the fact that E_{u0}, respectively E_{d0} only occur in the socle or the head of X, but not in both. □

Lemma 3.4. *Assume that X is liftable of Type II, associated to a path τ as in Figure 5. Then neither i nor j in Figure 5 is greater than zero.*

Proof. Suppose that $i \geq 1$. Then each of E_{u0} and E_{u1} occurs with multiplicity 1 as a composition factor of X. Let χ be the character of a lift of X. By Lemma 3.3, χ_{u0} is not contained in χ. Hence χ_{u1} occurs with multiplicity 1 in χ. In turn χ_{u2} (if $i \geq 2$) or χ_0 (if $i = 1$), does not occur in χ. This contradicts Lemma 2.3.
The same argument excludes the case $j \geq 1$. □

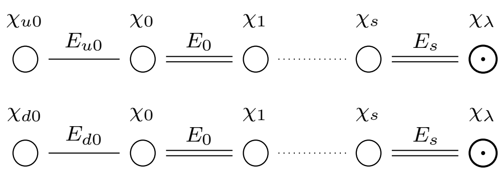


FIGURE 6. Special paths of Type II

Note that the sets of indecomposable modules (up to isomorphism) obtained from the two paths in Figure 6 are the same if E_{u0} is isomorphic to E_{d0} (see [Fei82, La. VII.12.5]). It therefore remains to discuss indecomposable liftable non-projective modules of Type II, which are associated to one of the three types of paths shown in Figure 2.

Lemma 3.5. *Let τ be one of the paths in Figure 2. Suppose that the module X constructed from τ is liftable. Assume that E_s occurs $t+1$ times as a composition factor of X for some $1 \leq t \leq m-1$.*
(a) If τ is not as in Figure 2, Case 1, with $s=0$, then the character of any lift of X equals $\chi_0+\chi_1+\cdots+\chi_s+\mu$, where μ is a sum of t distinct exceptional characters. In particular, χ_0 is a leaf of σ.
(b) If τ is as in Figure 2, Case 1, with $s=0$, then χ_0 or χ_λ is a leaf of the Brauer tree. Suppose that $e>1$, so that exactly one of χ_0, χ_λ is a leaf. Then in the first case, the character of any lift of X equals $\chi_0+\mu$, where μ is a sum of t distinct exceptional characters. In the latter case, the character of any lift of X is a sum of $t+1$ distinct exceptional characters.

Proof. (a) Assume that χ_0 is not a constituent of some lift $L(X)$ of X. Then τ is as in Case 1 by Lemma 3.3, since otherwise E_{u0} is a composition factor of X. Note that E_0 occurs twice as a composition factor in X in Case 1, namely in the socle and in the head of X. Hence χ_1 occurs twice in the character of $L(X)$ by assumption. Let Q be the projective cover of X. As $\mathrm{hd}(X) = E_0 \oplus E_1 \oplus \cdots \oplus E_s$, the character of the lift $L(Q)$ of Q equals $\chi_0 + 2\chi_1 + \ldots + 2\chi_s + \psi$, where ψ is the sum of the exceptional characters. We have a short exact sequence

$$0 \longrightarrow \Omega(X) \longrightarrow Q \longrightarrow X \longrightarrow 0,$$

which lifts to the short exact sequence

$$0 \longrightarrow L(\Omega(X)) \longrightarrow L(Q) \longrightarrow L(X) \longrightarrow 0,$$

where $L(\Omega(X))$ and $L(Q)$ are lifts of $\Omega(X)$ and of Q, respectively. By our assumption, the character of $L(\Omega(X))$ equals $\chi_0 + \vartheta$, where χ_1 is not a constituent of ϑ. This contradicts Lemma 2.3, since $\Omega(X)$ is indecomposable. Thus χ_0 occurs in the character of $L(X)$ with multiplicity one. By Remark 2.5(b), this implies that χ_0 is a leaf of σ in Case 1. Moreover, χ_i occurs in the character of $L(X)$ for $1 \leq i \leq s$. Finally, the character of $L(X)$ contains exactly t exceptional constituents, which are distinct by Lemma 2.7.
(b) The composition factor E_0 occurs $t+1$ times in X. Thus, if χ_0 is not a leaf, liftability is only possible if χ_λ is a leaf. If χ_0 is not a leaf of σ, then χ_0 cannot be a constituent of the character of a lift of X by Remark 2.5(b). If χ_λ is not a leaf of σ then χ_0 must be a constituent of the character of any lift of X, since X has composition factors occurring with multiplicity t. $\square$

