SCOPES REDUCTION AND MORITA EQUIVALENCE CLASSES OF BLOCKS IN FINITE CLASSICAL GROUPS

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1. Introduction

To what extent does the defect group of a block determine its structure and that of its module category? More specifically, how are certain invariants associated to a block controlled by its defect group? Answers to these questions are indicated by conjectures of Donovan and of Puig, which postulate a very strong influence of the defect group on some of these invariants: they should be restricted down to a finite number.

To state these conjectures accurately, let K be an algebraically closed field of characteristic $\ell > 0$. By an ℓ-block, or simply a block, we mean a finite-dimensional K-algebra which is isomorphic to a block of a group algebra KG for some finite group G. The determinant of the Cartan matrix $C(B)$ of a block B is a power of ℓ. We write $\ell^{d(B)}$ for the largest invariant factor of $C(B)$ and call $d(B)$ the defect of B.

Then Donovan's conjecture can be formulated as follows.

Conjecture 1.1. (Donovan, see [1, Conjecture M]) *Up to Morita equivalence, there are only finitely many blocks with a given defect.*

Usually this is stated with the defect replaced by the defect group, but the two forms are of course equivalent. The truth of Donovan's conjecture would imply that there are only finitely many Cartan matrices of blocks of a given defect. Since the defect of a block bounds the number of its simple modules, the latter statement is equivalent to an upper bound on the Cartan invariants of a block in terms of its defect. In [2, Problem 22], Brauer essentially asked for such a bound, although the one he suggested turned out to be too optimistic (cf. [26] for an example).

In [35], Puig has extended Donovan's conjecture considerably. To each block he associates an invariant, the source algebra, which is finer than the defect or the defect group. The source algebra of a block with defect group D is an interior D-algebra. Among other information, the source algebra determines the Morita equivalence class of the block. Hence the following conjecture of Puig is stronger than Donovan's.

Conjecture 1.2. (Puig, [35]) *Given a finite ℓ-group D, there are only finitely many interior D-algebras (up to isomorphism) which are source algebras of blocks with defect group D.*

For blocks of finite representation type, Puig's conjecture is known to be true (see [28]). In general, even the answer to Brauer's question is still open. A breakthrough was achieved by Joanna Scopes in [36], where she showed that there are only finitely many Morita equivalence classes among the blocks of the symmetric groups of a given defect. This result has later been extended by Thomas Jost [21] to cover the unipotent blocks of the groups $GL_n(q)$ for a fixed q not divisible by ℓ.

In a series of papers [22, 23, 24], the second author was able to show that Puig's conjecture is true for a large class of blocks, among them the blocks occurring in the double covers of the alternating and symmetric groups, the Weyl groups, and also the groups $GL_n(q)$ for fixed q.

In this paper we show that Puig's conjecture holds for the unipotent blocks of the classical groups in the unitary prime case (Theorem 8.3). In the even-dimensional orthogonal groups we have to exclude some series of blocks related to degenerate symbols. We hope to be able to treat these cases in a subsequent paper.

According to a conjecture of Broué, with the exception of so-called isolated blocks, every block of a finite group of Lie type should be Morita equivalent to some unipotent block. Given this conjecture it seems reasonable to restrict attention to unipotent blocks. It would have been possible to prove our results for non-unipotent blocks as well, at the cost of cumbersome and tedious computations, however.

The distinction into linear and unitary primes in the representation theory of classical groups (other than the general linear groups) has already been introduced by Fong and Srinivasan. Rather than giving the exact definition here, we just mention that the unitary primes constitute about two thirds of all the primes dividing the order of a classical group. The reason why we consider only unitary primes here is quite simple: the reduction method of Scopes, also used in our paper, does not give the desired finiteness result in the linear prime case. On the other hand, Jochen Gruber and the first author have shown in [16] that the computation of decomposition numbers of classical groups in the linear prime case can be reduced to the computation of decomposition numbers of general linear groups. This reduction, via q-Schur algebras, can be used to obtain results on Morita equivalence classes of blocks of classical groups in the linear prime case as well. This will be investigated in our subsequent paper.

We should like to emphasize the great assistance by GAP [13] in obtaining the results of this paper. Through innumerable experiments with GAP, in particular with the GAP share package CHEVIE [15], we convinced ourselves that the combinatorial results on the induction of characters in Weyl groups should be true.

The proof of our results falls into two parts. The first part follows the original approach of Scopes by establishing a bijection between the ordinary irreducible characters of two blocks, which are related in a certain way. This relation is defined through the Fong-Srinivasan classification of blocks [11] by cores. The bijection is obtained combinatorially, but the combinatorics is considerably more complicated than in the case of the symmetric groups. We refer to the method used in this part of the proof as *Scopes reduction*.

In the second part we show that the bijection is afforded by a certain bimodule with favorable properties. A result of Michel Broué [3] then gives a Morita equivalence between the two blocks in question. Using a result of Scott and Puig, we even obtain that their source algebras are equivalent (up to a twist). This part of the proof is new and can also be applied to the symmetric groups and the general linear groups.

We now comment on the contents of the individual sections of the paper. Section 2 contains the combinatorial notions on partitions, symbols and associated abacus diagrams. Section 3 recalls the branching rules in Weyl groups, thereby setting up our notation for the fundamental reflections and the irreducible characters. In Section 4 we introduce the classical groups we are going to consider, their natural matrix representations as well as some of their standard Levi subgroups. In Section 5 we recall the parametrization of the ordinary irreducible characters of the classical groups and comment on the computation of Harish-Chandra induction. Section 6 describes the unipotent blocks of the classical groups, according to Cabanes and Enguehard as well as to Fong and Srinivasan.

Section 7 contains our main technical result, called the reduction theorem. Our main theorem is contained in Section 8. Finally, in Section 9 we sketch some general ideas and a plan to approach Donovan's conjecture for the finite simple groups.

2. Notation and Combinatorics

2.1. **General Notation.** Our notation is mostly standard. Let H be a group, and n a positive integer. Then $[H, H]$ denotes the commutator subgroup of H. Also, H^n stands for the direct product of n copies of H. Occasionally, n also denotes a cyclic group of order n.

Suppose that H is finite and that K is a subgroup of H. If χ and ψ are class-function of H and K, respectively, we write $\mathrm{Ind}_K^H(\psi)$ and $\mathrm{Res}_K^H(\chi)$ for the induced and restricted class-functions. Analogous notation is used for modules. Further, if χ' is another class function of H, the usual inner product between χ and χ' is denoted by $\langle \chi, \chi' \rangle_H$, where the index H is omitted if there is no danger of confusion.

If $\mathbf{G}$ is a reductive algebraic group and F a Frobenius morphism of $\mathbf{G}$, we write $G := \mathbf{G}^F$ for the finite group of F-fixed points of $\mathbf{G}$. The identity component of $\mathbf{G}$ is denoted by $\mathbf{G}^\circ$.

Let V be a vector space and $t \in GL(V)$. We then write V_t for the set of vectors of V fixed by t, and we put $[t, V] := \{tv - v \mid v \in V\}$. Then V_t and $[t, V]$ are t-invariant. Moreover, $V = V_t \oplus [t, V]$, if t is a semisimple (i.e., diagonalizable) element of $GL(V)$.

Finally, ϕ_n denotes the cyclotomic polynomial of degree n over the rationals.

2.2. **Partitions and Symbols.** A *partition* is a sequence

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \tag{1}$$

of positive integers $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. The components α_i are called the *parts of* α. We write $|\alpha| := \sum_{i=1}^k \alpha_i$ for the sum of the parts of α. If $|\alpha| = m$, we say that α is a partition of m, and write $\alpha \vdash m$. The empty sequence, sometimes denoted by $(-)$, is the unique partition of 0.

The partition (1) is usually visualized via its *Young diagram*, an array of k rows of nodes of lengths $\alpha_1, \dots, \alpha_k$. A hook of partition α is a sub-diagram of its Young diagram consisting of a particular node, the corner of a hook, all nodes in the same row to the right of the corner and all nodes in the same column below the corner. The length of a hook is the number of nodes it contains.

Let $X \subset \mathbb{N}$ be finite and let $d \in \mathbb{N}$. The *d-shift* of X is the set

$$[0, d-1] \cup (X + d) := \{0, 1, \dots, d-1\} \cup \{x + d \mid x \in X\}.$$

A subset $\{x_1, x_2, \dots, x_m\}$ with $x_1 < x_2 \cdots < x_m$ of the non-negative integers is called a *β-set* for the partition (1), if

$$\alpha_i = x_{m-i+1} - m + i, \text{for } 1 \leq i \leq m$$

(with the convention that $\alpha_i = 0$ for $i = k+1, \dots, m$). Two β-sets for the same partiton are equivalent in the sense that, for some d, one is the d-shift of the other. There is a bijection between partitions and equivalence classes of β-sets, mapping the partition (1) to the β-set containing $\{\alpha_k, \alpha_{k-1}+1, \dots, \alpha_1+k-1\}$, the set of the hook lengths of the first column of the Young diagram of α. For example, $\{2, 5, 6, 10\}$ is a β-set for the partition $(7, 4, 4, 2)$.

Let X be a β-set for the partition α. There is a bijection between the set of hooks of α of length $t > 0$, and the set of pairs of integers (y, x) satisfying $0 \leq y < x$, $x \in X$, $y \notin X$, and $t = x - y$. Let (y, x) be a hook of α of length t. The partition α' with β-set $X' := (X \setminus \{x\}) \cup \{y\}$ is said to be obtained from α by *removing a t-hook*. The Young diagram of α' is obtained by removing the hook corresponding to (y, x), and moving the nodes to the south-east of this hook one position to the north-west. There is a dual notion of *adding a t-hook* to α.

A *symbol* $\Lambda = \{X, Y\}$ is an unordered pair of sets X and Y of non-negative integers. It is called *degenerate*, if $X = Y$. We consider the equivalence relation on the set of symbols given by simultaneous d-shift for non-negative integers d. That is, two symbols $\Lambda = \{X, Y\}$ and $\Lambda' = \{X', Y'\}$ are equivalent if there is a non-negative integer d such that

$$X' = [0, d-1] \cup (X + d) \quad \text{and} \quad Y' = [0, d-1] \cup (Y + d)$$

or

$$X = [0, d-1] \cup (X' + d) \quad \text{and} \quad Y = [0, d-1] \cup (Y' + d).$$

Let $\lambda = \{X, Y\}$ be a symbol. Then

$$||X| - |Y||$$

is called the *defect* and

$$\sum_{x \in X} x + \sum_{y \in Y} y - \left\lfloor \left(\frac{|X| + |Y| - 1}{2} \right)^2 \right\rfloor$$

the *rank* of λ, denoted by $\operatorname{rank}(\lambda)$. It is clear that the rank and the defect are constant on equivalence classes of symbols.

A hook ν of $\lambda = \{X, Y\}$ is a pair (y, x) of integers where $0 \leq y < x$ and

$$y \notin X \text{ (respectively } Y) \quad \text{and} \quad x \in X \text{ (respectively } Y).$$

If $x - y$ equals t we say that ν has length t or that ν is a t-hook. The symbol obtained by deleting x from X (respectively Y) and replacing it by y is said to be gotten from λ by removing ν.

A cohook ν of λ is a pair (y, x) of integers where $0 \leq y < x$ and

$$y \notin Y \text{ (respectively } X) \quad \text{and} \quad x \in X \text{ (respectively } Y).$$

Again, if $x - y$ equals t we say that ν has length t or that ν is a t-cohook. The symbol obtained by deleting x from X (respectively Y) and adding y to Y (respectively X) is said to be gotten from λ by removing ν.

2.3. **Abacus Diagrams.** Fix a positive integer e. By an e-abacus we will mean an abacus with e strings, numbered (from left to right) by $0, 1, \cdots, e-1$, and with rows numbered (from top to bottom) by $0, 1, 2, \dots$

For any β-set X, and hence for any partition, there is an associated e-abacus diagram Λ which is obtained by putting beads on some positions of the e-abacus according to the following rule. For integers t and i, $0 \leq t$, $0 \leq i \leq e-1$, there is a bead on string i and row t of Λ, if and only if $i + te$ is an element of X.

For an e-abacus diagram Λ and an integer i with $0 \leq i \leq e-1$, we write Λ_e^i for the set of positions on string i of Λ, which are occupied by beads. If Λ is a diagram for a β-set X, then Λ_e^i corresponds to the set of elements of X which are congruent to i modulo e. We put

$$w(\Lambda_e^i) := \sum_{x \in \Lambda_e^i} |\{y \in \mathbb{Z} \mid 0 \leq y < x, y \notin \Lambda_e^i\}|. \tag{2}$$

This is the number of moves necessary to move all beads on string i of Λ up as far as possible.

Let λ be a partition and Λ an e-abacus diagram associated to λ. We define

$$w_e(\lambda) := w_e(\Lambda) := \sum_{i=0}^{e-1} w(\Lambda_e^i).$$

This is independent of Λ, and is called the *e-weight of λ*. Furthermore, we put

$$s_e^i(\Lambda) := \begin{cases} |\Lambda_e^0| - |\Lambda_e^{e-2}| - 1, & \text{if } i = 0, \\ |\Lambda_e^1| - |\Lambda_e^{e-1}| - 1, & \text{if } i = 1, \\ |\Lambda_e^i| - |\Lambda_e^{i-2}|, & \text{if } 2 \leq i \leq e-1. \end{cases}$$

We finally write

$$s_e(\lambda) := \max\{s_e^i(\Lambda) \mid 0 \leq i \leq e-1\}.$$

Again, this is independent of Λ.

By a 2e-abacus we will mean an abacus with $2e$ strings, which are numbered (from left to right) by $0_g, 1_g, \ldots, (e-1)_g, 0_b, 1_b, \ldots, (e-1)_b$. The rows of the abacus are numbered (from top to bottom) by $0, 1, 2, \ldots$ To each symbol $\lambda = \{X, Y\}$, we associate two types of abacus diagrams called the 2e-linear diagram and the 2e-unitary diagram of λ, respectively. These are defined in the following way. First, we associate one of X or Y to g and the other to b, say we associate X to g and Y to b.

A 2e-linear diagram for λ is obtained by putting a bead on row t and string i_g of the abacus if and only if $i + te$ is an element of X, and a bead on row t and string i_b of the abacus if and only if $i + te$ is an element of Y. Thus in a 2e-linear diagram all elements of X go to the g-strings, whereas all elements of Y go to the b-strings.

A 2e-unitary diagram for the symbol $\lambda = \{X, Y\}$ is obtained as follows. For any non-negative integer t and any integer i between 0 and $e-1$, we put a bead on row t and string i_g of the abacus if and only if t is even and $i + te$ is an element of X, or t is odd and $i + te$ is an element of Y. Also, we put a bead on row t and string i_b of the abacus if and only if t is even and $i + te$ is an element of Y, or t is odd and $i + te$ is an element of X. Thus in a 2e-unitary diagram, the elements of X go to the even-numbered rows of the g-strings and to the odd-numbered rows of the b-strings. Note that if instead we associate Y to g and X to b, the diagram we get can be obtained from the first one by swapping the strings i_g and i_b for all i, $0 \leq i \leq e-1$.

Conversely, any 2e abacus diagram can be associated to a symbol for which it is the 2e-linear diagram as well as to a symbol for which it is the 2e-unitary diagram.

Clearly, if two diagrams are obtained from one another by swapping the strings i_g and i_b for all i, $0 \leq i \leq e-1$, then they give rise to the same symbol.

Note that if Λ is a 2e-linear or a 2e-unitary diagram of a degenerate symbol λ, then there is a bead on row t of string i_g of Λ if and only if there is a bead on row t of string i_b.

Example 2.3. Let $X = \{1,3,7\}$, $Y = \{0,2\}$ and let λ be the symbol $\{X,Y\}$. The defect of λ is 1 and the rank is 9. Let $e = 3$. The following is a 2e-linear diagram for λ (we associate X to g and Y to b, and we draw the beads corresponding to elements of X and Y as $\circ$ and $\bullet$, respectively).

$$\begin{array}{ccccccc} & 0_g & 1_g & 2_g & 0_b & 1_b & 2_b \\ 0 & \cdot & \circ & \cdot & \bullet & \cdot & \bullet \\ 1 & \circ & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & \circ & \cdot & \cdot & \cdot & \cdot \end{array}$$

Similarly, a 2e-unitary diagram for λ is the following.

$$\begin{array}{ccccccc} & 0_g & 1_g & 2_g & 0_b & 1_b & 2_b \\ 0 & \cdot & \circ & \cdot & \bullet & \cdot & \bullet \\ 1 & \cdot & \cdot & \cdot & \circ & \cdot & \cdot \\ 2 & \cdot & \circ & \cdot & \cdot & \cdot & \cdot \end{array}$$

Removing or adding a hook or a cohook to a symbol λ translates to moving a bead on a 2e-abacus diagram for λ to an empty spot on the abacus. Since we shall be interested in the removal and addition of 1-hooks as well as e-hooks and e-cohooks, we explain the effect this has on the corresponding 2e-diagrams.

