THE CLASSIFICATION OF THE INDECOMPOSABLE LIFTABLE MODULES IN BLOCKS WITH CYCLIC DEFECT GROUPS

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Abstract. Let G be a finite group, let k be an algebraically closed field of positive characteristic p and let $\mathbf{B}$ a block of kG with cyclic defect groups. We classify the indecomposable $\mathbf{B}$-modules which are liftable with respect to a splitting p-modular system with residue class field k. The indecomposable non-projective modules in $\mathbf{B}$ are constructed from certain paths in the Brauer tree of $\mathbf{B}$ (see [Ja69]). We determine those paths that give rise to liftable modules. We also find the characters of the lifts of these modules.

1. Introduction

We consider a finite group G, an algebraically closed field k of positive characteristic p and a block $\mathbf{B}$ of kG with cyclic defect groups. The aim of this note is to classify the indecomposable $\mathbf{B}$-modules which are liftable with respect to a splitting p-modular system with residue class field k.

The structure of a block with cyclic defect groups is encoded in its Brauer tree. This is a finite tree, together with a planar embedding and, possibly, a multiplicity assigned to one of its vertices. In [Ja69] and [Ku69], Janusz and Kupisch constructed the non-projective indecomposable modules in such a block. The construction is described in terms of a sequence of elementary modules in [Ku69], or, equivalently, in terms of certain paths on the Brauer tree in [Ja69]. Here, we determine those paths that correspond to the liftable modules, and describe the characters of their lifts.

Indecomposable direct summands of permutation kG-modules are trivial source modules and hence are liftable. One of the motivations for our work was the attempt to understand the trivial source modules in blocks with cyclic defect groups.

The exposition of this paper owes very much to the anonymous referee of a previous version, where we gave a more self-contained but less

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conceptual proof of the main result. It was the referee who suggested to start from the fact that the Green correspondence sets up a stable equivalence between the block $\mathbf{B}$ and its Brauer correspondent $\mathbf{b}$, compatible with Auslander-Reiten sequences and preserving liftability of indecomposable modules. The positions of the non-projective indecomposable liftable $\mathbf{b}$-modules on the stable Auslander-Reiten quiver of $\mathbf{b}$ are easily determined. One thus has to describe the structure of the indecomposable $\mathbf{B}$-modules located at a certain distance to the boundary of the stable Auslander-Reiten quiver of $\mathbf{B}$. As the referee suggested, this is most conveniently achieved by noticing that $\mathbf{B}$ modulo its socle is a string algebra in the sense of [BuRi87], which allows to describe the arrows in the stable Auslander-Reiten quiver by a process of adding hooks or removing cohooks. This construction is explicitly worked out in [BC02] for Brauer tree algebras, of which $\mathbf{B}$ is an example.

In this paper we proceed as follows. We produce a list of candidates for the non-projective indecomposable liftable $\mathbf{B}$-modules. Using [BC02, Theorem 3.5] we show that all our candidates lie on the positions of the liftable modules on the stable Auslander-Reiten quiver of $\mathbf{B}$, and that each orbit under the Auslander-Reiten translate of the non-projective liftable $\mathbf{B}$-modules contains at least one of our candidates. We then apply a result of Reiten [Rei77, Theorem 2.4] to prove that the set of our candidates is closed under the Auslander-Reiten translate. This shows that the set of our candidates contains all the liftable modules.

2. THE MAIN THEOREM

To describe our main result, let $(K, \mathcal{R}, k)$ be a p-modular system such that K has characteristic 0 and is large enough for G. A kG-module X is liftable, if there exists an $\mathcal{R}G$-lattice M such that $X \cong k \otimes_{\mathcal{R}} M$. Now let $\mathbf{B}$, as above, denote a kG-block with cyclic defect groups. The embedded Brauer tree of $\mathbf{B}$ is denoted by σ. The embedding is determined by specifying, for each vertex of σ, a cyclic ordering of the edges adjacent to this vertex. Consider two edges E and F of σ adjacent to the vertex χ. We say that F is a successor of E around χ, if F comes next to E in the cyclic ordering of the edges around χ. We use the convention that in a drawing of σ in the plane, the successor of an edge is the counter-clockwise neighbour of this edge.

