Morita equivalences between blocks of finite Chevalley groups

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1 Introduction

This is a worked out and slightly revised version of a talk I gave at the Joint Research Seminar "Representation Theory of Finite and Algebraic Groups", Osaka, March 13–19, 2000.

The motivation for the work I am going to present is Donovan's conjecture. This states that there are only finitely many blocks with a given defect group, up to Morita equivalence.

I have already given several talks on this and related topics, in particular one at Osaka in September 1998. Here I should like to report on the progress that has been made since then, but my main objective is to present a program to approach a solution to Donovan's conjecture.

I wish to thank Radha Kessar and Gunter Malle for helpful suggestions on the topics of this paper.

2 Some conjectures

Let me fix some notation which will be used throughout the paper. The set of positive integers is denoted by $\mathbb{N}$. Let p be a prime number and k an algebraically closed field of characteristic p. In the following, a *block* always means a block of a group algebra kG for some finite group G. If B is a block, we denote by $\delta(B)$ a defect group of B and by $c(B)$ the largest Cartan invariant of B.

Conjecture (D) (Donovan 1980, see [1], Conjecture M): For a fixed finite p-group P, there are only finitely many Morita equivalence classes in

$$\{B \mid B \text{ a block with } \delta(B) \cong P\}.$$

Conjecture (B) (Motivated by a question of Brauer, [3], Problem 22; see also [1], Conjecture K): Does there exist a function $f_p\colon \mathbb{N} \to \mathbb{N}$, such that $c(B) \leq f_p(|\delta(B)|)$ for all blocks B?

Remarks 2.1 (a) The truth of Conjecture (D) implies the truth of Conjecture (B) (since Morita equivalent algebras have the same Cartan matrix).

(b) Olaf Düvel [6] has reduced Conjecture (B) to quasi-simple groups (see [10] for more details).

(c) One might hope that Düvel's methods can be extended to yield a similar reduction for Donovan's conjecture.

The latter remark motivates the following variant of Donovan's conjecture.

Conjecture ($\mathbf{D^{qs}}$): For a fixed finite p-group P, there are only finitely many Morita equivalence classes in

$$\{B \mid B \text{ a block of a quasi-simple group with } \delta(B) \cong P\}.$$

3 First steps

What is known with respect to Donovan's conjecture if we restrict the class of groups among which the blocks are selected? To talk about this situation I use the following convention. If $\mathfrak{C}$ is a class of groups, I say that "(D) *is true for* $\mathfrak{C}$", if the set of Morita equivalence classes among the blocks of group algebras kG, $G \in \mathfrak{C}$, with a fixed defect group is finite. Then there are the following results.

Theorem 3.1 (Scopes [16], 1991): *Conjecture* (D) *is true for* $\{S_n \mid n \in \mathbb{N}\}$.

Using this result it is easy to prove, by a general argument, that (D) is also true for the alternating groups.

Corollary 3.2 *Conjecture* (D) *is true for* $\{A_n \mid n \in \mathbb{N}\}$.

Proof. Let B be a block of kA_n for some $n \in \mathbb{N}$ with $\delta(B) = P$. Let $\tilde{B}$ be a block of kS_n covering B with $\delta(\tilde{B}) = \tilde{P}$. Then $|P| \leq 2|\tilde{P}|$. Hence, given P, there are only finitely many possibilities for the isomorphism type of $\tilde{P}$. By Theorem 3.1 and Remark 2.1(a) there is a bound C, independent of B and n, such that $c(\tilde{B}) \leq C$. It follows from Clifford theory that $c(B) \leq 2c(\tilde{B})$ and so $c(B) \leq 2C$. By a result of Brauer and Feit [7, Theorem IV, 4.18], there are only finitely many possibilities for the Cartan matrix of B.

Every Brauer character of A_n has values in a quadratic extension of $\mathbb{Q}$. This implies that B is defined over the subfield $k_0 := \mathbb{F}_{p^2}$ of k, i.e., there is a k_0-algebra B_0 such that $B \cong k \otimes_{k_0} B_0$ and k_0 is a splitting field for B_0 (see [11, Proposition 9.2]).

Thus if A denotes the basic algebra of B, then $A \cong k \otimes_{k_0} A_0$ for some k_0-algebra A_0. Since the dimension of A is bounded above (the dimension of A is the sum of the entries of the Cartan matrix of B), there are only finitely many possibilities for the isomorphism type of A_0, and hence of A. Thus there are only finitely many possibilities for the Morita equivalence type for B. This completes the proof.