For instance, if Λ is the 2e-linear diagram for λ, and λ' is obtained from λ by removing a 1-hook, then we can obtain a 2e-linear diagram for λ' by making one of the following moves on Λ:

(i) For $i \geq 1$, move a bead on row t and string i_g one position to the left.

(ii) For $i \geq 1$, move a bead on row t and string i_b one position to the left.

(iii) Move a bead on row t ($t > 0$) and string 0_g to row $t-1$ and string $(e-1)_g$.

(iv) Move a bead on row t ($t > 0$) and string 0_b to row $t-1$ and string $(e-1)_b$.

Conversely making any such move on Λ gives a 2e-linear diagram for a symbol which is obtained by removing a 1-hook from λ.

Similary, if Λ is the 2e-unitary diagram for λ, and λ' is obtained from λ by removing a 1-hook, then we can obtain the 2e-unitary diagram for λ' by making either move (i) or (ii) as above or by doing the following:

(v) Move a bead on row t ($t > 0$) and string 0_g to row $t-1$ and string $(e-1)_b$.

(vi) Move a bead on row t ($t > 0$) and string 0_b to row $t-1$ and string $(e-1)_g$.

Again, making any such move on Λ gives the 2e-unitary diagram for a symbol which is obtained by removing a 1-hook from λ.

Also notice that removing (or adding) an e-hook from λ translates into moving a bead one position up (or down) on a 2e-linear diagram of λ and removing (or adding) an e-cohook from λ translates into moving a bead one position up (or down) on a 2e-unitary diagram of λ.

For a 2e-abacus diagram Λ and an integer i with $0 \leq i \leq e-1$, write $\Lambda_e^{i_g}$ for the set of beads on string i_g of Λ and write $\Lambda_e^{i_b}$ for the set of beads on string i_b of Λ. The numbers $w(\Lambda_e^{i_g})$ and $w(\Lambda_e^{i_b})$ ared defined by (2), with i replaced by i_g and i_b, respectively. The e-weight of Λ is denoted $w_e(\Lambda)$ and is defined by

$$w_e(\Lambda) := \sum_{0 \leq i \leq e-1} w(\Lambda_e^{i_g}) + \sum_{0 \leq i \leq e-1} w(\Lambda_e^{i_b}).$$

The core of an abacus diagram Λ is defined to be the diagram obtained from Λ by moving all beads up as far as possible on their strings. The number of moves required to obtain the core equals $w(\Lambda)$.

Now let λ be a symbol. We define the e-linear weight of λ to be the weight of a 2e-linear diagram of λ and denote it by $\check{w}_e(\lambda)$. The e-linear core of λ is defined to be the symbol which has as 2e-linear diagram the core of a 2e-linear diagram of λ. Similarly, the e-unitary weight of λ is defined to be the weight of a 2e-unitary diagram of λ. It is denoted by $\check{w}_e(\lambda)$. The e-unitary core of λ is defined to be the symbol which has as 2e-unitary diagram the core of a 2e-unitary diagram of λ. Clearly, the e-linear and e-unitary weights of λ are invariants of the equivalence class of λ. Also, equivalent symbols have equivalent e-linear and e-unitary cores.

Again, let Λ be a 2e-abacus diagram. We let

$$\check{s}_e^{i_g}(\Lambda) := \begin{cases} |\Lambda_e^{0_g}| - |\Lambda_e^{(e-1)_g}| - 1, & \text{if } i = 0, \\ |\Lambda_e^{i_g}| - |\Lambda_e^{(i-1)_g}|, & \text{if } 1 \leq i \leq e-1. \end{cases}$$

Similarly,

$$\check{s}_e^{i_b}(\Lambda) := \begin{cases} |\Lambda_e^{0_b}| - |\Lambda_e^{(e-1)_b}| - 1, & \text{if } i = 0, \\ |\Lambda_e^{i_b}| - |\Lambda_e^{(i-1)_b}|, & \text{if } 1 \leq i \leq e-1. \end{cases}$$

We also define

$$\hat{s}_e^{i_g}(\Lambda) := \begin{cases} |\Lambda_e^{0_g}| - |\Lambda_e^{(e-1)_b}| - 1, & \text{if } i = 0, \\ |\Lambda_e^{i_g}| - |\Lambda_e^{(i-1)_g}|, & \text{if } 1 \leq i \leq e-1. \end{cases}$$

$$\hat{s}_e^{i_b}(\Lambda) := \begin{cases} |\Lambda_e^{0_b}| - |\Lambda_e^{(e-1)_g}| - 1, & \text{if } i = 0, \\ |\Lambda_e^{i_b}| - |\Lambda_e^{(i-1)_b}|, & \text{if } 1 \leq i \leq e-1, \end{cases}$$

and

$$\check{s}_e(\Lambda) := \max\{\check{s}_e^{i_g}(\Lambda), \check{s}_e^{i_b}(\Lambda) \mid 0 \leq i \leq e-1\},$$

$$\hat{s}_e(\Lambda) := \max\{\hat{s}_e^{i_g}(\Lambda), \hat{s}_e^{i_b}(\Lambda) \mid 0 \leq i \leq e-1\}.$$

Finally, if λ is a symbol, we put

$$\check{s}_e(\lambda) := \check{s}_e(\Lambda_1),$$

and

$$\hat{s}_e(\lambda) := \hat{s}_e(\Lambda_2),$$

where Λ_1 is a 2e-linear diagram and Λ_2 a 2e-unitary diagram for λ. Clearly, $\check{s}_e(\lambda)$ and $\hat{s}_e(\lambda)$ are invariants of the equivalence class of λ.

3. Branching Rules In Weyl Groups

We fix non-negative integers m and k, with $0 \leq k \leq m$.

3.1. **Type A.** Partitions arise in the representation theory of the classical Weyl groups. For example, if S_m denotes the symmetric group on the set $\{1, 2, \dots, m\}$ (with the convention that S_0 is the trivial group), the ordinary irreducible representations of S_m are labeled by the partitions of m (see, for example [20, Section 2.1]). Let ζ^α denote the irreducible character of S_m labeled by $\alpha \vdash m$.

We write $S_{m-k} \times S_k$ for the subgroup of S_m fixing $\{1, \dots, m-k\}$. For partitions α, β, and γ of $m, m-k$, and k, respectively, let $g^\alpha_{\beta\gamma}$ denote the multiplicity of $\zeta^\beta \times \zeta^\gamma$, an irreducible character of $S_{m-k} \times S_k$, in the restriction of ζ^α. This *Littlewood-Richardson coefficient* $g^\alpha_{\beta\gamma}$ can be computed by a purely combinatorial rule, the *Littlewood-Richardson rule* [20, 2.8.13]. For example, if $k = 1$, this rule describes the restriction of the irreducible characters of S_m to the subgroup S_{m-1} fixing m. In this case, $\gamma = (1)$, and $g^\alpha_{\beta,(1)} \leq 1$. Moreover, $g^\alpha_{\beta,(1)} = 1$ if and only if the Young diagram of β is obtained from that of α by removing a node, i.e., a 1-hook. Frobenius reciprocity gives an analogous description for the induction of characters from S_{m-1} to S_m. These facts are known as *branching rules.*

3.2. **Type B.** Now let W_m denote the Weyl group of type B_m with Dynkin diagram labeled as follows.

(3)

The ordinary irreducible characters of W_m are parametrized by bi-partitions of m. (A bi-partition of m is an ordered pair of partitions whose parts sum up to m.) For example, the trivial character and the sign-character are labeled by $((m), (-))$ and $((-), (m))$, respectively. We write χ^α for the irreducible character of W_m labeled by the bi-partition α.

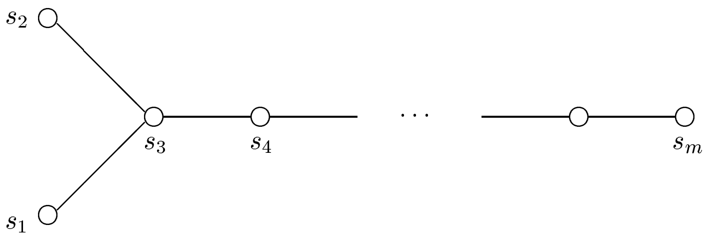
Let $W_{m-k} \times S_k$ denote the parabolic subgroup of W_m generated by $\{s_1, \dots, s_m\} \setminus \{s_{m-k+1}\}$. ($W_0$ is the trivial group.) We are interested in the restriction of the irreducible characters of W_m to $W_{m-k} \times S_k$. Let $\alpha = (\alpha^0, \alpha^1)$ and $\beta = (\beta^0, \beta^1)$ be bi-partitions of m and $m-k$, respectively, and let γ be a partition of k. Put $j := |\alpha^0| - |\beta^0|$. Then the multiplicity of $\chi^\beta \times \zeta^\gamma$ in the restriction of χ^α to $W_{m-k} \times S_k$ is equal to 0, if $j < 0$ or $j > k$. Otherwise this multiplicity is

$$\sum_{\delta^0 \vdash j} \sum_{\delta^1 \vdash k-j} g^{\alpha^0}_{\beta^0 \delta^0} g^{\alpha^1}_{\beta^1 \delta^1} g^{\gamma}_{\delta^0 \delta^1}. \tag{4}$$

This formula can be derived from [33, Theorem 6] and a further application of the Littlewood-Richardson rule for symmetric groups (see also [34, Lemma 3.5.2]).

Again, we consider the special case of $k = 1$. In this case we have $\gamma = (1)$ and the multiplicity of χ^β in the restriction of χ^α to W_{m-1} is at most 1. Moreover, χ^β occurs in the restriction of χ^α if and only if β^0 is obtained from α^0 by removing a node (and $\beta^1 = \alpha^1$) or β^1 is obtained from α^1 by removing a node (and $\beta^0 = \alpha^0$). Using Frobenius reciprocity, we can rephrase this *branching rule for Weyl groups of type B_m* as follows: The constituents of the induced character $\mathrm{Ind}_{W_{m-1}}^{W_m}(\chi^\beta)$ are the characters χ^α, where α is obtained from $\beta = (\beta^0, \beta^1)$ by adding a node either to β^0 or to β^1.

3.3. **Type D.** Suppose that $m \geq 2$ and let $\tilde{W}_m$ denote the Weyl group of type D_m with Dynkin diagram



(5)

We view $\tilde{W}_m$ as the subgroup of W_m of index 2 generated by $s_1 s_2 s_1$ and $s_2, \ldots, s_m$. Restricting an irreducible character χ^α from W_m to $\tilde{W}_m$ either results in an irreducible character, or in the sum of two distinct irreducible characters. The first case occurs if the two components α^0 and α^1 of α are distinct. In this case, $\chi^{\alpha'}$ has the same restriction to $\tilde{W}_m$ as χ^α, where $\alpha' := (\alpha^1, \alpha^0)$. The second case occurs if m is even and α is of the form $\alpha = (\beta, \beta)$. Thus the irreducible characters of $\tilde{W}_m$ are labeled by unordered pairs of partitions whose parts sum up to m, such that pairs with equal components label two irreducible characters (see [30, 4.6]). We write $\tilde{\chi}^\alpha$ for the irreducible character(s) of $\tilde{W}_m$ labeled by α.

From the embedding described above and the branching rule for Weyl groups of type B, we can now easily derive a branching rule for Weyl groups of type D. Assume that $k \leq m-2$, and write $\tilde{W}_{m-k} \times S_k$ for the parabolic subgroup of $\tilde{W}_m$ generated by $\{s_1, \ldots, s_m\} \setminus \{s_{m-k+1}\}$.

Again, we consider the special case of $k = 1$. Let $\alpha = (\alpha^0, \alpha^1)$ and $\beta = (\beta^0, \beta^1)$ be unordered pairs of partitions whose parts sum up to m and $m-1$, respectively. Then $\tilde{\chi}^\alpha$ occurs in the character $\mathrm{Ind}_{\tilde{W}_{m-1}}^{\tilde{W}_m}(\tilde{\chi}^\beta)$ if and only if α is obtained from β by adding a node to one of β^0 or β^1. If α has equal components, either both irreducible characters corresponding to α occur, or non of them. If β has equal components, the two irreducible characters of $\tilde{W}_{m-1}$ corresponding to β induce to the same character of $\tilde{W}_m$. The dual rule for the restriction of characters can be obtained from Frobenius reciprocity.

4. The Classical Groups

Let p be a prime number, q a power of p, and $\mathbb{F}_q$ the finite field with q elements.

4.1. **The Groups.** For a non-negative integer n, we let $G_n(q)$ denote one of the following groups.

(1) The unitary group $GU_n(q)$ of an n-dimensional vector space over $\mathbb{F}_{q^2}$ with a non-degenerate unitary form.
(2) The special orthogonal group $SO_n(q)$ of an n-dimensional vector space over $\mathbb{F}_q$ with a quadratic form of maximal Witt index, where n and q are odd.
(3) The conformal symplectic group $CSp_n(q)$ of an n-dimensional vector space over $\mathbb{F}_q$ with a non-degenerate symplectic form, where n is even.
(4) The conformal special orthogonal group $CSO_n^+(q)$ of an n-dimensional vector space over $\mathbb{F}_q$ with a quadratic form of maximal Witt index, where n is even and q is odd.

(5) The conformal special orthogonal group $CSO_n^-(q)$ of an n-dimensional vector space over $\mathbb{F}_q$ with a quadratic form of Witt index $n/2-1$, where n is even and q is odd.
(6) The special orthogonal group $SO_n^+(q)$ of a quadratic space as in (4), where q is even.
(7) The conformal special orthogonal group $CSO_n^-(q)$ of a quadratic space as in (5), where q is even.

By convention, $G_0(q)$ denotes the trivial group. We also introduce the parameter $\delta = \delta(G_n(q))$ by letting $\delta = 2$, if $G_n(q) = GU_n(q)$, and $\delta = 1$, otherwise; δ will have this meaning for the remainder of our paper.

4.2. **Matrix Representations.** In order to describe these groups and some of their properties, needed later on, more explicitly, we introduce their natural matrix representations. At the same time we describe the groups as finite reductive groups.

Matrices are usually denoted by boldface lower case letters, the transposed of a matrix $\mathbf{a}$ is written as $\mathbf{a}^t$. If n is a non-negative integer, E_n denotes the identity matrix of degree n, and J_n the $(n\times n)$-matrix with ones along the anti-diagonal and zeroes otherwise. (With the convention that E_0 and J_0 denote the empty matrix.)

We fix an algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$. For every positive integer n, we denote by F_0 the standard Frobenius endomorphism of $GL_n(\bar{\mathbb{F}}_q)$ obtained by raising the matrix entries to the qth power. Thus $F_0([a_{ij}]) = [a_{ij}^q]$ for $[a_{ij}] \in GL_n(\bar{\mathbb{F}}_q)$. This notation is extended to $GL(\mathbf{V})$ for finite dimensional vector spaces $\mathbf{V}$ over $\bar{\mathbb{F}}_q$, if a basis of $\mathbf{V}$ has been chosen.

4.2.1. Let V be a finite-dimensional vector space over a field K endowed with a non-degenerate form $\mathbf{f}$, which is either a hermitean form, a symplectic or symmetric bilinear form, or a quadratic form. We denote by $I(V) := I(V,\mathbf{f})$ the group of isometries of $(V,\mathbf{f})$ and by $I_0(V)$ the intersection of $I(V)$ with $SL(V)$.

If $\dim_K(V) = 2m$ is even we also consider the group of conformal transformations $J(V) := J(V,\mathbf{f})$ of $(V,\mathbf{f})$. By definition, $J(V) = \{\mathbf{x} \in GL(V) \mid \text{ there is } \lambda_{\mathbf{x}} \in K^* \text{ with } \mathbf{f}(\mathbf{x}v,\mathbf{x}w) = \lambda_{\mathbf{x}}\mathbf{f}(v,w), \text{ for all } v,w \in V\}$, if $\mathbf{f}$ is a bilinear form, and $J(V) = \{\mathbf{x} \in GL(V) \mid \text{ there is } \lambda_{\mathbf{x}} \in K^* \text{ with } \mathbf{f}(\mathbf{x}v) = \lambda_{\mathbf{x}}\mathbf{f}(v), \text{ for all } v \in V\}$, if $\mathbf{f}$ is a quadratic form. The scalar $\lambda_{\mathbf{x}}$ is uniquely determined by $\mathbf{x} \in J(V)$ and is called the *multiplier* of $\mathbf{x}$.

