On a question of Brauer in modular representation theory of finite groups

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1 Introduction

This note is based on talks I gave in Paris, Osaka, and Kyoto in 1998. Its aim is to discuss a question of Brauer and present a reduction theorem obtained by my student Olav Düvel in his PhD thesis [4]. In the final section, I shall relate Brauer's question to Donovan's conjecture.

Throughout this article, k denotes an algebraically closed field of characteristic $p > 0$. Moreover, G is a finite group, kG its group algebra over k and $\textbf{mod}\text{-}kG$ the category of finitely generated right kG-modules.

Brauer's question belongs to a whole class of problems, namely to relate numerical invariants of G with cohomological invariants of $\textbf{mod}\text{-}kG$. An example for a numerical invariant of G is $|G|_p$, the p-part of the order of G. A cohomological invariant of $\textbf{mod}\text{-}kG$ is $c(kG)$, the largest Cartan invariant of kG, i.e.,

$$c(kG) = \max\{\dim_k \operatorname{Hom}_{kG}(P, Q) \mid P, Q \in \textbf{mod}\text{-}kG \text{ PIMs}\},$$

where a PIM is a projective indecomposable kG-module. A relation between the two invariants is implied by Maschke's theorem: $|G|_p = 1$ if and only if $c(kG) = 1$. Next consider the case that $|G|_p = p$. This implies that a Sylow p-subgroup of G is cyclic. Then, by results of Brauer and others, kG is representation finite and $c(kG) \leq p = |G|_p$.

These results and further examples might have motivated the following question of Brauer.

Question 1.1 (Brauer [2], Problem 22) Is it true that $c(kG) \leq |G|_p$ for all finite groups G?

In [11], Landrock showed that the Suzuki group $Sz(8)$ for $p = 2$ provides a negative answer to Brauer's question.

Brauer's original question is asking for a *particular* bound for the Cartan invariants of kG in terms of $|G|_p$. It is natural to modify Brauer's original question by just asking for *some* bound.

Question 1.2 Is there a function $f_p : \mathbb{N} \to \mathbb{N}$ such that $c(kG) \leq f_p(\log_p |G|_p)$ for all finite groups G?

2 Blocks and some of their invariants

Brauer's question can be refined by looking at blocks. We have a direct sum decomposition

$$kG = \bigoplus_B B,$$

into indecomposable two-sided ideals B of kG, the *blocks* of kG. Correspondingly, we have a direct sum decomposition of the module category

$$\mathbf{mod}\text{-}kG = \bigoplus_B \mathbf{mod}\text{-}B.$$

The problem now is to compare numerical invariants of a block B with cohomological invariants of $\mathbf{mod}$-B. First one has to define the "right" numerical invariants of blocks. To a block B one associates a conjugacy class of p-subgroups of G, the *defect groups* of B. If D is such a defect group for B,

$$d(B) := \log_p(|D|)$$

is called the *defect* of B (replacing $\log_p(|G|_p)$ in the previous example).

In the following, a *block* is a finite dimensional k-algebra which is isomorphic to a block of kG for some finite group G. We consider the following invariants of a block B.

- $C(B) :=$ Cartan matrix of B (= matrix of Cartan invariants).
- $c(B) :=$ largest entry of $C(B)$.
- $\ell(B) :=$ number of simple B-modules (up to isomorphism).

- $LL(B) := \min\{0 \neq n \in \mathbb{N} \mid J(B)^n = 0\}$ (the Loewy length of B).
- $e(B) := \max\{\dim_k \operatorname{Ext}^1_B(S,T) \mid S, T$ simple B-modules$\}$.

Note that all these invariants are invariants of the Morita equivalence class of B. It is well known that $C(B)$ is a symmetric matrix whose determinant is a power of p. Moreover, $p^{d(B)}$ is the unique largest invariant factor of $C(B)$ [12, Corollary III.8.13], such that $d(B)$ can be recovered from **mod**-B.

3 More questions and Düvel's reduction

In view of Brauer's question one may ask whether $d(B)$ bounds the invariants of B introduced in the previous section. More precisely, do there exist functions $f_{i,p} : \mathbb{N} \to \mathbb{N}$, $i = 1, \ldots, 4$, such that the following hold for all blocks B?