Remark 3.6. Assume that G has a cyclic Sylow p-subgroup P which is not of prime order. Then any Scott module whose vertex is smaller than P arises from a path τ as in Figure 2, Case 1.

Lemma 3.7. *Let τ be one of the paths in Figure 2, Case 2, and assume that the module X constructed from τ is liftable.*
If E_{u0} is not marked, the succesor of E_{u0} around χ_0 is E_0. If E_{u0} is marked, the successor of E_0 around χ_0 is E_{u0}.

Proof. This follows from the construction of X. If E_{u0} is not marked, then a simple $\mathbf{B}$-module corresponding to an edge of σ adjacent to χ_0 and lying between E_{u0} and E_0 is not a composition factor of X. On the other hand, since χ_0 is a constituent of a lift of X, every simple $\mathbf{B}$-module corresponding to an edge of σ adjacent to χ_0 must occur in X as a composition factor. □

Lemma 3.8. *Let τ be one of the paths in Figure 2, Case 3, and assume that the module X constructed from τ is liftable. Then E_{u0} is not marked and the successor of E_{u0} around χ_0 is E_{d0}.*

Proof. Note that under the hypothesis that E_{u0} is marked, each of E_{u0} and E_{d0} occur twice as composition factors of X. As neither χ_{u0} nor χ_{d0} can be constituents of any lift of X by Lemma 3.7, χ_0 must occur twice as a constituent of a lift of X. This leads to a contradiction as in the proof of Lemma 3.5(a).
The second assertion is clear (see the proof of Lemma 3.7). □

4. CHARACTERISATION OF THE INDECOMPOSABLE LIFTABLE MODULES

So far we have described possible structures of indecomposable liftable non-projective $\mathbf{B}$-modules. The natural question that arises is which of them are indeed liftable. The answer is simple: As long as the Brauer tree admits a structure as described as above and as long as the respective restrictions are satisfied, the module is in fact liftable.
For the proof of this assertion we use a graph theoretic counting argument. Before doing so, we need some preliminaries. As before, we use the notation and assumptions in Hypothesis 2.1. Recall that the Brauer tree of $\mathbf{b}$ is a star with e edges and its exceptional vertex (if one exists) at the centre. Recall that m denotes the exceptional multiplicity of $\mathbf{b}$.

Lemma 4.1. *(a) If $e > 1$, there are $e(2m+1)$ indecomposable liftable $\mathbf{b}$-modules.*
(b) If $e = 1$, there are $m+1$ indecomposable liftable $\mathbf{b}$-modules.

Proof. (a) There are e simple and e projective indecomposable $\mathbf{b}$-modules, and these are all liftable. By Lemma 3.2, Lemma 3.5, and Lemma 3.7 we find em candidates for indecomposable liftable modules of Type I and $e(m-1)$ candidates for indecomposable liftable modules of Type II (only Case 1).
Thus we need to show that these candidates are in fact liftable. Let Y be such a module with socle V_0. Then Y is isomorphic to a liftable submodule of the projective cover of V_0 by Theorem 2.2. Thus we find the proposed number of liftable modules in total.
(b) The same argument as in (a) applies, except for the fact that there are only $m+1$ indecomposable $\mathbf{b}$-modules in total. □

Note that in the special case that the Brauer tree of $\mathbf{B}$ is a star with the exceptional vertex at the centre, then the preceding Lemma gives already a proof of the fact we are aiming at in this section. We will thus assume