The arguments in the last two paragraphs are given in more general form in [10, Section 5] and in [11, Section 9]. Radha Kessar has informed me that she has a more conceptual proof of the above result along the lines of her original proof of Donovan's conjecture for the double covers of the alternating groups.

Theorem 3.3 (Kessar [13], 1996): *Conjecture* (D) *is true for* $\{2S_n \mid n \in \mathbb{N}\}$ *as well as for the blocks of* $k2A_n$*,* $n \in \mathbb{N}$*, containing faithful characters.*

Kessar has proved the above theorem only for odd p, but when $p = 2$ it follows from the corresponding results for S_n and A_n from results on source algebras of p-blocks of central p-extensions.

In order to prove Conjecture (D^{qs}), it suffices, by the classification of the finite simple groups and in view of Corollary 3.2 and Theorem 3.3, to consider quasi-simple groups of Lie type.

So let's turn to the finite groups of Lie type. They can be distributed into finitely many classes, some of them depending on two parameters. First of all, we have the classical groups

$$SL_n(q), SU_n(q), SO_n(q)\ (n, q \text{ odd}), Sp_n(q)\ (n \text{ even}), SO_n^{\pm}(q)\ (n \text{ even}),$$

depending on the two parameters (n, q), and we have the exceptional groups of Lie type

$$G_2(q), \ldots, E_8(q), {}^2B_2(q), {}^2G_2(q), \ldots,$$

depending only on the single parameter q, which is a prime power.

Let's write $\hat{\mathbb{P}}$ for the set of powers of primes, i.e., $\hat{\mathbb{P}} = \{r^a \mid r \text{ a prime number}, a \in \mathbb{N}\}$. By fixing either the dimension of the natural representation or the size of the underlying field, we can sort the groups in a collection of classical groups of a fixed type into infinite subsets of two kinds.

Example 3.4 For the orthogonal groups in odd dimension we have

$$\begin{aligned}\{SO_n(q) \mid n \in \mathbb{N}, q \in \hat{\mathbb{P}}, n, q \text{ odd}\} &= \bigcup_{n \in \mathbb{N} \text{ odd}} \{SO_n(q) \mid q \in \hat{\mathbb{P}} \text{ odd}\} \\ &= \bigcup_{q \in \hat{\mathbb{P}} \text{ odd}} \{SO_n(q) \mid n \in \mathbb{N} \text{ odd}\}.\end{aligned}$$

In the following, the set of groups $\{SO_n(q) \mid q \in \hat{\mathbb{P}} \text{ odd}\}$ (n a fixed odd positive integer), will be called a *series*, and a set of the form $\{SO_n(q) \mid n \in \mathbb{N} \text{ odd}\}$ ($q \in \hat{\mathbb{P}}$ odd), a *family*.

To continue with this example, let P be a finite p-group. Then, for a fixed odd integer $n \in \mathbb{N}$ write

$$\mathcal{B}_n(P) := \{B \mid B \text{ a block of } kSO_n(q) \text{ for some odd } q \in \hat{\mathbb{P}}, \delta(B) \cong P\}.$$

Similarly, for fixed odd $q \in \hat{\mathbb{P}}$, let

$$\mathcal{B}^q(P) := \{B \mid B \text{ a block of } kSO_n(q) \text{ for some odd } n \in \mathbb{N}, \delta(B) \cong P\}.$$

In general, there are infinitely many groups contributing blocks to each of the sets $\mathcal{B}_n(P)$, and $\mathcal{B}^q(P)$. For example, let p be odd and suppose that $P \cong C_p \times C_p$ is elementary abelian of order p^2. There are infinitely many odd primes q such that $q \equiv p - 1(\textbf{mod}\ p^2)$. This condition is equivalent to the fact that p divides $q + 1$ exactly to the first power. For every such q, the principal p-block of $SO_5(q)$ has defect group isomorphic to P. In particular, $\mathcal{B}_5(C_p \times C_p)$ is infinite.

More generally, if $n = 2s(s + 1) + 5$, $s = 0, 1, \ldots$, and q is as above, then $SO_n(q)$ has a p-block with defect group isomorphic to $C_p \times C_p$. Thus $\mathcal{B}_n(C_p \times C_p)$ is infinite for all such n. In turn, $\mathcal{B}^q(C_p \times C_p)$ is infinite for all such q.