The vertices of the Brauer tree are labelled by irreducible K-characters of $\mathbf{B}$. If σ has an exceptional vertex, this is labelled by any one of the exceptional characters and indicated by a black circle in our drawings. The edges of σ are labelled by the simple $\mathbf{B}$-modules. A vertex of σ and the corresponding irreducible K-character are denoted by the same symbols, and the analogous convention is used for the edges of σ. By a leaf of σ we either mean a vertex of valency 1 or the edge adjacent to it. Thus the leaves of σ correspond to the simple liftable $\mathbf{B}$-modules. We use the same convention as in [Alp86, Section 17], i.e. the edge F is a successor of E around a vertex of σ, if and only if there is a nonsplit extension $0 \to F \to X \to E \to 0$ of $\mathbf{B}$-modules.

Let us assume that $\mathbf{B}$ has defect $n \geq 1$ and that the number of simple modules of $\mathbf{B}$ equals e. Put $m := (p^n - 1)/e$. If $m > 1$, it is called the exceptional multiplicity of $\mathbf{B}$. Let us briefly recall the description of the indecomposable $\mathbf{B}$-modules. A non-projective, non-zero indecomposable $\mathbf{B}$-module X determines a non-empty sequence $(E_1, E_2, \ldots, E_s)$ of simple modules such that for all $1 \leq i < s$, one of E_i, E_{i+1} is in the socle of X and the other one is in the head of X, and the edges E_i, E_{i+1} are incident on the Brauer tree. This is called the *top-socle sequence* of $\mathbf{B}$, and visualised by the corresponding path in the Brauer tree (see [Ja69, Ku69] and [Fei82, Section VII.12.]). In [BC02, Definition 2.1], Bleher and Chinburg also assign to X a pair $\varepsilon = (\varepsilon_1, \varepsilon_s)$ of signs, called the *direction* of X, and an integer $0 \leq \mu \leq m$, called the *multiplicty* of X as follows. If X is simple, define $\varepsilon = (-1, 1)$. Otherwise, for $i = 1, s$, let $\varepsilon_i = -1$, if E_i is in the socle of X, and $\varepsilon_i = 1$, if E_i is in the head of X. If none of the E_i is adjacent to the exceptional vertex (in particular, if $m = 0$), we set $\mu = 0$. If $m > 1$ and E_i is ajacent to the exceptional vertex for some $1 \leq i \leq s$, we let μ denote the number of composition factors of X isomorphic to E_i (this is independent of the chosen edge adjacent to the exceptional vertex). The module X is determined by its top-socle sequence, its direction and its multiplicty.

A projective indecomposable $\mathbf{B}$-module P has two uniserial submodules P_1 and P_2 such that $\mathrm{rad}(P) = P_1 + P_2$ and $\mathrm{soc}(P) = P_1 \cap P_2$. An indecomposable $\mathbf{B}$-module X is called a *hook*, if there is a projective indecomposable module P such that $X \cong P/P_1$ or $X \cong P/P_2$. Notice that hooks are uniserial. (Dually, the modules isomorphic to P_1 or P_2, as P runs through the projective indecomposable $\mathbf{B}$-modules are called *cohooks*, but every cohook is a hook and vice versa.)

The stable Auslander-Reiten quiver $\Gamma_s(\mathbf{B})$ of $\mathbf{B}$ is the finite tube $(\mathbb{Z}/e\mathbb{Z})A_{p^n-1}$ (see [Ben91, Theorem 6.5.5]). We will also give a characterisation of the non-projective indecomposable liftable $\mathbf{B}$-modules in terms of their minimal distance to the boundary of $\Gamma_s(\mathbf{B})$. It follows from Green's results in [Gr74], that the hooks and cohooks are exactly the modules located at the boundary of $\Gamma_s(\mathbf{B})$, and that these modules are liftable.

Theorem 2.1. *Suppose that* $\mathbf{B}$ *has defect* $n \geq 1$ *and* e *simple modules and put* $m = (p^n - 1)/e$.

(a) *The number of indecomposable liftable* $\mathbf{B}$*-modules equals* $m+1$ *if* $e = 1$, *and* $e(2m+1)$ *if* $e > 1$. *If* $e = 1$, *all indecomposable* $\mathbf{B}$*-modules are liftable. We thus assume that* $e > 1$ *in the following.*

(b) *Let* X *be a non-projective indecomposable* $\mathbf{B}$*-module. Then* X *is liftable, if and only if the minimal distance of* X *to the boundary of* $\Gamma_s(\mathbf{B})$ *is of the form* ei *for* $0 \leq i \leq \lfloor (m-1)/2 \rfloor$ *or* $ei - 1$ *for* $1 \leq i \leq \lfloor m/2 \rfloor$.