4.2.2. Let $\mathbf{V}$ be a finite-dimensional vector space over $\bar{\mathbb{F}}_q$ with (ordered) basis $e_1,\dots,e_n$ and put V to be the $\mathbb{F}_{q^2}$-span of $e_1,\dots,e_n$. Define the hermitean form $\mathbf{f}$ on V by

$$\mathbf{f}(\sum_{i=1}^n a_ie_i, \sum_{i=1}^n b_ie_i) := \sum_{i=1}^n a_ib_{n-i+1}^q.$$

Then $I(V)$ is isomorphic to the general unitary group $GU_n(q)$ over $\mathbb{F}_{q^2}$, i.e., to the matrix group

$$GU_n(q) = \{\mathbf{a} \in GL_n(q^2) \mid \mathbf{a}^tJ_nF_0(\mathbf{a}) = J_n\}.$$

Thus $GU_n(q)$ is the group of fixed points of the Frobenius morphism F of the algebraic group $GL_n(\bar{\mathbb{F}}_q)$ with $F(\mathbf{a}) = J_n^{-1}\left(F_0(\mathbf{a})^t\right)^{-1}J_n$.

4.2.3. Suppose that p is odd. Let $n = 2m+1$ be odd and let $\mathbf{V}$ be an n-dimensional vector space over $\bar{\mathbb{F}}_q$ with (ordered) basis $e_1, \dots, e_m, e_0, e'_m, \dots, e'_1$, endowed with the symmetric bilinear form $\mathbf{f}$ defined by $\mathbf{f}(e_0, e_0) = 1$, $\mathbf{f}(e_0, e_i) = \mathbf{f}(e_0, e'_i) = 0$ for $1 \leq i \leq m$, $\mathbf{f}(e_i, e_j) = \mathbf{f}(e'_i, e'_j) = 0$, for $1 \leq i, j \leq m$ and $\mathbf{f}(e_i, e'_j) = \delta_{ij}$.

Put $\mathbf{G} := I_0(\mathbf{V})$, and $F = F_0$. Then $\mathbf{G}$ is isomorphic to the special orthogonal group $SO_n(\bar{\mathbb{F}}_q)$, and $G := \mathbf{G}^F$ is isomorphic to $SO_n(q)$.

The groups $\mathbf{G}$ and G and may be explicitly described in matrix notation as follows. Let us denote by $\mathbf{A}$ the matrix

$$\begin{bmatrix} 0 & 0 & J_m \\ 0 & 1 & 0 \\ J_m & 0 & 0 \end{bmatrix}.$$

Then

$$\mathbf{G} \cong \{\mathbf{x} \in SL_{2m+1}(\bar{\mathbb{F}}_q) \mid \mathbf{x}^t\mathbf{A}\mathbf{x} = \mathbf{A}\}$$

and

$$G \cong \{\mathbf{x} \in SL_{2m+1}(q) \mid \mathbf{x}^t\mathbf{A}\mathbf{x} = \mathbf{A}\}.$$

4.2.4. Let $n = 2m$ be even and let $\mathbf{V}$ be an n-dimensional vector space over $\bar{\mathbb{F}}_q$ with (ordered) basis $e_1, \dots, e_m, e'_m, \dots, e'_1$, endowed with the symplectic bilinear form $\mathbf{f}$ defined by $\mathbf{f}(e_i, e_j) = 0 = \mathbf{f}(e'_i, e'_j)$, for $1 \leq i, j \leq m$ and $\mathbf{f}(e_i, e'_j) = \delta_{ij}$. Put $\mathbf{G} := J(\mathbf{V})$, and $F = F_0$. Then $\mathbf{G}$ is isomorphic to the conformal symplectic group $CSp_{2m}(\bar{\mathbb{F}}_q)$, and $G := \mathbf{G}^F$ is isomorphic to $CSp_{2m}(q)$.

Let $\mathbf{A}$ be the matrix

$$\begin{bmatrix} 0 & J_m \\ -J_m & 0 \end{bmatrix}.$$

Then

$$\mathbf{G} \cong \{\mathbf{x} \in GL_{2m}(\bar{\mathbb{F}}_q) \mid \mathbf{x}^t\mathbf{A}\mathbf{x} = \lambda_{\mathbf{x}}\mathbf{A} \text{ for some } \lambda_{\mathbf{x}} \in \bar{\mathbb{F}}_q\}$$

and

$$G \cong \{\mathbf{x} \in GL_{2m}(q) \mid \mathbf{x}^t\mathbf{A}\mathbf{x} = \lambda_{\mathbf{x}}\mathbf{A} \text{ for some } \lambda_{\mathbf{x}} \in \mathbb{F}_q\}.$$

4.2.5. Let $n = 2m$ be even and let $\mathbf{V}$ be an n-dimensional vector space over $\bar{\mathbb{F}}_q$ with (ordered) basis $e_1, \dots, e_m, e'_m, \dots, e'_1$, endowed with the quadratic form $\mathbf{f}$ defined by

$$\mathbf{f}(\sum_{i=1}^m x_i e_i + x'_i e'_i) = \sum_{i=1}^m x_i x'_i.$$

Let $F = F_0$. If q is odd, put $\mathbf{G} := J(\mathbf{V})^\circ = \{\mathbf{x} \in J(\mathbf{V}) \mid \det \mathbf{x} = \lambda_{\mathbf{x}}^m\}$. Then $\mathbf{G}$ is isomorphic to the conformal special orthogonal group $CSO_{2m}(\bar{\mathbb{F}}_q)$, and $G := \mathbf{G}^F$ is isomorphic to $CSO^+_{2m}(q)$. If q is even, put $\mathbf{G} := I(\mathbf{V})^\circ$. Then $\mathbf{G}$ is isomorphic to the special orthogonal group $SO_{2m}(\bar{\mathbb{F}}_q)$, and $G := \mathbf{G}^F$ is isomorphic to $SO^+_{2m}(q)$.

4.2.6. Let $\mathbf{V}$ and $\mathbf{G}$ be as in 4.2.5 and suppose that $m \geq 1$. Let ϵ be the automorphism of $\mathbf{V}$ which swaps e_m with e'_m and fixes all other basis vectors. Then conjugation with ϵ induces the graph automorphism of $\mathbf{G}$. Put $F := F_0 \circ \operatorname{ad} \epsilon$. Then F is a Frobenius morphism of $\mathbf{G}$ and $G := \mathbf{G}^F$ is isomorphic to the conformal special orthogonal group $CSO^-_{2m}(q)$, if q is odd, and to the special orthogonal group $SO^-_{2m}(q)$, if q is even.

4.2.7. Let $(\mathbf{V}, \mathbf{f})$ be one of the linear or quadratic spaces defined above. For an integer k, $1 \leq k < n/2$, let $\mathbf{V}_{n-2k}$ be the subspace of $\mathbf{V}$ spanned by the vectors $e_{k+1}, \cdots, e_{n-k}$ in Case 4.2.2, by the vectors $e_{k+1}, \cdots, e_m, e_0, e'_m, \cdots, e'_{k+1}$ in Case 4.2.3, and by the vectors $e_{k+1}, \cdots, e_m, e'_m, \cdots, e'_{k+1}$ in the remaining cases. Put $\mathbf{G}_{n-2k} := \mathbf{G}(\mathbf{V}_{n-2k})$, where $\mathbf{G}(\mathbf{V}_{n-2k})$ denotes the group induced on $\mathbf{V}_{n-2k}$ by the stabilizer of the decomposition $\langle e_1, \ldots, e_k \rangle \oplus \mathbf{V}_{n-2k} \oplus \langle e'_k, \ldots, e'_1 \rangle$ (with the convention $e'_i := e_{n-i+1}$, $1 \leq i \leq k$, in Case 4.2.2). Then $\mathbf{G}^F_{n-2k} \cong G_{n-2k}(q)$.

4.3. BN**-Pairs and Weyl Groups.** We have described each of the finite classical groups in 4.2.2–4.2.6 as groups of the form $G = \mathbf{G}^F$, where $\mathbf{G}$ is a connected reductive algebraic group with connected center, and F a Frobenius morphism of $\mathbf{G}$. We next describe split BN-pairs and the Weyl groups for the finite groups G. We assume $n \geq 2$, and $n \geq 3$, in case $\mathbf{G}$ is an orthogonal group.

Let $\mathbf{T}$ be the intersection of $\mathbf{G}$ with the subgroup of diagonal matrices of $GL(\mathbf{V})$, $\mathbf{B}$ the intersection of $\mathbf{G}$ with the subgroup of upper triangular matrices of $GL(\mathbf{V})$, and $\mathbf{N}$ the intersection of $\mathbf{G}$ with the group of monomial matrices of $GL(\mathbf{V})$. Then $\mathbf{B}$ and $\mathbf{N}$ form an F-stable BN-pair for $\mathbf{G}$ containing the F-stable maximal torus $\mathbf{T}$. Let $\mathbf{W} = \mathbf{N}/\mathbf{T}$, the Weyl group of $\mathbf{G}$. Taking F-fixed points yields a split BN-pair for G.

Let $m = \lfloor n/2 \rfloor$. In Cases 4.2.2, 4.2.3 and 4.2.4, the Weyl group $W := \mathbf{W}^F$ of G is of type B_m. Using the notation of the Dynkin diagram (3), the fundamental reflections are represented by the matrices:

$$s_1 = \begin{bmatrix} E_{m-1} & 0 & 0 \\ 0 & \tilde{J} & 0 \\ 0 & 0 & E_{m-1} \end{bmatrix},$$

and

$$s_i = \begin{bmatrix} E_{m-i} & 0 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 & 0 \\ 0 & 0 & \tilde{E} & 0 & 0 \\ 0 & 0 & 0 & J_2 & 0 \\ 0 & 0 & 0 & 0 & E_{m-i} \end{bmatrix}, \tag{6}$$

$2 \leq i \leq m$. Here, $\tilde{J} = J_2$, if n is even, $\tilde{J} = J_3$ in Case 4.2.2, and

$$\tilde{J} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

in Case 4.2.3. Also, $\tilde{E}$ equals E_{2i-4} for even n, and E_{2i-3} for odd n.

In Case 4.2.5, the Weyl group $W = \mathbf{W}^F$ is of type D_m. Using the notation of the Dynkin diagram (5), the fundamental reflection s_1 is represented by the matrix

$$s_1 = \begin{bmatrix} E_{m-2} & 0 & 0 & 0 \\ 0 & 0 & E_2 & 0 \\ 0 & E_2 & 0 & 0 \\ 0 & 0 & 0 & E_{m-2} \end{bmatrix},$$

and $s_2, \cdots, s_m$ by the matrices (6) with $\tilde{E} = E_{2i-4}$, $2 \leq i \leq m$.

In Case 4.2.6, $\mathbf{W}$ is as in Case 4.2.5, but now F induces the non trivial automorphism of the root system of type D_n, i.e., it interchanges s_1 and s_2, and so $W = \mathbf{W}^F$, the Weyl group of G, is of type B_{m-1}, with generators $s_1s_2, s_3, \ldots, s_{m-1}, s_m$.

4.4. **Some Levi Subgroups.** In this section the term *Levi subgroup* is used as in the theory of (algebraic) groups with split BN-pairs satisfying the commutator relations (see, for example [7, Sections 2.5 and 2.6]). In the more general notion of d-split Levi subgroups of algebraic groups introduced by Broué, Malle and Michel in [5, Section 1], the Levi subgroups considered here are (the F-fixed points of) 1-split Levi subgroups.

Let $G = G_n(q)$ be one of the groups of 4.1, where we impose the same conditions on n as in 4.3. Again, we put $m = \lfloor n/2 \rfloor$, and fix an integer k with $1 \leq k \leq m$ ($k \geq 2$ in case of the even-dimensional orthogonal groups, and $k \leq m-1$ for those of Witt index $m-1$). Let S denote the set of fundamental reflections of the Weyl group W of G.

We then let M_k and L_k denote the standard Levi subgroups of G corresponding to $S \setminus \{s_{m-k+1}\}$ and $S \setminus \{s_{m-k+1}, s_{m-k+2}, \dots, s_m\}$, respectively. Thus,

$$M_k \cong G_{n-2k}(q) \times GL_k(q^\delta), \tag{7}$$

where $G_{n-2k}(q)$ is a group of the same type as G (for the definition of δ see 4.1). Moreover, L_k is the direct product of $G_{n-2k}(q)$ with the maximally split torus $GL_1(q^\delta)^k$ of $GL_k(q^\delta)$.

In the matrix representation of $G_n(q)$ described in 4.2, the group M_k is represented by the set of matrices

$$\left\{ \begin{bmatrix} \mathbf{x} & 0 & 0 \\ 0 & \mathbf{y} & 0 \\ 0 & 0 & \lambda_{\mathbf{y}}\tilde{F}(\mathbf{x}) \end{bmatrix} \mid \mathbf{x} \in GL_k(q^\delta), \mathbf{y} \in G_{n-2k}(q) \right\},$$

where $\lambda_{\mathbf{y}}$ denotes the multiplier of $\mathbf{y}$ in case of the conformal groups (and 1 otherwise), and where $\tilde{F}(\mathbf{x}) = J_k(F_0(\mathbf{x})^t)^{-1}J_k^{-1}$ (note that $F_0(\mathbf{x}) = \mathbf{x}$ except in the unitary case).

The same notation is used for the algebraic group $\mathbf{G}$. In particular, $\mathbf{M}_k$ is the stabilizer of the decomposition $\mathbf{V} = \langle e_1, \dots, e_k \rangle \oplus \mathbf{V}_{n-2k} \oplus \langle e'_k, \dots, e'_1 \rangle$ (with the convention $e'_i := e_{n-i+1}$, $1 \leq i \leq k$, in Case 4.2.2). Then $\mathbf{G}_{n-2k}$ can naturally be considered as a subgroup of $\mathbf{M}_k$ via the embedding

$$\mathbf{y} \mapsto \begin{bmatrix} E_k & 0 & 0 \\ 0 & \mathbf{y} & 0 \\ 0 & 0 & \lambda_{\mathbf{y}}E_k \end{bmatrix},$$

for $\mathbf{y} \in \mathbf{G}_{n-2k}$. Restriction yields an embedding of $G_{n-2k}(q)$ into M_k and hence the isomorphism (7). In the following we shall identify $\mathbf{G}_{n-2k}$ and $G_{n-2k}(q)$ with their images in $\mathbf{G}$ and G, respectively.

5. Characters and Harish-Chandra Induction

Let $G = G_n(q)$ be one of the classical groups introduced in 4.1. We shortly describe the parameterization of the ordinary irreducible characters of G.

From now on we shall use the term Levi subgroup as in the theory of algebraic groups (see [8, Chapter 1]). In particular we do not assume that Levi subgroups are 1-split.

5.1. **Lusztig Series.** Recall that G is a group of the form $\mathbf{G}^F$, where $\mathbf{G}$ is a connected reductive linear algebraic group with connected center, and F is a Frobenius morphism of $\mathbf{G}$. The algebraic group dual to $\mathbf{G}$ is denoted by $\mathbf{G}^*$. The morphism F

acts on $\mathbf{G}^*$, and the finite group $\mathbf{G}^{*F}$ is denoted by G^*. The set of ordinary irreducible characters of G is partitioned into *Lusztig series* $\mathcal{E}(G,(s))$, where (s) runs through the G^*-conjugacy classes of semisimple elements of G^* (see [8, Chapter 13]). The elements in $\mathcal{E}(G,(1))$ are called the *unipotent characters* of G.