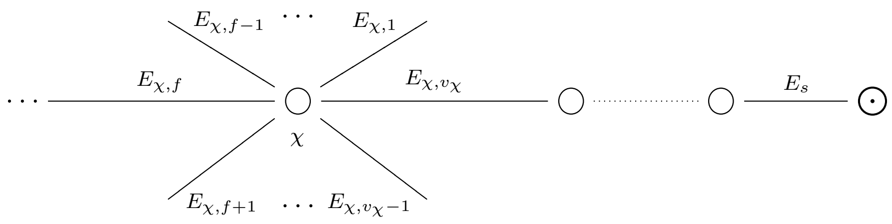


FIGURE 7. Labelling of edges on Brauer tree at χ_i

that $e > 1$ in the following. We need to show that the number of candidates for indecomposable liftable non-projective modules determined in Subsection 3.1 and Subsection 3.2 equals $2em$. Indeed, in this case, all possible candidates are then liftable by Lemma 2.4.

Lemma 4.2. *Suppose that $e > 1$. Then the number of indecomposable* **B**-*modules which are either simple or projective indecomposable or which satisfy the restrictions described in Subsection 3.1 and Subsection 3.2 equals $e(2m+1)$.*

Proof. There are e projective indecomposable modules and $2e$ indecomposable modules, which arise from Green's walk around the Brauer tree (see [Gr74]). The latter are simple or satisfy the restrictions derived in Subsection 3.1. Thus we are done if $m = 1$.

Suppose then that $m > 1$. Let χ be an arbitrary vertex of σ different from the exceptional one and let the edges emanating from χ in σ be labelled as in Figure 7. Denote the valency of the vertex χ by v_χ. Then there are exactly $v_\chi(m-1)$ indecomposable modules of Type II arising from paths as in Theorem 1.2, which start in one of the edges incident to χ. These $v_\chi(m-1)$ modules can be described by the starting edge $E_{\chi,i}$, the marking of the edges, and a multiplicity. First assume that $v_\chi \geq 2$. Then we have the following possible paths and edge markings:

- Type II, Case 2: Starting edge $E_{\chi,1}$ and $E_{\chi,1}$ marked;
- Type II, Case 3: Starting edge $E_{\chi,j}$, for $1 \leq j \leq v_\chi - 2$, and $E_{\chi,j}$ not marked;
- Type II, Case 2: Starting edge $E_{\chi,v_\chi-1}$ and $E_{\chi,v_\chi-1}$ not marked.

Finally, if χ is a leaf, i.e. $v_\chi = 1$, there is only one possible path of Type II, Case 1. Note that each of the above marked paths gives rise to $m-1$ indecomposable modules, according to the multiplicity associated to the marked edge incident to the exceptional vertex.

Now let χ be the exceptional vertex with valency v_χ. If $v_\chi > 1$, i.e. if χ is not a leaf, then there are exactly v_χ distinct marked paths of Type I as in Lemma 3.2 with χ as their middle vertex. Each of these paths gives rise to m indecomposable modules, one of which is contained in Green's walk. If χ is a leaf, the path of Type II as in Lemma 3.5(b) gives rise to m indecomposable

modules, one of which has already been counted. Thus we can associate to the exceptional vertex χ exactly $v_\chi(m-1)$ indecomposable modules.
We have now accounted for all the marked paths and their multiplicities arising in Subsections 3.1 and 3.2.
It is well known, that

$$\sum_{\chi \in \sigma} v_\chi = 2e.$$

As there are at most $v_\chi(m-1)$ indecomposable liftable modules associated to each vertex as above, we get at most $2e(m-1)$ indecomposable liftable modules which are not projective or which do not arise from Green's walk. Summing up, there are

$$e + 2e + 2e(m-1) = e(2m+1)$$

indecomposable liftable modules in total. □

It is clear that the above lemma together with the results of Section 3 proves Theorem 1.2.