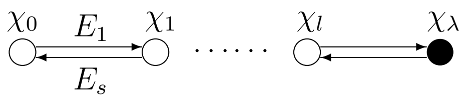
(c) *The indecomposable liftable* $\mathbf{B}$*-modules* X *are exactly those described in* (1)–(6) *below. The modules in* (3)–(6) *only occur if* $m > 1$.

(1) X *is projective.*

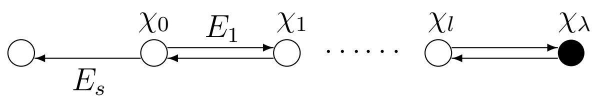
(2) X *is a hook; in particular,* X *is uniserial with descending composition series corresponding to a counter-clockwise walk around a vertex* χ *of* σ *(each composition factor of* X *occurring with multiplicity* m *if* χ *is the exceptional vertex). The character of any lift of* X *equals* χ *or the sum of the exceptional characters.*

The number of modules of this type is $2e$. *These are exactly the modules occurring in Green's walk around the Brauer tree* [Gr74] *and exactly the modules lying at the boundary of* $\Gamma_s(\mathbf{B})$.

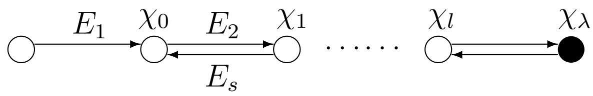
(3) X *corresponds to the following path with* $l \geq 0$; *in case* $l > 0$ *the vertex* χ_0 *is a leaf of the Brauer tree, in case* $l = 0$ *either* χ_0 *or* χ_λ *is a leaf. The direction of* X *is* $(1, -1)$.



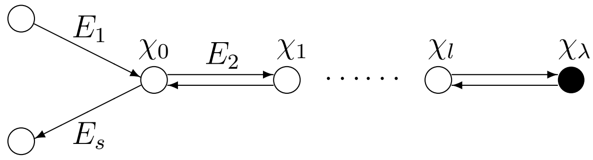
(4) X *corresponds to the following path, where* $l \geq 0$, *the successor of* E_1 *around* χ_0 *is* E_s, *and* E_1 *is in the head of* X, *i.e. the direction of* X *is* $(1, 1)$.



(5) X *corresponds to the following path, where* $l \geq 0$, *the successor of* E_1 *around* χ_0 *is* E_s, *and* E_1 *is in the socle of* X, *i.e. the direction of* X *is* $(-1, -1)$.



(6) *X corresponds to the following path, where $l \geq -1$, the successor of E_1 around χ_0 is E_s and E_1 is in the socle of X, i.e. the direction of X is $(-1, 1)$. (If $\chi_0 = \chi_\lambda$ is the exceptional vertex, we put $l = -1$.)*



For any of the paths in (3)–(6)*, there are $m - 1$ modules X distinguished by their multiplicity μ with $2 \leq \mu \leq m$ (or $1 \leq \mu \leq m - 1$ in Case* (6) *with $l = -1$).*

(d) *If X is as in* (c)*, the character of any lift of X equals $\sum_{i=0}^{l} \chi_i + \Xi$, where Ξ is a sum of $\mu - 1$ (or μ in Case* (6) *with $l = -1$) distinct exceptional characters.*

3. PRELIMINARIES

We keep the notation introduced in the previous section. Let D be a defect group of $\mathbf{B}$ of order p^n with $n \geq 1$. Recall that e denotes the number of simple $\mathbf{B}$-modules and $m = (p^n - 1)/e$. The unique subgroup of order p of D is denoted by D_1 and the normaliser of D_1 in G by N. If X is an indecomposable non-projective kG-module with vertex contained in D, we denote its Green correspondent in kN by $f(X)$. If X belongs to $\mathbf{B}$, then $f(X)$ belongs to the Brauer correspondent $\mathbf{b}$ of $\mathbf{B}$ (compare [Alp86, Corollary 14.4]). Likewise, if Y is an indecomposable kN-module with vertex in D, then $g(Y)$ denotes its Green correspondent in kG. The Brauer tree of $\mathbf{b}$ is a star with the exceptional vertex (if $m > 1$) at the centre. The number of simple kN-modules belonging to $\mathbf{b}$ is e.