(1) $\ell(B) \leq f_{1,p}(d(B))$,

(2) $LL(B) \leq f_{2,p}(d(B))$,

(3) $e(B) \leq f_{3,p}(d(B))$,

(4) $c(B) \leq f_{4,p}(d(B))$?

Of course, Question (4) is just the block version of Question 1.2.

Remarks 3.1 (a) By a result of Brauer and Feit (see [5, Theorem IV.4.18]) Question (1) has a positive answer, namely $\ell(B) \leq \frac{1}{4}p^{2d(B)} + 1$.

(b) In view of this, a positive answer to Question (4) is equivalent to a simultaneous positive answer to Questions (2) and (3).

(c) If B is replaced by kG (and $d(B)$ by $\log_p(|G|_p)$) in Question (1), then this question does not have a positive answer. To reduce Question (4) to the two Questions (2) and (3), it is therefore essential to turn to blocks.

(d) It has been conjectured by Kiyota and Wada [9], that $LL(B) \leq \rho(C(B))$, where $\rho(C(B))$ is the largest eigenvalue of $C(B)$.

Donovan has formulated a conjecture which appears to go far beyond Brauer's question.

Conjecture 3.2 (Donovan, see [1], Conjecture M) The number of Morita equivalence classes of blocks of a given defect is finite.

It is obvious that the truth of Donovan's conjecture would imply a positive answer to Questions (1)–(4) above. To weaken Donovan's conjecture, let $\mathcal{C}$ be a class of finite groups. Say that Donovan's conjecture holds for $\mathcal{C}$, if the number of Morita equivalence classes among the blocks of a given defect of the group algebras kG for $G \in \mathcal{C}$, is finite. Scopes has proved in [15] that Donovan's conjecture holds for the symmetric groups, the results of Kessar [8] imply that Donovan's conjecture holds for the alternating groups. In [10] Külshammer has proved that in order to verify Donovan's conjecture, it suffices to consider blocks of groups generated by the defect groups of the blocks.

In the final section of this note a condition will be introduced allowing to derive the truth of Donovan's conjecture for a class of groups from a positive answer to Brauer's question for this class.

We are now going to state Düvel's reduction theorem, but need to introduce some more notation. First of all, put

$$\widehat{\mathcal{E}}_{p'} := \{\hat{G} \mid \hat{G} \text{ perfect, finite group, } Z := Z(\hat{G}) \text{ cyclic }, p \not| \, |Z|, \hat{G}/Z \text{ simple}\}.$$

The following definitions are due to Düvel.

Definition 3.3 (a) For $d \in \mathbb{N}$ let

$$\overline{LL}_p(d) := \sup\{LL(B) \mid B \text{ block of } k\hat{G}, \hat{G} \in \widehat{\mathcal{E}}_{p'}, d(B) \leq d\} \in \mathbb{N} \cup \{\infty\},$$

and

$$\bar{e}_p(d) := \sup\{e(B) \mid B \text{ block of } k\hat{G}, \hat{G} \in \widehat{\mathcal{E}}_{p'}, d(B) \leq d\} \in \mathbb{N} \cup \{\infty\}.$$

(b) Define the maps $LL_p, e_p : \mathbb{N} \to \mathbb{N}$ recursively by

$$LL_p(0) := 1, LL_p(n) := \max\{\overline{LL}_p(n), LL_p(i)LL_p(n-i) \mid 0 < i < n\}, n > 0,$$

and

$$e_p(0) := 0, e_p(n) := \max\{\bar{e}_p(n), e_p(i) + e_p(n-i) \mid 0 < i < n\}, n > 0.$$

With these notations we can now formulate Düvel's reduction theorem.