For semisimple $s \in G^*$, put $\mathbf{G}(s) := C_{\mathbf{G}^*}(s)^*$, and $G(s) := \mathbf{G}(s)^F$. There is a one to one correspondence between $\mathcal{E}(G,(s))$ and $\mathcal{E}(G(s),(1))$, which we are going to describe in a special case. Suppose that $C_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$. Then we may identify $\mathbf{G}(s)$ with an F-stable Levi subgroup of $\mathbf{G}$, and Lusztig induction (see [8, Chapter 11]) $R^G_{G(s)}$ induces a bijection

$$\mathcal{E}(G(s),(1)) \to \mathcal{E}(G,(s)), \qquad \vartheta \mapsto \varepsilon_G \varepsilon_{G(s)} R^G_{G(s)}(\hat{s}\vartheta),$$

where $\hat{s}$ is a suitable linear character of $G(s)$, and $\varepsilon_G = (-1)^{\mathbb{F}_q\text{-rank of }\mathbf{G}}$ (see the last paragraph of [8, Chapter 13]). The irreducible character $\varepsilon_G \varepsilon_{G(s)} R^G_{G(s)}(\hat{s}\vartheta)$ will be denoted by $\chi_{s,\vartheta}$. We shall not need to consider the case when $C_{\mathbf{G}^*}(s)$ is not a Levi subgroup of $\mathbf{G}$.

5.2. **Unipotent Characters in Harish-Chandra Series.** The unipotent characters of G are parametrized (independently of q) by partitions or (equivalence classes of) symbols. A unipotent character labeled by a partition or a symbol λ, will be denoted by χ_λ. We now describe this parameterization in the individual cases, and, at the same time, the distribution of the unipotent characters into Harish-Chandra series. Additional information is collected in [7, Section 13.8].

To begin with, let $G = GU_n(q)$ be the unitary group. Then the unipotent characters of G are parametrized by partitions of n. A unipotent character χ_λ of G is cuspidal if and only if λ is a triangular partition $(s, s-1, \ldots, 1)$ (and n is the triangular number $s(s+1)/2$). The only general linear group which does have a cuspidal unipotent character is $GL_1(q)$. Hence a Levi subgroup L of G has a cuspidal unipotent character if and only if $L = L_{m'} = GU_{n-2m'}(q) \times GL_1(q^2)^{m'}$ for an integer m' such that $n-2m'$ is a triangular number $(s^2+s)/2$. The endomorphism ring of the corresponding Harish-Chandra induced module is an Iwahori-Hecke algebra of type $B_{m'}$. The unipotent characters of G in the corresponding Harish-Chandra series are thus labeled by bi-partitions of m'. This gives a labeling of the unipotent characters of G by triples (s,μ,ν) where $s \in \mathbb{N}$ is such that $(s^2+s)/2 \leq n$, $n - (s^2+s)/2 = 2m'$ is even, and (μ,ν) is a bi-partition of m'. We thus have two labelings for the unipotent characters of G which are matched as follows. For a partition $\lambda \vdash n$, the unipotent character χ_λ has Harish-Chandra label (s,μ,ν) such that the 2-core of λ is a partition of s and (μ,ν) is the 2-quotient of λ. For a proof of these facts see the appendix of [12].

Now let G be one of the groups $SO_n(q)$ (with $n \geq 3$ odd), or $CSp_n(q)$ (with $n \geq 2$ even), and put $m = \lfloor n/2 \rfloor$. Then the unipotent characters of G are labeled by the equivalence classes of symbols of rank m and odd defect. A unipotent character χ_λ of G is cuspidal if and only if λ is equivalent to the symbol

$$\{\{0,1,\ldots,2s\},\{\}\}$$

for some $s \in \mathbb{N}$ (and m equals s^2+s). Hence a Levi subgroup L of G has a cuspidal unipotent character if and only if $L = L_{m'}$ for an integer $m' \leq m$ such that $m - m' = s^2 + s$ for some $s \in \mathbb{N}$. The endomorphism ring of the corresponding Harish-Chandra induced module is an Iwahori-Hecke algebra of type $B_{m'}$. The unipotent characters of G in the corresponding Harish-Chandra series are thus labeled by bi-partitions of m'. This gives a labeling of the unipotent characters of G by triples (s, μ, ν) where $s \in \mathbb{N}$ such that $s^2 + s \leq m$, and (μ, ν) is a bi-partition of $m - (s^2 + s)$. We thus have two labelings for the unipotent characters of G which are matched as follows.

If the symbol λ has defect $2s + 1$ and contains $\{X, Y\}$ with $|X| > |Y|$, then χ_λ lies in the Harish-Chandra series of $L_{m-(s^2+s)}$ and corresponds to the bi-partition (μ, ν) of $m - (s^2 + s)$, such that X and Y are β-sets of μ and ν respectively.

Next let $G = CSO_n^+(q)$ or $SO_n^+(q)$, with $n = 2m \geq 4$. Then the unipotent characters of G are labeled by the equivalence classes of symbols of rank m and defect divisible by 4, with the convention that degenerate symbols label two unipotent characters. A unipotent character χ_λ of G is cuspidal if and only if λ is equivalent to the symbol

$$\{\{0, 1, \ldots, 2s - 1\}, \{\}\}$$

for some even $s \in \mathbb{N}$ (and m equals s^2). Hence a Levi subgroup L of G has a cuspidal unipotent character if and only if $L = L_{m'}$ for an integer $m' \leq m$ such that $m - m' = s^2$ for some even $s \in \mathbb{N}$. The endomorphism ring of the corresponding Harish-Chandra induced module is an Iwahori-Hecke algebra of type D_m if $s = 0$, and of type $B_{m'}$, otherwise. In the latter case, the unipotent characters of G in the corresponding Harish-Chandra series are thus labeled by bi-partitions of m'. In the former case, they are labeled by unordered pairs of partitions whose parts sum up to m (with the convention that (β, β) labels two unipotent characters). This gives a labeling of the unipotent characters of G by triples (s, μ, ν) where $s \in \mathbb{N}$ is even and nonzero, such that $s^2 \leq m$, and (μ, ν) is a bi-partition (unordered in case $s = 0$) of $m - s^2$. The two labelings for the unipotent characters of G arising in this way are matched as follows.

If the symbol λ has defect 0 and contains $\{X, Y\}$, then χ_λ lies in the principal series and corresponds to the unordered pair of partitions (μ, ν), such that X and Y are β-sets of μ and ν respectively. If λ has defect $2s$ for some even nonzero $s \in \mathbb{N}$ with $s^2 \leq m$, then χ_λ lies in the Harish-Chandra series of L_{m-s^2} and corresponds to the bi-partition (μ, ν) of $m - s^2$, such that X and Y are β-sets of μ and ν respectively.

Finally, let $G = CSO_n^-(q)$ or $SO_n^-(q)$, with $n = 2m \geq 4$. Then the unipotent characters of G are labeled by the equivalence classes of symbols of rank m and defect congruent to 2 modulo 4. A unipotent character χ_λ of G is cuspidal if and only if λ is equivalent to the symbol

$$\{\{0, 1, \ldots, 2s - 1\}, \{\}\}$$

for some odd $s \in \mathbb{N}$ (and m equals s^2). Hence a Levi subgroup L of G has a cuspidal unipotent character if and only if $L = L_{m'}$ for an integer $m' \leq m$ such that $m - m' = s^2$ for some odd $s \in \mathbb{N}$. The endomorphism ring of the corresponding Harish-Chandra induced module is an Iwahori-Hecke algebra of type $B_{m'}$. The unipotent characters of G in the corresponding Harish-Chandra series are thus labeled by bi-partitions of m'. This gives a labeling of the unipotent characters of G by triples (s, μ, ν) where $s \in \mathbb{N}$ is odd, such that $s^2 \leq m$, and (μ, ν) is a bi-partition of $m - s^2$. The two labelings for the unipotent characters of G arising in this way are matched as follows.

If the symbol λ has defect $2s$ and contains $\{X, Y\}$ with $|X| > |Y|$, then χ_λ lies in the Harish-Chandra series of L_{m-s^2} and corresponds to the bi-partition (μ, ν) of $m - s^2$, such that X and Y are β-sets of μ and ν respectively.

5.3. **The Comparison Theorem.** Harish-Chandra induction is the special case of Lusztig induction for 1-split Levi subgroups. Harish-Chandra induction and restriction of unipotent characters in a finite classical group corresponds to ordinary induction in certain Weyl groups. This is, roughly speaking, the content of the comparison theorem [18, Theorem 5.9]. To be more specific, let G be one of the finite classical groups introduced in 4.1. Let M be a 1-split Levi subgroup of G and let ψ be a unipotent character of M. We wish to describe the constituents of the Harish-Chandra induced character $R_M^G(\psi)$. Since Harish-Chandra induction preserves Lusztig series of characters, all these constituents are unipotent.

By Harish-Chandra theory, ψ is a constituent of some $R_L^M(\vartheta)$, where $L \leq M$ is a 1-split Levi subgroup of M and ϑ is a cuspidal unipotent character of L. (The possibilities for the pairs (L, ϑ) are described in the previous subsection.) The irreducible constituents of $R_L^M(\vartheta)$ and $R_L^G(\vartheta)$ are labeled by the irreducible characters of the relative Weyl groups $W_M(L)$ and $W_G(L)$, respectively. Note that $W_M(L)$ is embedded in $W_G(L)$ in a natural way. Let us write ψ_β for the constituent of $R_L^M(\vartheta)$ corresponding to the irreducible character β of $W_M(L)$, and χ_α for the constituent of $R_L^G(\vartheta)$ corresponding to the irreducible character α of $W_G(L)$. Thus $\psi = \psi_\gamma$ for some irreducible character γ of $W_M(L)$. The comparison theorem now states that the multiplicity of χ_α in $R_M^G(\psi)$ equals the multiplicity of α in the induced character $\mathrm{Ind}_{W_M(L)}^{W_G(L)}(\gamma)$. Dually, of course, Harish-Chandra restriction corresponds to restriction in Weyl groups.

5.4. **Harish-Chandra Induction.** In all cases of interest to our paper, Harish-Chandra induction of arbitrary characters can be reduced to Harish-Chandra induction of unipotent characters. Namely, let $\mathbf{M}$ be a 1-split F-stable Levi subgroup of $\mathbf{G}$. Then $\mathbf{M}^*$ is a 1-split F-stable Levi subgroup of $\mathbf{G}^*$. Let s be an F-stable semisimple element of $\mathbf{M}^*$ and suppose that $C_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$. Then $C_{\mathbf{M}^*}(s) = C_{\mathbf{G}^*}(s) \cap \mathbf{M}^*$ is a Levi subgroup of $\mathbf{M}^*$. Using properties of Lusztig induction [8, 11.5, 12.6], we obtain the following commutative diagram (notation as in 5.1).

$$\begin{tikzcd}
\mathcal{E}(G,(s)) & & \mathcal{E}(G(s),(1)) \arrow[ll, "\varepsilon_G\varepsilon_{G(s)}R_{G(s)}^G(\hat{s}-)"'] \\
\mathcal{E}(M,(s)) \arrow[u, "R_M^G"] & & \mathcal{E}(M(s),(1)) \arrow[ll, "\varepsilon_M\varepsilon_{M(s)}R_{M(s)}^M(\hat{s}-)"] \arrow[u, "R_{M(s)}^{G(s)}"']
\end{tikzcd} \qquad (8)$$

This allows to compute R_M^G from $R_{M(s)}^{G(s)}$, applied to unipotent characters. Note that $M(s)$ is a Levi subgroup of $G(s)$.

5.5. **Branching Rules.** Harish-Chandra induction (of unipotent characters) can be described combinatorially with the help of the labels of the unipotent characters.

To begin with, let $G = GU_n(q)$ and let $\rho \vdash n-2$, such that χ_ρ denotes a unipotent character of $L_1 = GU_{n-2}(q) \times GL_1(q^2)$. Then the constituents of $R_{L_1}^G(\chi_\rho)$

are exactly those unipotent characters χ_λ of G, such that λ is obtained from ρ by adding a 2-hook.

Now let $G = G_n(q)$ be one of the other groups, and ρ a symbol labeling a unipotent character of $L_1 = G_{n-2}(q) \times GL_1(q)$. Then $R_{L_1}^G(\chi_\rho)$ contains exactly those unipotent characters χ_λ of G, such that λ is obtained from ρ by adding a 1-hook. (In case ρ is degenerate, none of the λs is degenerate, and the two characters labeled by ρ have the same Harish-Chandra induction to G. If λ is degenerate, then the two characters labeled by λ occur in $R_{L_1}^G(\chi_\rho)$.)

These facts can be seen as follows. Suppose that χ_ρ lies in a Harish-Chandra series arising from the cuspidal unipotent character of the Levi subgroup $L_{m'} \leq L_1$ of G. Let (s, μ, ν) be the Harish-Chandra label (see 5.2) of χ_ρ, such that (μ, ν) labels an irreducible character of the Weyl group $W_{m'-1}$ or $\tilde{W}_{m-1}$, respectively. By the comparison theorem (see the previous paragraph), we have to induce this character from $W_{m'-1}$ to $W_{m'}$ (or from $\tilde{W}_{m-1}$ to $\tilde{W}_m$) and find its constituents. This is done by adding a node to either μ or ν in all possible ways, as described in Section 3. Translating the labels obtained this way back into partitions or symbols according to the rules in 5.2, we obtain the claimed results.

5.6. **Restriction of Unipotent Characters.** We will also need the following fact relating unipotent characters of finite reductive groups to the characters of certain subgroups.

Lemma 5.7. *Suppose that* $\mathbf{M} \subseteq \mathbf{N}$ *are connected reductive groups defined over* $\mathbb{F}_q$ *with corresponding Frobenius endomorphism* F. *If* $[\mathbf{N}, \mathbf{N}] \subseteq \mathbf{M}$, *then restriction induces a bijection between* $\mathcal{E}(\mathbf{N}^F, (1))$ *and* $\mathcal{E}(\mathbf{M}^F, (1))$. *Moreover, if* $\mathbf{L}$ *is an* F*-stable Levi subgroup of an* F*-stable parabolic subgroup of* $\mathbf{N}$, *then for any unipotent character* ζ *of* $\mathbf{L}^F$ *we have*

$$\mathrm{Res}^{\mathbf{N}^F}_{\mathbf{M}^F}(R^{\mathbf{N}^F}_{\mathbf{L}^F}(\zeta)) = R^{\mathbf{M}^F}_{(\mathbf{L}\cap\mathbf{M})^F}(\mathrm{Res}^{\mathbf{L}^F}_{(\mathbf{L}\cap\mathbf{M})^F}(\zeta)).$$

In particular, if η *is a unipotent character of* $\mathbf{N}^F$, *then*

$$\langle R^{\mathbf{N}^F}_{\mathbf{L}^F}(\zeta), \eta\rangle_{\mathbf{N}^F} = \langle R^{\mathbf{M}^F}_{(\mathbf{L}\cap\mathbf{M})^F}(\mathrm{Res}^{\mathbf{L}^F}_{(\mathbf{L}\cap\mathbf{M})^F}(\zeta)), \mathrm{Res}^{\mathbf{N}^F}_{\mathbf{M}^F}(\eta)\rangle_{\mathbf{M}^F}$$

Proof. See [8, Propositions 13.20 and 13.22]. □

6. Unipotent blocks of finite classical groups

From now on, $G = G_n(q)$ will denote one of the groups introduced in 4.1. Let ℓ be an odd prime not dividing q.

6.1. **Linear and Unitary Primes.** Let d denote the order of q modulo ℓ. If G is a unitary group, we let e denote the order of $-q$ modulo ℓ. In the remaining cases, e is the order of q^2 modulo ℓ. Then, if G is unitary, we have

$$e = \begin{cases} 2d, & \text{if } d \text{ is odd,} \\ d/2, & \text{if } d \text{ is even and } d/2 \text{ is odd,} \\ d, & \text{otherwise.} \end{cases}$$

If G is one of the other classical groups we have

$$e = \begin{cases} d, & \text{if } d \text{ is odd,} \\ d/2, & \text{if } d \text{ is even.} \end{cases}$$

We say that ℓ is *unitary* for G, if $e = d/2$. Otherwise, ℓ is *linear* for G.

We will adopt the following convention. If λ is a partition, by a diagram of λ we will mean an e-abacus diagram for a β-set of λ. If λ is a symbol and ℓ is linear (respectively unitary) for G, a diagram for λ will mean a 2e-linear diagram (respectively a 2e-unitary diagram) for λ. We say that λ is an e-core, if the diagram for λ is an e-core.