The following lemma allows us to transfer the analysis of the indecomposable liftable $\mathbf{B}$-modules to the respective Green correspondent in $\mathbf{b}$ (and vice versa).

Lemma 3.1. *If X is an indecomposable liftable non-projective* $\mathbf{B}$-*module, then $f(X)$ is an indecomposable liftable non-projective* $\mathbf{b}$-*module.*

If Y is an indecomposable liftable non-projective $\mathbf{b}$*-module, then* $g(Y)$ *is an indecomposable liftable non-projective* $\mathbf{B}$*-module.*

Proof. The assertion that $g(Y)$ is an indecomposable liftable kG-module, if Y is an indecomposable liftable $\mathbf{b}$-module, has been proven by Peacock in [Pea77, Thm. 2.8] (with the help of [Tho67, Lemma 1]). Considering [Alp86, Thm. 17.3], the other assertion is proven similarly. □

Our next task is the determination of the indecomposable liftable $\mathbf{b}$-modules. This makes use of a theorem of Zassenhaus.

Theorem 3.2 (Zassenhaus, [La83, Lemma I.17.3])**.** *Let M be an $\mathcal{R}G$-lattice and let χ be the character of $K \otimes_{\mathcal{R}} M$. Let $\chi = \chi_1 + \chi_2$ for two KG-characters χ_1, χ_2. Then M contains $\mathcal{R}$-pure submodules N_1, N_2, such that χ_i is the character of $K \otimes_{\mathcal{R}} N_i$ for $i = 1, 2$. In particular, $k \otimes_{\mathcal{R}} M$ contains submodules isomorphic to $k \otimes_{\mathcal{R}} N_i$, $i = 1, 2$.*

Lemma 3.3. *An indecomposable* $\mathbf{b}$*-module Y is liftable, if and only if the composition length of Y is congruent to 0 or 1 modulo e.*

In particular, the number of liftable $\mathbf{b}$*-modules equals $m+1$, if $e = 1$, and $e(2m + 1)$, if $e > 1$.*

Proof. Suppose first that Y is liftable. Since Y is uniserial, a lift of Y can contain at most one non-exceptional character by Theorem 3.2. The number of composition factors of the reduction modulo p of an exceptional character equals e. Thus the composition length of Y is congruent to 0 or 1 modulo e. The converse follows from Theorem 3.2 by starting with the projective indecomposable characters. □

Lemma 3.4. *Suppose that $m > 1$. Let X denote an indecomposable* $\mathbf{B}$*-module. If X is liftable, then the multiplicity of an exceptional character in any lift of X is at most one.*

Proof. Let P be a projective cover of X. By construction (see [Fei82, Section VII.12.])), the head of X contains at most one simple module corresponding to an edge of σ connected to the exceptional vertex. Thus P contains at most one indecomposable summand which is the projective cover of such a simple module. Thus the lift of P either contains no exceptional character at all, or else the sum of all exceptional characters. By [Gr74, (3.6b)], any lift of X is an epimorphic image of the lift of P. This proves the assertion. □

4. PROOF OF THE THEOREM

Let Ω denote the Heller operator. Green correspondence sets up an Ω^2-equivariant graph isomorphism between $\Gamma_s(\mathbf{b})$ and $\Gamma_s(\mathbf{B})$ (see

[Ben91, Page 203]). Let Γ be one of these Auslander Reiten quivers. The boundary of Γ consists of two Ω^2-orbits, called the two *sides* of the boundary in the following. The Ω^2-orbits on Γ are the sets of vertices of fixed distance to either side of the boundary. If an orbit has distance i to one side of the boundary, it has distance $p^n - 2 - i = em - 1 - i$ to the other side.