4 Related conjectures

Let me now introduce some variants of Donovan's conjecture for classical groups.

Conjecture ($\mathbf{D_1}$): Fix a finite p-group P and a type G of classical group. Then there is a bound $M = M(P, G)$ such that for any q (and hence any family $\{G_n(q) \mid n\}$) the following holds:

If B is a block of $kG_n(q)$ with $\delta(B) \cong P$, then B is Morita equivalent to a block B' of $kG_m(q)$ with $\delta(B') \cong B$ for some $m \leq M$.

I next formulate a conjecture analogous to Conjecture (D_1), this time for series of classical groups rather than families. The intention is to fix the dimension of the natural representation and vary q. In order to formulate the conjecture for all groups of Lie type, not just the classical groups, I replace their natural dimension by their Lie rank.

Conjecture ($\mathbf{D_2}$): Fix a finite p-group P and an $m \in \mathbb{N}$. Let $\{G(q) \mid q\}$ be a series of quasi-simple groups of Lie type of Lie rank m. Then there is a bound $C(P, m) \in \mathbb{N}$ such that the following holds:
If B is a block of $kG(q)$ with $\delta(B) \cong P$ then $c(B) \leq C(P, m)$.

Remarks 4.1 (a) In the above conjecture $C(P, m)$ is allowed to depend on m.
(b) The truth of Conjecture (D_2) would follow from the truth of Conjecture (D) (with a bound independent of m).

I shall also need a variant of Conjecture (D_2), where the term "block" is replaced by "isolated block". This will be called Conjecture (D_2^i).

To explain the term "isolated block", let G be any finite group of Lie type. Then the set of ordinary irreducible characters of G is partitioned into so-called Lusztig series $\mathcal{E}(G, s)$, where s runs through representatives of conjugacy classes of semisimple elements of the dual group G^* of G. For example, $\mathcal{E}(G, 1)$ is the set of *unipotent characters* of G.

Now suppose that p is different from the characteristic of the field underlying G, and let $s \in G^*$ be a semisimple p'-element. Then, by a result of Broué and Michel, the set

$$\mathcal{E}_p(G, s) := \bigcup_{t \in C_{G^*}(s)_p} \mathcal{E}(G, st)$$

is a union of p-blocks of G. A p-block of G contained in $\mathcal{E}_p(G, s)$ is *unipotent* if $s = 1$, and it is *isolated*, if s is an isolated element of G^*. A semisimple element is called *isolated*, if its centralizer (in the underlying algebraic group) has the same rank as this group.

For example, if $G = SO_n(q)$ with $n \geq 3$ and q odd, then $G^* = Sp_{n-1}(q)$. The isolated elements in G^* are the semisimple elements whose eigenvalues are ± 1. The centralizer of such an element is isomorphic to $Sp_{2l}(q) \times Sp_{2m}(q)$ with $2l + 2m = n - 1$. In quasi-simple groups of type A, the unipotent blocks are the only isolated blocks.

The latter two conjectures can also be formulated in terms of decomposition numbers rather than Cartan invariants.

Conjecture ($\overline{\mathbf{D_2}}$): Let the assumptions be as in Conjecture (D_2). Then there is a bound $D(m) \in \mathbb{N}$ which does neither depend on q nor on p such that the following holds:

If B is a block of $kG(q)$ with $\delta(B) \cong P$ then the decomposition numbers of B are bounded above by $D(m)$.

Of course, Conjecture $(\overline{\mathrm{D}_2^i})$ designates the variant of Conjecture ($\overline{\mathrm{D}_2}$) where again "block" is replaced by "isolated block".

Remark 4.2 The truth of Conjecture (D_2^i) follows from the truth of Conjecture $(\overline{\mathrm{D}_2^i})$, and the theorem of Brauer and Feit [7, Theorem IV, 4.18], already mentioned above.

5 Back to Conjecture (D)

In this section I want to sketch the proof of a theorem which allows to reduce Donovan's conjecture for groups of Lie type to some of the variants described in the previous section.

Theorem 5.1 *If Conjectures* (D_1) *and* $(\overline{\mathrm{D}_2^i})$ *are true, then Donovan's conjecture is true for quasi-simple groups, i.e., Conjecture* (D^{qs}) *holds.*

Proof (Sketch). Let P be a finite p-group and let B be a block of a finite quasi-simple group with $\delta(B) \cong P$. By the remarks in Section 3 we may assume that B is a block of a group of Lie type.