6.2. **Unipotent Blocks.** To describe the unipotent ℓ-blocks of G, we follow Cabanes and Enguehard [6] and Fong and Srinivasan [10, 11]. Let $\mathbf{G}$ be an algebraic group as in 4.2 and F a Frobenius morphism of G such that $G = \mathbf{G}^F$. We recall the definition of d-split Levi subgroups of $\mathbf{G}$ given in [5, Section 1]. Such a Levi subgroup is, by definition, the centralizer of a ϕ_d-torus of $\mathbf{G}$. The notion of a ϕ_d-torus is introduced in [4, Section 3]. A ϕ_d-torus is an F-stable torus of $\mathbf{G}$ whose polynomial order is a power of ϕ_d, the dth cyclotomic polynomial.

The unipotent ℓ-blocks of G are in one to one correspondence with the unipotent d-cuspidal pairs $(\mathbf{K}, \psi)$ (taken up to conjugacy in G). Such a pair consists of a d-split Levi subgroup $\mathbf{K}$ of $\mathbf{G}$ and a unipotent d-cuspidal character ψ of $\mathbf{K}^F$. If $\psi = \chi_\gamma$ for a partition or symbol γ, then the d-cuspidality of ψ is equivalent to γ being an e-core.

If $(\mathbf{K}, \psi)$ is a d-cuspidal pair, the corresponding unipotent ℓ-block of G is denoted by $b_G(\mathbf{K}, \psi)$. The unipotent characters in $b_G(\mathbf{K}, \psi)$ are exactly the constituents of the Lusztig induced character $R_K^G(\psi)$. The other irreducible characters in $b_G(\mathbf{K}, \psi)$ are of the form $\chi_{t^*,\rho}$ for an ℓ-element t^* of G^* and a unipotent character ρ of $G(t^*) := \mathbf{G}(t^*)^F$ (recall from 5.1 that $\mathbf{G}(t^*)$ is an F-stable Levi subgroup of $\mathbf{G}$ in duality with $C_{\mathbf{G}^*}(t^*)$) with the following properties: There is a d-cuspidal pair $(\mathbf{K}_{t^*}, \psi_{t^*})$ of $\mathbf{G}(t^*)$ with $[\mathbf{K}_{t^*}, \mathbf{K}_{t^*}] = [\mathbf{K}, \mathbf{K}]$, the restrictions of ψ and ψ_{t^*} to $[\mathbf{K}, \mathbf{K}]^F$ agree, and ρ its a constituent of $R_{K_{t^*}}^{G(t^*)}(\psi_{t^*})$. Any Sylow ℓ-subgroup of $C^\circ_{\mathbf{G}}([\mathbf{K}, \mathbf{K}])^F$ is a defect group of $b_G(\mathbf{K}, \psi)$. This description of the unipotent ℓ-blocks of G and their defect groups is due to Cabanes and Enguehard [6, Theorem 4.4].

To obtain the description of Fong and Srinivasan, one uses the classification of the d-cuspidal pairs (see e.g., [5, p. 49 and p. 52]). If $(\mathbf{K}, \psi)$ is a d-cuspidal pair, then

$$\mathbf{K} = C_{\mathbf{G}}(\mathbf{S}) = Z(\mathbf{G})(\mathbf{H} \times \mathbf{T}), \tag{9}$$

where $\mathbf{S}$, $\mathbf{T}$ and $\mathbf{H}$ are F-stable, $\mathbf{T}$ is a torus containing the ϕ_d-torus $\mathbf{S}$, and $\mathbf{H}$ is a classical group, such that $[\mathbf{H}, \mathbf{H}]$ is trivial or simple. Clearly, $[\mathbf{K}, \mathbf{K}] = [\mathbf{H}, \mathbf{H}]$, and the restriction of ψ to $\mathbf{H}^F$ is unipotent and d-cuspidal (cf. Lemma 5.7). The order of $\mathbf{S}^F$ equals $\phi_d(q)^w$ for some non-negative integer w, and $\mathbf{T}^F$ is isomorphic to $GU_1(q^e)^w = (q^e+1)^w$, if $e = d/2$, and to $GL_1(q^e)^w = (q^e-1)^w$, otherwise. If r is the degree of the natural representation of $\mathbf{H}$ as a classical group, then $n = r + ew$, if $G = GU_n(q)$, and $n = r + 2ew$, otherwise. A d-split Levi subgroup of the form $C_{\mathbf{G}}(\mathbf{S}') = Z(\mathbf{G})(\mathbf{H}' \times \mathbf{T}')$, where $\mathbf{S}'$ is a ϕ_d-torus and $[\mathbf{H}', \mathbf{H}']$ is trivial or simple, is conjugate to $\mathbf{K}$ in G if and only if $|\mathbf{S}'^F| = |\mathbf{S}^F|$. This is the case if and only if the degrees of the natural representations of $\mathbf{H}$ and $\mathbf{H}'$ are equal.

We shortly describe how to construct the d-split Levi subgroups of the form (9). One chooses an orthogonal decomposition (defined over $\mathbb{F}_q$) $\mathbf{V} = \mathbf{V}' \oplus \mathbf{V}''$ such that $\dim_{\mathbb{F}_q}(\mathbf{V}')$ equals $(3-\delta)ew$, and $I_0(\mathbf{V}')$ has a Sylow ϕ_d-torus $\mathbf{S}$ with $|\mathbf{S}^F| = \phi_d^w$. Viewing $\mathbf{S}$ as a subgroup of $\mathbf{G}$, we put $\mathbf{K} = C_{\mathbf{G}}(\mathbf{S})$. Then $\mathbf{S}$ acts trivially on $\mathbf{V}''$ and $[\mathbf{K}, \mathbf{K}]$ acts trivially on $\mathbf{V}'$.

Let $(\mathbf{K}, \psi)$ be a d-cuspidal pair with $\mathbf{K}$ as in (9). Then w is called the *weight* of the block $B := b_G(\mathbf{K}, \psi)$, and denoted by $w(B)$. By the remarks above, every ℓ-block of G of weight w is of the form form $b_G(\mathbf{K}, \psi')$ for some d-cuspidal unipotent character ψ' of $\mathbf{K}^F$. In particular, any two ℓ-blocks of G of the same weight have a defect group in common.

Let $\psi = \psi_\gamma$ for a partition or symbol γ. The constituents of $R_K^G(\psi)$ are exactly those unipotent characters of G whose label is obtained from γ by adding a sequence of e-hooks, if ℓ is linear, and e-cohooks, if ℓ is unitary. Similarly, the labels of the constituents of $R_{K_{t^*}}^{G(t^*)}(\psi_{t^*})$ are obtained from the label of ψ_{t^*} (which equals γ by Lemma 5.7) by adding a sequence of e-hooks (or e-cohooks). The label γ of ψ is called the *unipotent label* of $B = b_G(\mathbf{K}, \psi)$, and denoted by $\gamma(B)$.

6.3. **Remarks on ℓ-Elements.** If $\mathbf{G} = GL_n(\bar{\mathbb{F}}_q)$, i.e., if G is the unitary group, we identify $\mathbf{G}$ and $\mathbf{G}^*$. In the other cases, we use [14, Proposition 4.2] to observe that the set of Levi subgroups of $\mathbf{G}$ of the form $\mathbf{G}(t)$, where t is an F-stable ℓ-element of $\mathbf{G}^*$ is equal to the set of Levi subgroups of $\mathbf{G}$ of the form $C_{\mathbf{G}}(t)$, where t is an F-stable ℓ-element of $\mathbf{G}$. (Note that all the centralizers we are considering here are connected.) More precisely, there is a bijection between the G^*-conjugacy classes of ℓ-elements of G^* and the G-conjugacy classes of ℓ-elements of G, preserving element orders and inducing duality on the centralizers.

7. The Reduction Theorem

Let us keep all the notation introduced in the previous sections. In particular, $G = G_n(q)$ is one of the groups introduced in 4.1, $m = \lfloor n/2 \rfloor$, and ℓ is an odd prime not dividing q. For the definition of the integers d and e we refer to 6.1. In this section we also assume that ℓ does not divide $q^\delta - 1$.

7.1. **The Block B.** Let B be a unipotent ℓ-block of G of positive weight $w := w(B)$, and let $\gamma := \gamma(B)$ be the unipotent label of B. First we construct a particular d-cuspidal pair for B, which is most suitable for our purposes. Let k' be as large as possible such that $\mathbf{G}_{n-2k'}$ contains a Sylow ϕ_d-torus $\mathbf{S}$ with $|\mathbf{S}^F| = \phi_d^w$. Then, by [4, Theorem 3.4(2)], $k' = (n - ew)/2$, if G is unitary. If G is symplectic or odd-dimensional orthogonal, $k' = m - ew$. In the remaining cases, $k' = m - ew$ or $k' = m - ew - 1$. Identifying $\mathbf{G}_{n-2k'}$ with a subgroup of $\mathbf{L}_{k'}$, we have $\mathbf{S} \leq \mathbf{G}$. Let $\mathbf{K} := C_{\mathbf{G}}(\mathbf{S})$. Then $\mathbf{K}$ is a d-split Levi subgroup of the form (9), with $\mathbf{T} \leq \mathbf{G}_{n-2k'} \leq \mathbf{L}_{k'}$ and $\mathbf{T}^F = (q^e \pm 1)^w$. Moreover, $\psi := \chi_\gamma$ is a d-cuspidal unipotent character of $\mathbf{H}^F$ and hence of $\mathbf{K}^F$ by Lemma 5.7. Thus $(\mathbf{K}, \psi)$ is a d-cuspidal pair with $B = b_G(\mathbf{K}, \psi)$.

7.2. **The Scopes Number $s(B)$.** We are going to associate a non-negative integer $s(B)$ to B, called the *Scopes number* of B. To define it we consider the following cases.

Case 1: G is a unitary group, i.e., γ is a partition.

Case 2: G is a non-unitary group and γ is an equivalence class of non-degenerate symbols.

In particular, we do not consider the case where γ is degenerate. Case 2 has the following subcases.

Case 2.1: ℓ is a linear prime for G, i.e., d is odd and $e = d$.

Case 2.2: ℓ is a unitary prime for G, i.e., d is even and $e = d/2$.

By our assumption that ℓ does not divide $q^\delta - 1$, we have $e > 2$ in Case 1, and $e > 1$ in Case 2.1. We divide Case 2.2 into two further subcases.

Case 2.2.1: $e > 1$.

Case 2.2.2: $e = 1$.

We now define $s(B)$ to be $s_e(\gamma)$ in Case 1, $\check{s}_e(\gamma)$ in Case 2.1, and $\hat{s}_e(\gamma)$ in Case 2.2 (see 2.3 for the definition of these numbers).

7.3. **The Diagram for γ.** Let $k := s(B)$. In Case 1 choose an e-abacus diagram Γ of γ and an integer i, $2 \le i \le e-1$, such that $k = s_e^i(\Gamma)$. In Case 2.1 choose a 2e-linear diagram Γ for the equivalence class of γ and an integer i, $1 \le i \le e-1$, such that $k = \check{s}_e^{i_g}(\Gamma)$. In Case 2.2.1 choose a 2e-unitary diagram Γ for the equivalence class of γ and an integer i, $1 \le i \le e-1$, such that $k = \hat{s}_e^{i_g}(\Gamma)$. In Case 2.2.2, choose a 2e-unitary diagram Γ for the equivalence class of γ such that $k = \hat{s}_e^{0_g}(\Gamma)$. Notice that such a Γ can always be found, by replacing, if necessary, γ by a suitable shift and in Case 2 by swapping the strings j_g and j_b for all j, $0 \le j \le e-1$.

Lemma 7.4. *With the notation of* 7.3*, let Λ be a diagram with core Γ which satisfies $w(\Lambda) \le k$. Then, in Case* 1*, there are exactly k beads on string i of Λ for which there is no bead on the corresponding row of string $i-2$. Analogous statements hold in the other cases.*

Proof. Since Λ is obtained from Γ by moving certain beads down on their strings, the number of beads on any string of Λ is equal to the number of beads on the same same string of Γ. So, we have

$$|\Lambda^i| = |\Gamma^i| = |\Gamma^{i-2}| + k = |\Lambda^{i-2}| + k,$$

i.e., there are k more beads on string i of Λ than on string $i-2$. Let $u \ge 0$ be such that the lowest bead of string $i-2$ of Λ is on row $|\Lambda^{i-2}| - 1 + u$, and suppose that $v \ge 0$ is such that the highest empty row on string i of Λ is row $|\Lambda^i| - v$. Then, clearly, $w(\Lambda)$ is greater than or equal to $u + v$. Hence

$$|\Lambda^i| - v - (|\Lambda^{i-2}| - 1 + u) = k + 1 - (u + v) \ge w(\Lambda) + 1 - w(\Lambda) > 0.$$

From this it follows that for every bead on string $i-2$ of Λ there is a bead on the corresponding row of string i and that there are exactly k beads on string i of Λ such that there is no bead on the corresponding row of string $i-2$. Cases 2.1 and 2.2.1 are proved similarly.

In Case 2.2.2, we have

$$|\Lambda^{0_g}| = |\Gamma^{0_g}| = |\Gamma^{0_b}| + k + 1 = |\Lambda^{0_b}| + k + 1,$$

i.e. there are $k+1$ more beads on string 0_g of Λ than on string 0_b. On the other hand, suppose that $u \ge 0$ is such that the lowest bead of string 0_b of Λ is on row $|\Lambda^{0_g}| - 1 + u$, and suppose that $v \ge 0$ is such that the highest empty row on string 0_g of Λ is row $|\Lambda^{0_g}| - v$. Then, again, $w(\Lambda)$ is greater than or equal to $u + v$. It follows that

$$|\Lambda^{0_g}| - v - (|\Lambda^{0_g}| - 1 + u) = k + 2 - (u + v) > 1.$$

Thus for every z such that there is a bead on row z and string 0_b of Λ there is a bead on row $z+1$ of string 0_g. Since there is clearly a bead on row 0_g of Λ (otherwise, $w \ge |\Lambda^{0_g}|$, but $|\Lambda^{0_g}| > k$), this means that there are exactly k beads on string 0_g of Λ with no bead on the one higher row of string 0_b. $\square$

7.5. **The Unipotent Label $\bar{\gamma}$.** In Case 1, let $\bar{\Gamma}$ be the diagram obtained from Γ by swapping strings i and $i-2$. Notice that since there are exactly k more beads on string i of Γ than on string $i-2$ and since all the beads of Γ are as high as possible on their strings, we can also obtain $\bar{\Gamma}$ from Γ by moving the bottom k beads on string i two positions to the left. In Cases 2.1 and 2.2.1 let $\bar{\Gamma}$ be the diagram obtained from Γ by swapping strings i_g and $(i-1)_g$. Again, notice that $\bar{\Gamma}$ may be obtained from Γ by moving the bottom k beads on string i_g one position to the left. In Case 2.2.2, let $\bar{\Gamma}$ be the diagram obtained from Γ by removing each of the bottom k beads from string 0_g of Γ and placing it one row higher (i.e., decreasing its row index by 1) on string 0_b. Notice that this makes sense since for each of the bottom $k+1$ beads on string 0_g there are no beads on the corresponding row of string 0_b.

Let $\bar{\gamma}$ denote the partition or symbol for which $\bar{\Gamma}$ is a diagram (in the case of non-unitary groups $\bar{\Gamma}$ is a 2e-linear diagram of $\bar{\gamma}$, if ℓ is linear, and a 2e-unitary diagram of $\bar{\gamma}$ otherwise). Note that in Case 1, $k = \frac{1}{2}(|\gamma| - |\bar{\gamma}|)$ and in Case 2, $k = \mathrm{rank}(\gamma) - \mathrm{rank}(\bar{\gamma})$.

If we are in Case 2, let us assume that $\bar{\Gamma}$ also is non-degenerate. Also, if G is an even-dimensional orthogonal group we assume that $\mathrm{rank}(\bar{\gamma}) \geq 2$.

7.6. **The Block $\bar{B}$.** Let $\bar{\mathbf{G}} := \mathbf{G}_{n-2k}$ and $\bar{G} := \bar{\mathbf{G}}^F$ (for the notation see 4.2.7.) We define a d-cuspidal pair for $\bar{G}$. Recall that $B = b_G(\mathbf{K}, \psi)$, where $\mathbf{K} = Z(\mathbf{G})(\mathbf{H} \times \mathbf{T})$ for some F-stable torus $\mathbf{T}$ of $\mathbf{L}_{k'}$ with $\mathbf{T}^F = (q^e \pm 1)^w$.