For $0 \leq i \leq p^n - 1$, the indecomposable $\mathbf{b}$-modules with composition length i constitute the Ω^2-orbit with distance $i-1$ from the orbit of the simple modules (see [Ben91, Page 203]), which forms one side of the boundary of $\Gamma_s(\mathbf{b})$. Lemma 3.3 now implies Theorem 2.1(a) and (b) for the block $\mathbf{b}$. Using Green correspondence together with Lemma 3.1, we obtain Theorem 2.1(a) and (b) for $\mathbf{B}$. As already remarked, the modules described in Theorem 2.1(2) are exactly the modules lying at the boundary of $\Gamma_s(\mathbf{B})$. If $m = 1$, we have accounted for all the non-projective indecomposable liftable modules.

Let us assume that $m > 1$ in the following. Using [BC02, Theorem 3.5], one can show that the indecomposable modules described in Theorem 2.1 lie in Ω^2-orbits whose distance to one of the two sides of the boundary is divisible by e.

Lemma 4.1. *Let X be one of the modules described in Theorem 2.1(3)–(6). Then the distance of X to one of the two sides of the boundary of $\Gamma_s(\mathbf{B})$ is divisible by e.*

Proof. This follows from [BC02, Theorem 3.5]. To give the reader a flavour of the arguments involved, we work out a special case. Let us assume that X is as in Theorem 2.1(3) with $l \geq 1$, i.e. χ_1 is not the exceptional vertex.

Using the labelling and notation of [BC02, Definition 3.2], we have $v_a = \chi_1$, $v_z = \chi_0$, and $S_a = E_1 = S_z$. Thus, the walk W_X of [BC02, Definition 3.4] is of length $q = 2e + 1$. By [BC02, Theorem 3.5] the distance of X to one of the two sides of the boundary equals $d = (q-1)/2 + \eta e = (1+\eta)e$, where η is as defined in [BC02, Definition 3.4]. The other cases are treated similarly. $\square$

In fact, the parameter η of the above proof is given by

$$\eta = \begin{cases} m - \mu, & \text{if } l \text{ is odd,} \\ \mu - 2, & \text{if } l \text{ is even,} \end{cases}$$

and $d = (1+\eta)e$ gives the distance of X to the boundary containing E_1.

The above lemma shows that the modules described in our theorem are liftable. To show that no other liftable modules exist, we proceed as follows. First we prove that each Ω^2-orbit of liftable modules contains at least one of the modules described in Theorem 2.1. Finally we prove that this set of modules is invariant under Ω^2.

Lemma 4.2. *Each Ω^2-orbit of liftable modules on $\Gamma_s(\mathbf{B})$ contains at least one of the modules described in Theorem 2.1.*

Proof. The result is clear if the Brauer tree of $\mathbf{B}$ is a star with the exceptional vertex at its centre. Let us assume that this is not the case. The modules of Theorem 2.1(2) constitute the boundary of $\Gamma_s(\mathbf{B})$. Consider a path τ as in Theorem 2.1(3). Assume that $l \geq 1$ (i.e. that the distance of χ_0 to the exceptional vertex is at least 2), and that the successor of the edge E_1 around χ_1 is E_s (i.e. there are no edges emanating from χ_1 strictly between E_1 and E_s).

By [BC02, Theorem 3.5], the indecomposable module with multiplicity μ corresponding to τ has distance $d = er$ to the side of the boundary containing E_1, where $r = m - (\mu - 1)$ or $\mu - 1$ if l is odd or even, respectively (see the remark after Lemma 4.1). Let τ' be the path arising from τ by omitting the last edge. By [BC02, Theorem 3.5], the indecomposable $\mathbf{B}$-module X corresponding to τ' with direction $(1, 1)$ (i.e. X is as in Theorem 2.1(4)) and multiplicity μ has distance $d = er' - 1$ to the side of the boundary containing E_1, where $r' = \mu - 1$ or $m - (\mu - 1)$ if l is odd or even, respectively. Since μ can vary between 2 and m, we obtain the claimed result. □

To finish the proof of Theorem 2.1(c), we show that Ω^2 sends a module of Theorem 2.1 to another such.

Lemma 4.3. *The set of modules described in Theorem 2.1 is invariant under Ω^2.*

Proof. This follows, e.g. from [Rei77, Theorem 2.4]. □

One can also use a graph theoretical counting argument to show that the number of modules described in Theorem 2.1 equals $e(2m + 1)$, if $e > 1$. This gives an alternative proof of the fact that our theorem describes all the liftable indecomposable modules.

Part (d) of Theorem 2.1 follows from the shape of the paths together with Lemma 3.4.

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