5. The Green correspondents

In this section we determine the Green correspondents of the indecomposable liftable **B**-modules. The main tool is the Heller operator, which commutes with the Green correspondence. The orbits of Ω on the indecomposable non-projective **b**-modules are well known and easy to describe. There are m orbits of Ω on this set, each of length $2e$. One of these orbits consists of the simple modules $V_0, V_1 := \Omega^2(V_0), \ldots, V_{e-1} := \Omega^{2e-2}(V_0)$, together with $\Omega(V_0), \Omega(V_1) = \Omega^3(V_0), \ldots, \Omega(V_{e-1}) = \Omega^{2e-1}(V_0)$. The Green correspondents of the elements in this orbit are exactly the indecomposable modules occurring in Green's walk around the Brauer tree [Gr74].
If $m = 1$, i.e. $d = 1$ and $e = p - 1$, then the above modules are all of the indecomposable liftable non-projective modules of **B**.
We may and will thus assume in the following that $m > 1$. Also, X denotes an indecomposable liftable non-projective module in **B**.

Lemma 5.1. *The number of exceptional characters is the same in any lift of X.*

Proof. This is a consequence of our classification. □

Definition 5.2. We write $m(X)$ for the number of exceptional characters in a lift of X.

Lemma 5.3. *Assume that the exceptional vertex is contained in the path τ associated to X. Then $m(X) + m(\Omega(X)) = m$.*

Proof. The assumptions imply that the projective cover Q of X has an indecomposable direct summand corresponding to an edge connected to the exceptional vertex. The lifts of the other direct summands of Q have no exceptional constituents. Thus the exceptional characters are constituents of the lift of Q, each with multiplicity one. This implies the result. □

Theorem 5.4. *Suppose that $m > 1$. Fix an edge E of σ connected to the exceptional vertex. For each $1 \leq t \leq m$ let X_t denote a uniserial liftable module with head E and $m(X_t) = t$. (If the exceptional vertex of σ is a leaf, then X_t is simple or of Type II and has all composition factors isomorphic to E. Otherwise, X_t is of Type I. In either case, E has multiplicity t as a composition factor of X_t.)*
(a) The Ω-orbit containing X_m is Green's orbit.
(b) Suppose that $t < m$ and let Y be a member of the Ω-orbit containing X_t. Then

$$m(Y) = \begin{cases} t, & \text{if } Y = \Omega^{2i}(X_m) \text{ for some } i, \\ m-t, & \text{if } Y = \Omega^{2i+1}(X_m) \text{ for some } i. \end{cases}$$

(c) We have

$$m(f(X_t)) = \begin{cases} t, & \text{if } m(f(X_m)) = m, \\ m-t, & \text{if } m(f(X_m)) = 0. \end{cases}$$

Moreover, all Green correspondents $f(X_t)$ for $1 \leq t \leq m$ have the same head.
(d) Each Ω-orbit on the set of (isomorphism classes of) indecomposable liftable non-projective $\mathbf{B}$-modules contains exactly one of the X_t.

Proof. (a) This follows from the results of Green [Gr74, Theorem 2].
(b) This is a consequence of Lemma 5.3 as any indecomposable liftable non-projective module of $\mathbf{B}$ not contained in Green's orbit satisfies the hypothesis of that lemma.
(c) By [Fei82, Theorem VII.2.14], the composition length of $f(X_1)$ equals e or $p^d - e$. In the first case, $m(f(X_1)) = 1$, and in the second case $m(f(X_1)) = m$. In particular, $f(X_1)$ is either short or long in the terminology of [Pea75]. Thus [Pea75, Corollary 3.7] carries over to the case where the Green correspondent of a simple $\mathbf{B}$-module is replaced by $f(X_1)$. Next, the proof of [Pea75, Theorem 3.10] carries over to the situation $W = X_t$ $(1 \leq t \leq m-1)$, and the simple module being replaced by X_1. In the first case, [Pea75, Theorem 3.10 (a)] yields $m(f(X_{t+1})) = t+1$, and in the second case, this yields $m(f(X_{t+1})) = m - (t+1)$. In particular, the first case is equivalent to $m(f(X_m)) = m$. The last assertion also follows along these lines.
(d) This follows from the corresponding fact for $\mathbf{b}$ together with (c). □

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