Let r denote the degree of the natural representation of $\mathbf{H}$. Since ψ is a unipotent character of $\mathbf{H}^F$, and γ is the label of ψ, in Case 1, we have that $r = |\gamma| \geq 2k$ and in Case 2, $\lfloor r/2 \rfloor = \mathrm{rank}(\gamma) \geq k$. On the other hand, $r + ew = n$, if G is unitary. Hence in this case, $2k \leq |\gamma| = r = n - ew = 2k'$. If G is a non-unitary group, we have $r + 2ew = n$. Thus, if G is a symplectic or an odd-dimensional orthogonal group, $k = \mathrm{rank}(\gamma) - \mathrm{rank}(\bar{\gamma}) \leq \mathrm{rank}(\gamma) = \lfloor r/2 \rfloor = m - ew = k'$. Finally, if G is an even-dimensional orthogonal group, $k = \mathrm{rank}(\gamma) - \mathrm{rank}(\bar{\gamma}) \leq \mathrm{rank}(\gamma) - 2 = \lfloor r/2 \rfloor - 2 < m - ew - 1 \leq k'$.

Therefore, in any case, $\mathbf{G}_{n-2k'}$ is contained in $\mathbf{G}_{n-2k} = \bar{\mathbf{G}}$. In particular, $\mathbf{T} \leq \bar{\mathbf{G}}$. Put $\bar{\mathbf{K}} := C_{\bar{\mathbf{G}}}(\mathbf{T})$. Then $\bar{\mathbf{K}} = Z(\bar{\mathbf{G}})(\bar{\mathbf{H}} \times \mathbf{T})$ with $\bar{\mathbf{H}} = \mathbf{H} \cap \bar{\mathbf{G}}$.

Since, in Case 2, we assume that $\bar{\Gamma}$ is a non-degenerate symbol, in all cases there is a unique unipotent character of $\bar{\mathbf{K}}^F$ with label $\bar{\gamma}$. We denote this character by $\bar{\psi}$. Then $(\bar{\mathbf{K}}, \bar{\psi})$ is clearly a d-cuspidal pair of $\bar{G}$ and we let $\bar{B}$ be the unipotent ℓ-block $b_{\bar{G}}(\bar{\mathbf{K}}, \bar{\psi})$ of $\bar{G}$.

Lemma 7.7. *Any Sylow ℓ-subgroup of $G_{n-2k'}(q)$ is a defect group for B and for $\bar{B}$.*

If $\chi_{t^*,\rho}$ is an irreducible character in B, there is an ℓ-element $t \in G_{n-2k'}(q)$ such that the G-conjugacy class of t corresponds to the G^*-conjugacy class of t^* by the bijection described in 6.3. *In particular $\mathbf{G}(t^*)$ is conjugate in G to $C_{\mathbf{G}}(t)$.*

Proof. By [6], any Sylow ℓ-subgroup of $C^\circ_{\mathbf{G}}([\mathbf{K}, \mathbf{K}])^F$ is a defect group of B. To prove the first assertion, we show that $C^\circ_{\mathbf{G}}([\mathbf{K}, \mathbf{K}]) \leq \mathbf{G}_{n-2k'}$. Since $\mathbf{T} \subseteq \mathbf{G}_{n-2k'}$, it follows that that $[\mathbf{K}, \mathbf{K}]$ contains $SL_{k'}(\mathbb{F}_q)$, embedded as

$$\mathbf{x} \mapsto \begin{bmatrix} \mathbf{x} & 0 & 0 \\ 0 & E_{n-2k'} & 0 \\ 0 & 0 & \tilde{F}(\mathbf{x}) \end{bmatrix}.$$

If t centralizes all of these matrices, then $t \in \mathbf{G}_{n-2k'}$. Hence $C^\circ_{\mathbf{G}}([\mathbf{K}, \mathbf{K}]) \leq C^\circ_{\bar{\mathbf{G}}}([\mathbf{K}, \mathbf{K}]) \leq C^\circ_{\bar{\mathbf{G}}}([\bar{\mathbf{K}}, \bar{\mathbf{K}}]) \leq \mathbf{G}_{n-2k'}$, where the latter inclusion follows as above.

It remains to show that the order of a defect group for B and for $\bar{B}$ equals the ℓ-part of $|G_{n-2k'}(q)|$. By [40], the latter is the product of the ℓ-part of $|\mathbf{T}^F|$ and the ℓ-part of $w!$. Taking ordinary irreducible characters of B and of $\bar{B}$ of height 0 and applying the formula for the ℓ-part of their degrees given by Olsson [32, Propositons 5 and 13], we obtain the desired result.

To prove the second assertion, we may assume, by the remarks in 6.3, that $\mathbf{G}(t^*)$ is of the form $C_{\mathbf{G}}(t)$ for some $t \in G$. Since $\chi_{t^*,\rho}$ lies in $b_G(\mathbf{K}, \psi)$ we have $[\mathbf{K}, \mathbf{K}] \subseteq [C_{\mathbf{G}}(t), C_{\mathbf{G}}(t)]$, i.e., t centralizes $[\mathbf{K}, \mathbf{K}]$, and so $t \in G_{n-2k'}(q)$. □

Lemma 7.8. *Let t and t' be ℓ-elements of $G_{n-2k'}(q)$. Then t and t' are conjugate in $\bar{G}$ if and only if they are conjugate in G.*

Proof. If G is unitary, this follows from looking at the rational Jordan normal forms of t and t'.

Let G be non-unitary and suppose that t and t' are conjugate in G. Since the centralizer of t in $\bar{\mathbf{G}}$ is connected, it suffices to show that t and t' are conjugate in $\bar{\mathbf{G}}$. In turn, it suffices to show that the two elements are conjugate in $[\bar{\mathbf{G}}, \bar{\mathbf{G}}]$. In the even-dimensional orthogonal groups we have $k < k'$ (see 7.2), and so $[\bar{\mathbf{G}}, \bar{\mathbf{G}}] = [\mathbf{G}_{n-2k}, \mathbf{G}_{n-2k}]$ induces the full (conformal) orthogonal group on $[\mathbf{G}_{n-2k'}, \mathbf{G}_{n-2k'}]$.

The result now follows from the description of the conjugacy classes in classical groups given by Wall (see [39, Corollary to Theorem 1.3.1 and Theorem 1.5.2]). □

7.9. **The Reduction Theorem.** We are now ready to state the main theorem of this section. Recall that we are assuming that $e > 2$ in Case 1 and $e > 1$ in Case 2.1. Also, in Case 2, we assume that $\bar{\Gamma}$ is non-degenerate and that $\operatorname{rank}(\bar{\gamma}) \geq 2$ for the even-dimensional orthogonal groups.

Recall the definition of the Levi subgroups L_k and M_k of G in 4.4. Recall also that $\bar{G} = G_{n-2k}(q)$ is naturally isomorphic to a direct factor of L_k and of M_k. We may thus view characters of $\bar{G}$ as characters of L_k and of M_k via inflation.

Theorem 7.10. *Suppose that $k = s(B) \geq w(B) = w$. Then there is a bijection $\bar{\chi} \mapsto \chi$ between the ordinary irreducible characters in $\bar{B}$ and those in B, such that for every ordinary irreducible character $\bar{\chi}$ of $\bar{B}$, $R_{L_k}^G(\bar{\chi})_B = k!\chi$ and $R_{M_k}^G(\bar{\chi})_B = \chi$. (Here, the subscript B on a character indicates its restriction to the block B).*

Proof. Let D be a Sylow ℓ-subgroup of $G_{n-2k'}(q) \leq G_{n-2k}(q)$. Then, by Lemma 7.7, D is a defect group for B and for $\bar{B}$.

Put $\bar{\mathbf{V}} := \mathbf{V}_{n-2k}$ (see 4.2.7) and let $t \in D$. Notice that our assumptions on e along with the fact that t is an ℓ-element, imply that t has trivial determinant and multiplier. In other words, t is an element of $I_0(\bar{\mathbf{V}})$. Hence $[t, \bar{\mathbf{V}}] = [t, \mathbf{V}]$ and $\bar{\mathbf{V}}_t$ is a subspace of $\mathbf{V}_t$ of codimension $2k$ (for the notation consult 2.1). Since $C_{\mathbf{G}}(t)$ fixes $\bar{\mathbf{V}}_t$ and $[t, \mathbf{V}]$, we have

$$[C_{\mathbf{G}}(t), C_{\mathbf{G}}(t)] \leq I_0(\mathbf{V}_t) \times \mathbf{C}$$

with $\mathbf{C} = C_{I_0([t,\mathbf{V}])}(t)$. (Here, we have identified $I_0(\mathbf{V}_t)$ and $I_0([t, \mathbf{V}])$ with subgroups of $\mathbf{G}$ acting as the identity on $[t, \mathbf{V}]$ and $\mathbf{V}_t$, respectively.) Similarly,

$$[C_{\bar{\mathbf{G}}}(t), C_{\bar{\mathbf{G}}}(t)] \leq I_0(\bar{\mathbf{V}}_t) \times \mathbf{C}.$$

Let V denote the $\mathbb{F}_{q^\delta}$-span of the basis elements of $\mathbf{V}$ described in 4.2 and put $\bar{V} := V \cap \bar{\mathbf{V}}$. Then $\bar{V}_t$ is a subspace of V_t of codimension $2k$. Also, $I_0(\mathbf{V}_t)^F$ and $I_0(\bar{\mathbf{V}}_t)^F$ are isomorphic to $I_0(V_t)$ and $I_0(\bar{V}_t)$, respectively.

Let t^* and $\bar{t}^*$ be F-stable ℓ-elements of $\mathbf{G}^*$ and $\bar{\mathbf{G}}^*$, respectively, which correspond to t under the bijection described in 6.3. We want to describe the set of irreducible characters of B which lie in $\mathcal{E}(G, t^*)$. By 6.2 and Lemma 5.7, it suffices to describe the unipotent constituents of

$$R_{[\mathbf{K},\mathbf{K}]^F}^{I_0(V_t)\times C}(\psi_\gamma). \tag{10}$$

Note that $[\mathbf{K},\mathbf{K}] \leq I_0(\mathbf{V}_t)$. This follows from the construction of $\mathbf{K}$ described in 6.2. With the notation used there, D is contained in $I_0(\mathbf{V}')$, and hence acts trivially on $\mathbf{V}''$. In turn, $[t, \mathbf{V}] \leq \mathbf{V}'$, and thus $[\mathbf{K},\mathbf{K}]$ acts trivially on $[t, \mathbf{V}]$. Thus the constituents of (10) are the unipotent characters ρ of $I_0(V_t)$ (inflated to $I_0(V_t) \times C$), such that the label of ρ has e-core γ. It follows that an irreducible character $\chi_{t^*,\rho} \in \mathcal{E}(G, t^*)$ lies in B, if and only if ρ restricts to a unipotent character of $I_0(V_t)$ whose label has e-core γ.

Similarly, an irreducible character $\chi_{\bar{t}^*,\sigma} \in \mathcal{E}(\bar{G}, \bar{t}^*)$ lies in $\bar{B}$, if and only if σ restricts to a unipotent character of $I_0(\bar{V}_t)$ whose label has e-core $\bar{\gamma}$.

Let ρ be a unipotent character of $C_G(t)$ such that $\chi := \chi_{t^*,\rho}$ lies in B. Let Λ be an abacus diagram for the partition or symbol labelling ρ (viewed as unipotent character of $I_0(V_t)$). Then the e-core Γ of Λ is an abacus diagram for γ. Let $\bar{\Gamma}$ be the diagram obtained from Γ as in 7.5 by swapping two strings of Γ (or, in Case 2.2.2, by moving the last k beads of string 0_g to string 0_b and lifting them up by one position). Then, let $\bar{\Lambda}$ denote the diagram obtained from Λ by applying the corresponding moves to Λ.

It is clear that in all cases, $\bar{\Gamma}$ is the core of $\bar{\Lambda}$. Moreover, $\bar{\Gamma}$ is an abacus diagram of $\bar{\gamma}$ of the appropriate type. Also, since in Case 2, $\bar{\Gamma}$ is assumed to be a non-degenerate symbol, $\bar{\Lambda}$ is also non-degenerate and $\bar{\Lambda}$ and Λ have the same defect. Hence in all cases $\bar{\Lambda}$ defines a unique unipotent irreducible character, $\bar{\rho}$ of $I_0(\bar{V}_t)$. We view $\bar{\rho}$ as a unipotent character of $C_{\bar{G}}(t)$ via extension (using Lemma 5.7), and put $\bar{\chi} := \chi_{\bar{t}^*,\bar{\rho}}$. Then $\bar{\chi}$ lies in $\bar{B}$. Also, the map $\chi_{t^*,\rho} \mapsto \chi_{\bar{t}^*,\bar{\rho}}$ is a bijection between the irreducible characters of B in $\mathcal{E}(G, t^*)$ and the irreducible characters of $\bar{B}$ in $\mathcal{E}(\bar{G}, \bar{t}^*)$. Lemma 7.8 implies that the map $\chi \mapsto \bar{\chi}$ is a bijection between the irreducible characters in B and those in $\bar{B}$. This proves the first assertion of the theorem.

Now, let $\chi = \chi_{t^*,\rho}$ be as above and suppose that $\tau = \chi_{t_1^*,\theta}$ is an irreducible character of $\bar{G}$ in $\bar{B}$ such that χ is a constituent of $R_{L_k}^G(\tau)$. We want to show that $\tau = \bar{\chi}$ and that $\langle R_{L_k}^G(\tau), \chi\rangle_G = k!$. Since Harish-Chandra induction preserves Lusztig series, we may assume that θ is a unipotent character of $C_{\bar{G}}(t)$. By the comparison theorem (see the commutative diagram (8)), we have to compute the multiplicity

$$\left\langle R_{C_{L_k}(t)}^{C_G(t)}(\theta), \rho \right\rangle_{C_G(t)}.$$

From $L_k = \bar{G} \times GL_1(q^\delta)^k$, we get

$$C_{L_k}(t) = C_{\bar{G}}(t) \times GL_1(q^\delta)^k.$$

Using Lemma 5.7, we see that we have to compute

$$\left\langle R_{I_0(\bar{V}_t)\times C\times GL_1(q^\delta)^k}^{I_0(V_t)\times C}(\theta), \rho \right\rangle_{I_0(V_t)\times C},$$

where we denote again by θ and ρ their restrictions to the relevant subgroups.

Suppose that θ restricts to the character $\theta_0 \times \theta_1$ of $I_0(\bar{V}_t) \times C$. Then the above inner product is non-zero only if θ_1 is the trivial character of C; we assume from now on that this is the case and write θ for the restriction θ_0 of θ to $I_0(V_t)$. The inner product then becomes equal to

$$\left\langle R^{I_0(V_t)}_{I_0(\bar{V}_t)\times GL_1(q^\delta)^k}(\theta), \rho \right\rangle_{I_0(V_t)}$$

and, again by Lemma 5.7, this is equal to

$$\left\langle R^{G_{n_t}(q)}_{G_{n_t-2k}(q)\times GL_1(q^\delta)^k}(\theta), \rho \right\rangle_{G_{n_t}(q)},$$

where n_t is the $\bar{\mathbb{F}}_q$-dimension of $\mathbf{V}_t$.

Let Δ and Λ be abacus diagrams for the labels of the unipotent characters θ and ρ of $G_{n_t-2k}(q)$ and $G_{n_t}(q)$, respectively.

Then by the branching rules given in Section 3, in Case 1, the above inner product equals the number of sequences of 2-hooks that can be added to Δ to obtain Λ or conversely, this is equal to the number of sequences of 2-hooks that can be removed from Λ to obtain Δ. Notice that since τ is in the block $\bar{B}$, Δ has e-core $\bar{\Gamma}$. Therefore,

$$|\Lambda^i_e| = |\Gamma^i_e| = |\bar{\Gamma}^i_e| + k = |\Delta^i_e| + k,$$

where i, as in 7.5, is the string with $k = s^i_e(\Gamma)$. So, in order to get to Δ from Λ, we must remove k beads from string i of Λ. On the other hand, $|\Lambda| - |\Delta|$ is equal to $2k$, so in order to get from Λ to Δ by removing a sequence of 2-hooks, we can move at most k beads from their positions. Thus, the only way to get from Λ to Δ by removing a sequence of 2-hooks is to move k beads on string i of $\bar{\Lambda}$ two positions to the left to the corresponding row of string $i-2$.

Note that $w(\Lambda) \leq w$. Then, by Lemma 7.4, there are exactly k beads on string i of Λ for which there is no bead on the corresponding row of string $i-2$, thus each of these must be moved. But then the resulting diagram is $\bar{\Lambda}$. This shows that $\Delta = \bar{\Lambda}$, and thus that $\tau = \bar{\chi}$. Also, since the beads on string i of Λ may be moved in any order, we get that the number of possible sequences of 2-hooks that can be removed from Λ to get Δ is $k!$.