Step 1. Since Conjecture (D_1) is supposed to be true, there is a bound $M'(P)$ such that B is Morita equivalent to a block with defect group P occurring in a quasi-simple group of Lie type $G(q)$ of Lie rank at most $M'(P)$.

Step 2. By a recent deep result of Bonnafé and Rouquier [2], either q is less than or equal to the Coxeter number of $G(q)$, or B is Morita equivalent to an isolated block of $G(q)$. Thus it suffices to assume that B is an isolated block of a group of Lie rank at most $M'(P)$.

Step 3. Since Conjecture $(\overline{\mathrm{D}_2^i})$ is assumed to hold, Conjecture (D_2^i) is also true by Remark 4.2. The conclusion of Step 2 together with the theorem of Brauer and

Feit cited in Remark 4.2 now imply that the Cartan matrix of B is one of finitely many matrices.

Step 4. Since B is an isolated block, there is a finite field $k_0 \subseteq k$, and a k_0-algebra B_0 such that $B \cong k \otimes_{k_0} B_0$ and k_0 is a splitting field for B_0. The degree of k_0 over its prime field is independent of B. This follows from a knowledge of the character values of Brauer characters contained in isolated blocks. (For example, in classical groups, the Brauer characters of unipotent blocks are rational). The proof is now completed exactly as the proof of Corollary 3.2.

More details concerning these arguments are given in in [11, Section 9].

6 Results on Conjectures (D_1) and (D_2)

Let Conjecture (D_1^u) denote the variant of Conjecture (D_1) obtained by replacing the word "block" by "unipotent block" throughout.

Theorem 6.1 (Jost [12]): *Conjecture* (D_1^u) *is true for* $\{GL_n(q) \mid n \in \mathbb{N}\}$.

By a result of Cabanes and Enguehard [4, Theorem 12], the above theorem implies that Conjecture (D_1^u) is also true for $\{SL_n(q) \mid n \in \mathbb{N}\}$.

A similar result holds for the other classical groups under further restrictions.

Theorem 6.2 (Hiss/Kessar [11]:) *Conjecture* (D_1^u) *is true for families of classical groups for which* p *is unitary (with some exceptions for the groups* $SO_{2m}^{\pm}(q)$*).*

The unitary primes for a classical group $G(q)$ form a subset of the primes dividing $|G(q)|$. For example, if n is odd and p is an odd prime dividing $|SO_n(q)| = q^{(n-1)^2/4}(q^2-1)(q^4-1)\cdots(q^{n-1}-1)$, then $p \mid q$ or else $p \mid q^{2e}-1$ for some $1 \leq e \leq (n-1)/2$. In the latter case either $p \mid q^e - 1$ or $p \mid q^e + 1$. If e is minimal with the property that $p \mid q^{2e}-1$, then p is linear for $G(q)$ if $p \mid q^e - 1$ and *unitary*, otherwise.

Let me now present some results towards Conjecture D_2.

Theorem 6.3 (Dipper/James [5]:) *Conjecture* $(\overline{\mathrm{D}_2})$ *is true for* $\{GL_n(q) \mid q \in \hat{\mathbb{P}}\}$.

Theorem 6.4 (Gruber/Hiss [9]:) *Conjecture* $(\overline{\mathrm{D}_2})$ *is true for series of classical groups and linear primes.*

Theorem 6.5 (Landrock/Michler [14], Okuyama/Waki [15, 17], Geck [8], White [18]:) *Conjecture* $(\overline{\mathrm{D}_2})$ *is true for* ${}^2G_2(q)$, $SU_3(q)$ *and* $Sp_4(q)$.

Let me close this article with a list of problems.

Problems 6.6 (a) Prove Conjecture (D_1) for classical groups, unitary primes and all blocks, not just the unipotent ones.

(b) Prove Conjecture (D_1) for classical groups and linear primes. Here, different methods than in the case of unitary primes are needed.

(c) Remove the exceptions in the case of the groups $SO_{2m}^{\pm}(q)$ left over in Theorem 6.2.

(d) Prove Conjecture $(\overline{\mathrm{D}_2})$ or $(\overline{\mathrm{D}_2^i})$ for all series of quasi-simple groups of Lie type.

The last of these problems appears to be the hardest one. In joint work with Radha Kessar we are trying to solve the first three of the above list.

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