Theorem 3.4 ([4, Theorem 3.2]) *Let G be a finite group, B a block of kG of defect d. Then $LL(B) \leq LL_p(d)$ and $e(B) \leq e_p(d)\ell(B)$.*

Remark 3.5 Let $d \in \mathbb{N}$. It follows from results of many authors that

$$\sup\{LL(B), e(B) \mid B \text{ block of } kG, d(B) \leq d, G \in \mathcal{C}\}$$

is finite for a large class $\mathcal{C}$ of finite Chevalley groups containing, for example, all general linear groups [3], the symmetric and alternating groups [15, 8], the 4-dimensional symplectic groups [14], the Suzuki groups (cyclic defect) and the "small" Ree groups [13]. We hope to be able to extend the class $\mathcal{C}$ in the future. The aim is, of course, to show that if we replace $\mathcal{C}$ by $\widehat{\mathcal{E}}_{p'}$ then the above number is still finite.

4 Idea of Düvel's reduction

It is perhaps not surprising to the experts that Düvel uses a generalized version of Fong reduction—with control of defects of blocks—based on Clifford theory à la Dade, to obtain his theorem. Without going into details, some of the main points of Düvel's reduction will now be sketched. This section is based on Düvel's thesis [4].

Let G be a finite group and let N be a normal subgroup of G.

(1) Define a full subcategory $\mathcal{S}_G(N)$ of **mod**-kG consisting of those finite dimensional kG-modules whose restriction to N is semisimple. Then every kG-module M has a filtration, of length at most $LL(kN)$, with factors in $\mathcal{S}_G(N)$.

(2) Let $V_1, \dots, V_t$ represent the G-orbits of the isomorphism classes of the simple kN-modules. For $1 \leq i \leq t$ let G_i denote the stabilizer of V_i in G. We then have a direct sum decomposition

$$\mathcal{S}_G(N) = \bigoplus_{i=1}^{t} \mathcal{S}_G(N, V_i),$$

where $\mathcal{S}_G(N, V_i)$ is the full subcategory of $\mathcal{S}_G(N)$ consisting of those kG-modules whose restriction to N is a direct sum of modules G-conjugate to V_i.

(3) Fix an integer i with $1 \leq i \leq t$. Then there is a cyclic central p'-extension

$$1 \to Z_i \to \widehat{G_i/N} \to G_i/N \to 1$$

and a simple kZ_i-module X_i such that $\mathcal{S}_G(N, V_i)$ is Morita equivalent to $\mathcal{S}_{\widehat{G_i/N}}(Z_i, X_i)$.

(4) Let M be an indecomposable kG-module and B the block of kG containing M. Next, let M' be an indecomposable subquotient (a factor module of a submodule) of M contained in $\mathcal{S}_G(N)$.

Since M' is indecomposable, it is contained in $\mathcal{S}_G(N, V_i)$ for a unique i. By (3), M' determines a block B_i of $\widehat{G_i/N}$. Finally, let b_i denote the kN-block containing V_i. Then

$$d(B_i) + d(b_i) \leq d(B).$$

(5) The proof bounding the Loewy length proceeds as follows. Let B be a block of kG and let $b_1, \ldots, b_s$ be the blocks of kN covered by B. Then, since these are conjugate in G, all of the b_is have the same Loewy length. Now let M be a PIM of B of maximal Loewy length. Among all indecomposable subquotients of M' of M which lie in $\mathcal{S}_G(N)$, choose one such that $LL(B_i)$ is maximal, where B_i is the block of $\widehat{G_i/N}$ determined by M' as in (4). Then, by (4), $d(B_i) + d(b_i) \leq d(B)$.

By induction on the order of $G/O_{p'}(G)$ one may assume that either G is in the class $\widehat{\mathcal{E}}_{p'}$, or else that b_i and B_i belong to groups H with $|H/O_{p'}(H)| < |G/O_{p'}(G)|$. Then

$$\begin{aligned} LL(B) &\leq LL(b_i)LL(B_i) \\ &\leq LL_p(d(b_i))LL_p(d(B_i)) \\ &\leq LL_p(d(b_i))LL_p(d(B) - d(b_i)) \\ &\leq LL_p(d(B)). \end{aligned}$$

Here, the first inequality follows by filtering M with at most $LL(b_i)$ factors of $\mathcal{S}_G(N)$ and the choice of M', the second by induction, and the third and fourth by properties of the function LL_p.

5 Towards Donovan's conjecture

In the final section a few ideas will be sketched which allow to approach Donovan's conjecture for certain classes of groups, provided Brauer's question has a positive answer for these groups.