In Cases 2.1 and 2.2.1, there are k more beads on string i_g of Λ than on string i_g of Δ so at least k beads must be removed from string i_g of Λ (again, i_g is the string with $\check{s}^{i_g}_e(\Gamma)$, respectively $\hat{s}^{i_g}_e(\Gamma)$). Arguing as in Case 1, one sees immediately that $\Delta = \bar{\Lambda}$ and also that the total number of sequences of 1-hooks which can be added to $\bar{\Lambda}$ to get Λ is $k!$.

In Case 2.2.2, also, we see that there are k more beads on string 0_g of Λ than on string 0_b of Δ, so at least k beads must be removed from string 0_g of Λ. Also, since every move we make in order to go from Λ to Δ must translate into the removal of a 1-hook for the corresponding symbols, when we remove a bead from string 0_g of Λ we must place it on string 0_g but one row higher than it was previously. But we showed before that there are exactly k beads on string 0_g of Λ for which there is no bead on one higher row of string 0_b. This shows immediately that $\Delta = \bar{\Lambda}$ and also that the total number of sequences of 1-hooks which can be added to $\bar{\Lambda}$ to get Λ is $k!$.

To summarize, we have shown that $\bar{\chi}$ is the only character of $\bar{G}$ in the block $\bar{B}$ such that $R^G_{L_k}(\bar{\chi})$ contains χ and that the multiplicity of χ in $R^G_{L_k}(\bar{\chi})$ is $k!$.

We now consider Harish-Chandra induction of characters from M_k. Let $\chi = \chi_{t^*,\rho}$ and $\tau = \chi_{t^*,\theta}$ as before, consider τ as character of M_k via inflation, and suppose that $R_{M_k}^G(\tau)$ contains χ. Then, χ is contained in $R_{L_k}^G(\tau)$, and hence $\tau = \bar{\chi}$, by what we have already shown. It only remains to show that $\langle R_{M_k}^G(\bar{\chi}), \chi\rangle_G = 1$. As above, the multiplicity on the left hand side equals

$$\left\langle R_{G_{n_t-2k}(q)\times GL_k(q^\delta)}^{G_{n_t}(q)}(\bar{\rho}), \rho \right\rangle_{G_{n_t}(q)}. \tag{11}$$

This can be computed inside Weyl groups as explained in Section 3. Namely, there is a Weyl group $\hat{W}_{m'}$ of type $B_{m'}$ or $D_{m'}$, and (unordered) pairs of partitions (α^0, α^1) and $(\bar{\alpha}^0, \bar{\alpha}^1)$, such that (α^0, α^1) labels an irreducible character of $\hat{W}_{m'}$ corresponding to ρ, and $(\bar{\alpha}^0, \bar{\alpha}^1)$ labels an irreducible character of $\hat{W}_{m'-k}$ corresponding to $\bar{\rho}$. Then the inflation of $\bar{\rho}$ to $G_{n_t-2k}(q)\times GL_k(q^\delta)$ corresponds to the inflation of the latter character to $\hat{W}_{m'-k}\times S_k$. By the comparison theorem and the branching rules, the multiplicity (11) is zero unless $|\bar{\alpha}^0| \leq |\alpha^0|$ and $|\bar{\alpha}^1| \leq |\alpha^1|$ in which case, by (4), it equals

$$\sum_{\delta^0\vdash j}\sum_{\delta^1\vdash k-j} g_{\bar{\alpha}^0\delta^0}^{\alpha^0} g_{\bar{\alpha}^1\delta^1}^{\alpha^1} g_{\delta^0\delta^1}^{(k)}, \tag{12}$$

where $j = |\alpha^0| - |\bar{\alpha}^0|$. But $g_{\delta^0\delta^1}^{(k)}$ is only non-zero if $\delta^0 = (j)$ and $\delta^1 = (k-j)$, in which case $g_{\delta^0\delta^1}^{(k)} = 1$. Thus the above multiplicity equals

$$g_{\bar{\alpha}^0(j)}^{\alpha^0} g_{\bar{\alpha}^1(k-j)}^{\alpha^1}. \tag{13}$$

We have to show that this number equals 1. Let $\{x_1, \dots, x_r\}$ and $\{\bar{x}_1, \dots, \bar{x}_r\}$ be β-sets for α^0 and $\bar{\alpha}^0$, respectively, such that $x_1 < x_2 < \cdots < x_r$ and $\bar{x}_1 < \bar{x}_2 < \cdots \bar{x}_r$. Then, by Young's rule [20, 2.8.2], $g_{\bar{\alpha}^0(j)}^{\alpha^0} \in \{0,1\}$, and $g_{\bar{\alpha}^0(j)}^{\alpha^0} = 1$ if and only if $\bar{x}_j \leq x_j \leq \bar{x}_{j+1}$ for $1 \leq j \leq r-1$. The same rule applies, of course, to $g_{\bar{\alpha}^1(k-j)}^{\alpha^1}$. Now the way the unipotent label of $\bar{\rho}$ is obtained from the unipotent label of ρ, together with the rules for converting these labels into $(\bar{\alpha}^0, \bar{\alpha}^1)$ and (α^0, α^1), respectively (see 5.2), imply easily that $g_{\bar{\alpha}^0(j)}^{\alpha^0} g_{\bar{\alpha}^1(k-j)}^{\alpha^1} = 1$. This proves the theorem. □

8. Morita equivalence classes and source algebras

This section contains our main results. We keep all the notation of the previous section. Let D be a Sylow ℓ-subgroup of $G_{n-2k'}(q)$. Then, by Lemma 7.7, D is a defect group of B and of $\bar{B}$. Moreover, let $\mathcal{O}$ be a complete discrete valuation ring with field of fractions $\mathcal{K}$, which is large enough for G, and residue field of characteristic ℓ. The two centrally primitive idempotents of $\mathcal{O}\bar{G}$ and $\mathcal{O}G$ corresponding to $\bar{B}$ and B are denoted by $\bar{f}$ and f, respectively. Choose a primitive idempotent $\bar{j}$ of the algebra $(\mathcal{O}\bar{G}\bar{f})^D$ of D-fixed points of $\mathcal{O}\bar{G}\bar{f}$, and a primitive idempotent j of $(\mathcal{O}Gf)^D$ such that $\bar{j}\mathcal{O}\bar{G}\bar{j}$ is a source algebra for $\bar{B}$ and $j\mathcal{O}Gj$ is a source algebra for B. As usual, we consider these source algebras as interior D-algebras (for the definition and properties of source algebras and interior D-algebras see [38]).

We introduce the following terminology for interior D-algebras. Two interior D-algebras A_1 and A_2 are called equivalent if there is an isomorphism of algebras

$$\psi : A_1 \mapsto A_2$$

such that for all $x \in D$, $\psi(x)1_{A_1} = x1_{A_2}$.

Also, we say that A_1 and A_2 are equivalent up to a twist of D, if there is an automorphism σ of D such that A_1 and $\mathrm{Res}_\sigma(A_2)$ are isomorphic as interior D-algebras, i.e., if there is an isomorphism of algebras

$$\psi : A_1 \mapsto A_2$$

such that for all $x \in D$, $\psi(x)1_{A_1} = \sigma(x)1_{A_2}$.

The reduction theorem has the following consequence.

Theorem 8.1. *The blocks B and $\bar{B}$ of the reduction theorem are Morita equivalent. Furthermore, the source algebras $j\mathcal{O}Gj$ and $\bar{j}\mathcal{O}\bar{G}\bar{j}$ are equivalent up to a twist of D, i.e., there is an isomorphism of algebras*

$$\psi : j\mathcal{O}Gj \to \bar{j}\mathcal{O}\bar{G}\bar{j}$$

and an automorphism σ of D such that for every element x of D, $\psi(xj) = \sigma(x)\bar{j}$.

Proof. We use a result of Broué to see that the two blocks $\bar{B}$ and B are Morita equivalent. We put

$$u := \frac{1}{|U|}\sum_{v\in U} v \in \mathcal{O}G,$$

where U is the unipotent radical of Q, the standard parabolic subgroup of G with Levi complement M_k. We let

$$t := \sum_{g\in L} g \in \mathcal{O}G,$$

where L is the "linear component" of M_k isomorphic to $GL_k(q^\delta)$.

Then

$$X := \bar{f}tu\mathcal{O}Gf$$

is a $\bar{B}$-B-bimodule which is free and finitely generated as $\mathcal{O}$-module. Moreover, for any $\bar{B}$-lattice Y we have $Y \otimes_{\bar{B}} X \cong R^G_{M_k}(Y)_B$ (we again use the convention that Y also denotes the inflation of Y to an $\mathcal{O}M_k$-lattice).

We are now going to show that ${}_{\bar{B}}X$ and X_B are projective. Let $\bar{\Phi}$ be the character of the $\mathcal{K}\bar{G}$-module $\mathcal{K}\bar{B}$. Then the character of $\mathcal{K}X_B$ equals $\Phi := R^G_{M_k}(\bar{\Phi})_B$. By the reduction theorem, $k!\Phi = R^G_{L_k}(\bar{\Phi})_B$. Since ℓ does not divide $q^\delta - 1$, the inflation to L_k of $\bar{B}$ is a projective $\mathcal{O}L_k$-lattice, and thus $k!\Phi$ is the character of a projective $\mathcal{O}G$-lattice. In particular, Φ vanishes on all non-trivial ℓ-elements of G.

Since X_B is a direct summand of $t\mathcal{O}G$, it is a direct sum of indecomposable $\mathcal{O}G$-lattices with trivial source. Let Y be such an indecomposable direct summand of X_B, and let x be a non-trivial ℓ-element of G. Then the character of Y on x equals 0, since the value of the character of a trivial source lattice on an ℓ-element is a non-negative integer (see e.g. [27, Lemma II.16.2(ii)]). But then x can not be contained in any vertex of Y since the restriction of Y to its vertex contains a trivial direct summand (and thus the character of Y is non-zero for all elements of its vertex). It follows that Y has a trivial vertex and thus is projective.

The projectivity of ${}_{\bar{B}}X$ is proved similarly. Namely, $u\mathcal{O}Gf$ is a projective $\mathcal{O}M_k$-lattice, hence a direct summand of a free $\mathcal{O}M_k$-module, and so $tu\mathcal{O}Gf$ is a direct sum of indecomposable $\mathcal{O}M_k$-lattices with trivial source and vertices in L. Therefore, the restriction of $tu\mathcal{O}Gf$ to $\bar{G}$ is a projective module. This implies that ${}_{\bar{B}}X = \bar{f}tu\mathcal{O}Gf$ is a projective $\bar{B}$-module.

Now, by the reduction theorem the functor

$$- \otimes_{\mathcal{K}\bar{B}} \mathcal{K}X : \text{mod-}\mathcal{K}\bar{B} \mapsto \text{mod-}\mathcal{K}B$$

is a Morita equivalence. Thus Broué's theorem [3, Théorème 2.4], applies and $\bar{B}$ and B are Morita equivalent.

We may consider $t\mathcal{O}G$ as a left $\mathcal{O}[\bar{G} \times G]$ module via the action $(g_1 \times g_2)x := g_1 x {g_2}^{-1}$. Choose a transversal Y for the right cosets of $D \times L$ in G. Then the set $\{txy \mid x \in D, y \in Y\}$ is an $\mathcal{O}$-basis of $t\mathcal{O}G$, and this basis is stabilized by the action of the group $D \times D$, since D commutes with L. In other words, $\mathrm{Res}^{\bar{G}\times G}_{D\times D}\, t\mathcal{O}G$ is a permutation module. Since X is clearly a direct summand of $t\mathcal{O}G$ as $\mathcal{O}[D \times D]$-modules, it follows from [38, Corollary (27.2)], that $\mathrm{Res}^{\bar{G}\times G}_{D\times D} X$ is a permutation module.

Put $X^* := \mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})$, a B-$\bar{B}$-bimodule. Then $X^* \otimes_{\bar{B}} X$ is isomorphic to B, as B-B-bimodule (see [3, Remarque (3) and Démonstration du Théorème 2.3]). It follows that X is an indecomposable $\mathcal{O}[\bar{G} \times G]$-module belonging to the block $\bar{B} \otimes_{\mathcal{O}} B$ of $\mathcal{O}[\bar{G} \times G]$. Hence there is a vertex, say P of X (as $\mathcal{O}[\bar{G} \times G]$-modules) which is contained in $D \times D$. In particular, $\mathrm{Res}^{\bar{G}\times G}_{P}\, X$ is a permutation module. The second assertion of the theorem now follows from a result of Scott [37] and Puig [35, Remark 7.5]. □

For a given positive integer w, the reduction theorem allows us to compare a unipotent block of a finite classical group $G_n(q)$ of weight w, whose unipotent label γ satisfies a certain combinatorial condition depending on w, to a unipotent block of a smaller classical group with the same weight. We next determine an upper bound for the number of unipotent labels which fail to satisfy this combinatorial condition in case ℓ is a unitary prime for G.

Proposition 8.2. *Let w and e be positive integers.*

(i) *If e is odd, there are at most*

$$\prod_{i=1}^{(e-1)/2} \left[\left(\frac{e-1}{2}i + i^2\right)(w-1)^2 + \left(\frac{e-1}{2} + 3i\right)(w-1) + 2\right] \tag{14}$$

partitions γ which are their own e-cores and such that $s_e(\gamma) < w$. Moreover, such γ satisfy

$$|\gamma| \leq \frac{1}{4}(e-1)^2(e(w-1)+1)^2. \tag{15}$$

(ii) *There are at most*

$$[e(w-1)+2]\prod_{i=1}^{e-1}\left[(ei+i^2)(w-1)^2 + (e+3i)(w-1) + 2\right] \tag{16}$$

equivalence classes of symbols γ which are their own e-unitary cores and for which $\hat{s}_e(\gamma) < w$. Moreover, such γ satisfy

$$\mathrm{rank}(\gamma) \leq e^2((2e-1)(w-1)+1)^2. \tag{17}$$

(iii) *Suppose that $e > 2$. There are at most*

$$[e(w-1)+2]\,(\prod_{i=1}^{e-1}\left[(ei+i^2)(w-1)^2 + (e+3i)(w-1)+2\right])$$

$$+ [(e-1)(w-1)+2]\prod_{i=1}^{e-2}[i(w-1)+1] \tag{18}$$

equivalence classes of symbols γ which are their own e-unitary cores and which have the following property:

If Γ is a 2e-unitary abacus diagram for some element of γ, then for all i, $1 \leq i \leq e-1$ and for any $a \in \{g, b\}$ such that $\hat{s}_e^{i_a}(\Gamma) > 0$, either $\hat{s}_e^{i_a}(\Gamma) < w$ or the diagram obtained by swapping strings i_a and $(i-1)_a$ is degenerate. Moreover, such γ satisfy (17).

Proof. (i) Let γ be a partition such that γ is its own core and with $s_e(\gamma) < w$. As a β-set for γ we choose the first column hook length β-set Γ so that $\Gamma_e^0 = 0$. From $s_e(\gamma) < w$ it follows that, for $1 \leq i \leq (e-1)/2$, string $2i$ of the abacus diagram for Γ contains at most $i(w-1)$ beads. In particular, string $e-1$ contains at most $(e-1)(w-1)/2$ beads, and so string 1 contains at most $(e-1)(w-1)/2+w$ beads. Hence, for $1 \leq i \leq (e-1)/2$, string $2i-1$ contains at most $(e-1)(w-1)/2+1+i(w-1)$ beads. The number of abacus diagrams satisfying all these conditions equals the number given in (14).

The above proof also shows that the lowest row of Γ on which there can be a bead, is row $(e-1)(w-1)$ (on string $e-2$). Therefore, the greatest hook length of γ is less than or equal to $e(e-1)(w-1)+e-2$. It is easy to see that if the hook lengths of a partition are bounded above by x, then the magnitude of the partition is bounded by $(x+1)^2/4$. This gives the required bound on $|\gamma|$.