Proposition 5.1 *Let $\mathcal{B}$ be a set of blocks. Suppose there exist positive integers d and c and a finite field $k_0 \subset k$ such that for all blocks $B \in \mathcal{B}$ the following conditions are satisfied.*

(a) $d(B) \leq d$,

(b) $c(B) \leq c$,

(c) $B \cong B_0 \otimes_{k_0} k$ *for some split k_0-algebra B_0.*

Then there are only finitely many Morita equivalence classes among the blocks in $\mathcal{B}$.

Proof. Let $B \in \mathcal{B}$. By the results of Brauer and Feit, $\ell(B) \leq p^{2d}/4 + 1$. By Assumption (b), the sum of the entries of $C(B)$ is at most $c(p^{2d}/4 + 1)^2$.

Let $P_1, \ldots, P_n$ denote a set of representatives for the isomorphism classes of the PIMs of B_0 and let $M := P_1 \oplus \cdots \oplus P_n$. Put $A_0 := \mathrm{End}_{B_0}(M)$. Since k_0 is a splitting field for B_0 by assumption, the projective B-module $P_i \otimes_{k_0} k$ is indecomposable. Thus $P_1 \otimes_{k_0} k, \ldots, P_n \otimes_{k_0} k$ is a complete set of representatives for the PIMs of B. It follows that $A := A_0 \otimes_{k_0} k \cong \mathrm{End}_B(M \otimes_{k_0} k)$ is a basic algebra for B. By the first paragraph of the proof, $\dim_{k_0} A_0 = \dim_k A \leq c(p^{2d}/4 + 1)^2$.

Since there are only finitely many isomorphism classes among the k_0-algebras of fixed, bounded dimension, the above implies that there are only finitely many isomorphism classes among the basic algebras of the blocks in $\mathcal{B}$. Since two k-algebras are Morita equivalent if and only if their basic algebras are isomorphic, the result follows.

Proposition 5.2 *Let G be a finite group and B a block of kG. Let $\varphi_1, \ldots, \varphi_n$ denote the k-characters of the simple B-modules. If k_0 is the finite field containing the $\phi_i(g)$, $i = 1, \ldots, n$, $g \in G$, then there is a k_0-algebra B_0 with $B \cong B_0 \otimes_{k_0} k$. Moreover, k_0 is a splitting field for B_0.*

Proof. Let $e \in Z(kG)$ be the central idempotent with $B = kGe$. By Osima's theorem and Brauer reciprocity it follows that $e \in k_0G$ (see [12, Theorem III.2.9]). Hence $B = B_0 \otimes_{k_0} k$ with $B_0 := k_0Ge$. Also, by [6, Corollary 9.23], every B-module is realizable over k_0 and hence k_0 is a splitting field for B_0.

We conclude this section with an example.

Example 5.3 (Unipotent blocks of $GL_n(q)$). Let $d \in \mathbb{N}$.

(1) For $0 \neq n \in \mathbb{N}$ let $\mathcal{B}_{n,d}$ denote the set of unipotent blocks of $kGL_n(q)$ of defect d, where q runs through the prime powers not divisible by p. Then Assumption (a) of Proposition 5.1 is trivially satisfied. By results of Dipper and James [3] on the decomposition numbers of the general linear groups, (b) is also satisfied. All unipotent characters of $GL_n(q)$ are rational valued. The same is true for the irreducible Brauer characters of the unipotent blocks. By Proposition 5.2, Assumption (c) is satisfied with $k_0 = \mathbb{F}_p$.

(2) Next, fix a prime power q not divisible by p. The results of Jost [7, Theorem 6.2] imply that there is a bound N, depending only on d and p, but not on q, such that every unipotent block of defect d of $kGL_m(q)$ for some $0 \neq m \in \mathbb{N}$ is Morita equivalent to a unipotent block of the same defect of $kGL_n(q)$ for some $n \leq N$.

(3) By first using (2) and then applying (1) for $1 \leq n \leq N$, it follows that there are only finitely many Morita equivalence classes of unipotent blocks of defect d among the unipotent blocks of $kGL_n(q)$, where n runs through the positive integers and q through the prime powers not divisible by p.

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