(ii) Let γ be an equivalence class of symbols such that γ is its own e-unitary core and such that $\hat{s}_e(\gamma) < w$. We choose an e-unitary diagram Γ for γ such that there are no beads on string 0_g of Γ. From $\hat{s}_e(\gamma) < w$ it follows that, for $1 \leq i \leq (e-1)$, string i_g of the abacus diagram for Γ contains at most $i(w-1)$ beads. In particular, string $(e-1)_g$ contains at most $(e-1)(w-1)$ beads and this in turn implies that string 0_b contains at most $e(w-1)+1$ beads. Consequently, for $1 \leq i \leq (e-1)$, string i_b of the abacus diagram for Γ contains at most $(e+i)(w-1)+1$ beads. The number of such abacus diagrams equals the number given in (16). Now the lowest row of Γ on which there can be a bead is row $(2e-1)(w-1)$ (on string $(e-1)_b$). Therefore if Γ represents the symbol $\{X, Y\}$ then the largest element of X or Y is less than or equal to $(2e-1)(w-1)e+(e-1)$. On the other hand, if the maximum element of a set X is less than or equal to x and if the maximum element of a set Y is less than or equal to y, then the rank of the symbol $\{X, Y\}$ is less than or equal to $x(x+1)/2+y(y+1)/2$. This proves the inequality (17).

(iii) Let γ be an equivalence class of symbols satisfying the conditions in (iii) and let Γ be an 2e-unitary diagram for γ which has no beads on string 0_g. We will show that either Γ is a diagram of the type described in (ii), or Γ satisfies the following: $|\Gamma^{0_g}| = |\Gamma^{1_b}| = 0$, $w \leq |\Gamma^{1_g}| = |\Gamma^{0_b}| \leq (e-1)(w-1)+1$ and for all j, $2 \leq j \leq e-1$, $|\Gamma^{j_g}| = |\Gamma^{j_b}| \leq (j-1)(w-1)$.

Note that the number of diagrams of the above description is

$$[(e-1)(w-1)+2]\prod_{i=1}^{e-2}[i(w-1)+1],$$

thus proving the first part of (iii). Also, it is easy to see that if $\{X, Y\}$ is a symbol whose diagram answers to the above description, then the maximum elements of X and Y are bounded above by $(2e-1)(w-1)e+(e-1)$ and therefore the inequality (17) is satisfied.

If $\hat{s}_e(\gamma) < w$, then it is immediate that Γ is as in (ii), therefore we assume that $\hat{s}_e(\gamma) \geq w$.

By assumption, $|\Gamma^{0_g}| = 0$, therefore $\hat{s}_e^{0_g}(\Gamma) \leq 0 < w$. We claim that $\hat{s}_e^{0_b}(\Gamma) < w$ and $\hat{s}_e^{1_b}(\Gamma) < w$. Indeed, suppose if possible, that $\hat{s}_e^{0_b}(\Gamma) \geq w$ and let Γ' be the 1-shift of Γ. Then $\hat{s}_e^{1_b}(\Gamma') = \hat{s}_e^{0_b}(\Gamma) \geq w$ and $|\Gamma'^{1_g}| = |\Gamma^{0_g}| = 0$. On the other hand, $|\Gamma'^{0_b}| \geq 1$ since Γ' is the 1-shift of Γ. So the diagram obtained by swapping strings 1_b and 0_b of Γ' is non-degenerate, and this is a contradiction since Γ' is a diagram for γ. Now suppose that $\hat{s}_e^{1_b}(\Gamma) \geq w$. Then there are at least w beads on string 1_b of Γ. On the other hand, there are no beads on string 0_g of Γ. Hence the diagram obtained by swapping strings 1_b and 0_b of Γ is non-degenerate and this is again a contradiction. This proves our claim.

Let i , $0 \leq i \leq e-1$ and $a \in \{g, b\}$ be such that $\hat{s}_e^{i_a}(\Gamma) \geq w$. We have shown above that either $i_a = 1_g$ or $i > 1$. Since $i \geq 1$, the diagram obtained by swapping string i_a and $(i-1)_a$ is degenerate. Explicitly, this means that $|\Gamma^{(i-1)_g}| = |\Gamma^{i_b}|$, $|\Gamma^{(i-1)_b}| = |\Gamma^{i_g}|$ and for j different from i and $i-1$, $|\Gamma^{j_g}| = |\Gamma^{j_b}|$. Also, $|\Gamma^{i_a}| \geq |\Gamma^{(i-1)_a}| + w$, so in particular, we have that $|\Gamma^{(i-1)_g}| \neq |\Gamma^{(i-1)_b}|$ and $|\Gamma^{i_b}| \neq |\Gamma^{i_g}|$. Hence for any string $i'_{a'}$, such that $i' \geq 1$ and $i' \neq i$, the diagram obtained by swapping strings $i'_{a'}$ and $(i-1)'_{a'}$ of Γ is non-degenerate. But then by our assumptions on γ, $\hat{s}_e^{i'_{a'}}(\Gamma) < w$ for all strings $i'_{a'}$ different from i_a.

We first consider the case $i > 1$. Then $0 \notin \{i, i-1\}$, and thus $|\Gamma^{0_b}| = |\Gamma^{0_g}| = 0$. So we may assume that $a = b$. By the remarks in the previous paragraph, for all j, $1 \leq j \leq e-1$, $\hat{s}_e^{j_g}(\Gamma) < w$. But then arguing as in (ii), we see immediately that $|\Gamma^{j_g}| \leq j(w-1)$ for all j, $1 \leq j \leq e-1$ and hence if j is not equal to i or $i-1$, $|\Gamma^{j_b}| = |\Gamma^{j_g}| \leq j(w-1) \leq (e+j)(w-1)+1$. Also, clearly $|\Gamma^{i_b}| = |\Gamma^{(i-1)_g}| \leq (i-1)(w-1) \leq (e+i)(w-1)+1$ and $|\Gamma^{(i-1)_b}| = |\Gamma^{i_g}| \leq i(w-1) \leq (e+i-1)(w-1)+1$. Thus Γ is a diagram of the type described in (ii).

It remains to consider the case that $i_0 = 1_a$. In this case, $|\Gamma^{1_b}| = |\Gamma^{0_g}| = 0$ and for all j, $1 \leq j \leq e-1$, $\hat{s}_e^{j_b}(\Gamma) < w$. Thus for all j, $2 \leq j \leq e-1$, $|\Gamma^{j_g}| = |\Gamma^{j_b}| \leq (j-1)(w-1)$. The fact that $e > 2$ now implies that $|\Gamma^{(e-1)_g}| = |\Gamma^{(e-1)_b}| \leq (e-2)(w-1)$. On the other hand, we showed above that $\hat{s}_e^{0_b}(\Gamma) < w$. So, $|\Gamma^{1_g}| = |\Gamma^{0_b}| \leq |\Gamma^{(e-1)_g}| + w \leq (e-1)(w-1)+1$. Finally, the assumption that $\hat{s}_e^{1_g}(\Gamma) \geq w$ implies that $|\Gamma^{1_g}| \geq w$. This shows that Γ is of the type described at the beginning of the proof of (iii). □

Notice that (i) does not hold if e is odd, (ii) and (iii) do not hold for 2e-linear diagrams of symbols and (iii) does not hold if $e = 2$.

We can now prove that the Donovan and Puig conjectures hold for certain families of unipotent blocks of finite classical groups. We will use the following two remarks in our proof.

First, suppose that A_1 and A_2 are interior D-algebras that are equivalent up to a twist σ of D. If σ is an inner automorphism of D which is induced by the element y, then composing ad $y \circ \psi$ is an interior D-algebra isomorphism between A_1 and A_2. Thus the number of equivalence classes of interior D-algebras that are equivalent up to a twist of D, to a given interior D-algebra is at most $|\mathrm{Out}(D)|$, where $\mathrm{Out}(D)$ denotes the group of outer automorphisms of D.

Secondly, if B is a block of a finite group G and if D is a defect group of B, then any two source algebras of B are equivalent up to a twist of D.

Theorem 8.3. *Let ℓ be a prime, q be a prime power not divisible by ℓ, and w a positive integer. Let $\{G_n(q) \mid n = 1, 2, \dots\}$ be one of the families in* 4.1 *and suppose that ℓ is a unitary prime for this family.*

(i) *If $G_n(q) = GU_n(q)$, then the number of Morita equivalence classes of unipotent ℓ-blocks of weight w occurring in the group algebras of $\{GU_n(q) \mid n = 1, 2, \dots\}$ is finite. In fact, there are at most*

$$\prod_{i=1}^{(\ell-3)/2} \left[\left(\frac{\ell-3}{2}i + i^2\right)(w-1)^2 + \left(\frac{\ell-3}{2} + 3i\right)(w-1) + 2\right] \tag{19}$$

such blocks. Furthermore, any such block is Morita equivalent to a unipotent block of $GU_n(q)$, for some n which is less than or equal to

$$\frac{1}{4}(\ell-3)^2\left[(\ell-2)(w-1)+1\right]^2 + (\ell-2)w.$$

Let D be a common defect group for these blocks (see 6.2). The number of isomorphism classes of interior D-algebras which are source algebras for these blocks is at most $|\mathrm{Out}(D)|$ times the number given in (19); in fact the corresponding source algebras are equivalent up to a twist of D.

(ii) *If the groups $G_n(q)$ are as in 4.2.3 and 4.2.4, then the number of Morita equivalence classes of unipotent ℓ-blocks of weight w occurring in the group algebras of $\{G_n(q) \mid n = 1, 2, \dots\}$ is finite. There are at most*

$$\left[\frac{\ell-1}{2}(w-1)+2\right] \prod_{i=1}^{(\ell-3)/2} \left[(\frac{\ell-1}{2}i + i^2)(w-1)^2 + (\frac{\ell-1}{2} + 3i)(w-1) + 2\right] \tag{20}$$

such blocks. Every unipotent block is Morita equivalent to a block of $G_n(q)$ for some n where n is less than or equal to

$$\frac{1}{2}(\ell-1)^2\left[(\ell-2)(w-1)+1\right]^2 + (\ell-1)w.$$

If D is a common defect group for these blocks, the number of isomorphism classes of interior D-algebras which are source algebras for these blocks is at most $|\mathrm{Out}(D)|$ times the number given in (20); the corresponding source algebras are equivalent up to a twist of D.

(iii) *If the groups $G_n(q)$ are as in 4.2.5 and 4.2.6, and if $e \neq 2$, i.e., if ℓ does not divide q^2+1, then the number of Morita equivalence classes of unipotent ℓ-blocks with non-degenerate cores and of weight w occurring in the group algebras of $\{G_n(q) \mid n = 2, 4, \dots\}$ is finite. There are at most*

$$\left[\frac{\ell-1}{2}(w-1)+2\right] \prod_{i=1}^{(\ell-3)/2} \left[(\frac{\ell-1}{2}i + i^2)(w-1)^2 + (\frac{\ell-1}{2} + 2i)(w-1) + 2\right]$$
$$+ \left[(\frac{\ell-5}{2})(w-1)+1\right] \prod_{i=1}^{(\ell-5)/2} \left[i(w-1)+1\right] + 4 \tag{21}$$

such blocks. Every unipotent block is Morita equivalent to a block of $G_n(q)$ for some n where n is less than or equal to

$$\max\{8 + (\ell-1)w, \frac{1}{2}(\ell-1)^2\left[(\ell-2)(w-1)+1\right]^2 + (\ell-1)w\}.$$

If D is a common defect group for these blocks, the number of isomorphism classes of interior D-algebras which are source algebras for these blocks is at most $|\mathrm{Out}(D)|$ times the number given in (21), and the corresponding source algebras are equivalent up to a twist of D.

Proof. (i) First, notice that if $e = 1$, then there is only one partition which is its own e-core, namely the empty partition, and hence there is only one group $G_n(q)$ which has a unipotent block of weight w. Hence we may assume that $e > 1$. Thus the hypotheses of the reduction theorem are satisfied. By inductively applying the reduction theorem and Theorem8.1, we see that every unipotent ℓ-block is Morita equivalent to one whose e-core γ satisfies $s_e(\gamma) < w$. Thus the number of such blocks is bounded above by the number given in (14). But it also follows from Theorem 8.1 that the source algebras of any such block are equivalent up to a twist of D, to the source algebras of one whose e-core γ satisfies $s_e(\gamma) < w$. The bound on the number of isomorphism classes of source algebras now follows from (14) and from the two remarks given before the statement of the theorem.

Also, if a block of a finite unitary group $G_n(q)$ has weight w and a core whose magnitude is r, then $n = r + ew$. Thus, by (15), every unipotent ℓ-block is Morita equivalent to a block of $G_n(q)$ where n satisfies

$$n \leq \frac{1}{4}(e-1)^2(e(w-1)+1)^2 + ew$$

and the corresponding source algebras are equivalent up to a twist of D. Now $\ell \neq 2$ since $e > 1$, and thus $e \leq \ell - 2$ since e is odd. The result follows.

(ii) The unipotent irreducible characters of the groups $G_n(q)$ are labelled by equivalence classes of symbols of odd defect. Removing an e-cohook from a symbol changes the defect by 2. Hence, the unipotent labels of unipotent blocks have odd defect and in particular, are non-degenerate. Therefore, by inductively applying the reduction theorem we see that every unipotent block is Morita equivalent to to one whose e-unitary core γ satisfies $\hat{s}_e(\gamma) < w$. The result now follows from (16) and (17) and the fact that $e \leq (\ell-1)/2$, since ℓ is a unitary prime for $G_n(q)$. The source algebra can be treated similarly.

(iii) Let B be a block with unipotent label γ. If $e = 1$, the reduction theorem can be applied to B if $\hat{s}_e(\gamma) \geq w$ and the symbol $\bar{\gamma}$ as defined for Case 2.2.2 of the reduction theorem has rank at least two, since in this case $\bar{\gamma}$ is non-degenerate. If $\mathrm{rank}(\bar{\gamma}) \geq 2$, the result thus follows as in (ii). If $\bar{\gamma}$ is non-degenerate but $\mathrm{rank}(\bar{\gamma}) < 2$, then γ is equivalent to $\{\{1,3\},\{0,2\}\}$.

If $e > 2$ and the symbol $\bar{\gamma}$ as defined for Case 2.2.1 is non-degenerate and has rank less than two, then γ is equivalent to one of $\{\{2\},\{0\}\}$, $\{\{0,2\},\{\}\}$ or $\{\{0,1,2\},\{0\}\}$. If $\mathrm{rank}(\bar{\gamma}) > 2$, the reduction theorem may be applied an the result follows from (17) and (18).

The summand 4 at the end of Formula (21) accounts for the four cores γ with $\mathrm{rank}(\bar{\gamma}) < 2$. The largest rank of such a γ is 4. □

Corollary 8.4. *Let ℓ be a prime, D a finite ℓ-group and q be a prime power not divisible by ℓ. Let $\mathcal{B}$ be the set of unipotent ℓ-blocks with defect group D occurring in the groups $\{G_n(q) \mid n = 1, 2, \dots\}$ of the families in* 4.1 *satisfying the following conditions: A unipotent ℓ-block of a group $G = G_n(q)$ is in $\mathcal{B}$ if and only if ℓ is a unitary prime for G, and if G is an even-dimensional orthogonal group, then ℓ does not divide q^2+1 and the unipotent label of B is non-degenerate.*

Then the number of isomorphism classes of the source algebras of the blocks in $\mathcal{B}$ is finite. In particular, there are only finitely many Morita equivalence classes among these blocks. Moreover, every such block is Morita equivalent to a block of some $G_n(q)$, where n is less than a bound $N(|D|)$ not depending on q.

Proof. Since $\ell^{w(B)} \leq |D|$ for $B \in \mathcal{B}$, this follows directly from Theorem 8.3. □

Example 8.5. Let ℓ be a prime, q be a prime power not dividing ℓ such that e, the order of $-q$ modulo ℓ is odd. Thus ℓ is a unitary prime for the groups $GU_n(q)$, $n = 1, 2, \ldots$

(1) There are exactly $(e+1)/2$ unipotent ℓ-blocks B with $w(B) = 1$ and $s(B) = 0$, whereas the bound (19) equals $2^{(\ell-3)/2}$ in this case. The e-cores of these blocks are the triangular partitions $(s, s-1, \ldots, 1)$, and the blocks occur in the groups $GU_{s(s+1)/2+e}(q)$, $0 \leq s \leq (e-1)/2$.

Since e is odd, $e < (\ell-1)/2$, such that there is an exceptional vertex on the Brauer trees of these blocks. The blocks are distinguished by the location of the exceptional vertex (see [12, **(6A)**]).

All other unipotent ℓ-blocks of weight 1 occurring in the groups $GU_n(q)$ are Morita equivalent to one of these. (Of course this is was known to be true by [11] as well, since two cyclic blocks with the same Brauer tree are Morita equivalent.)

(2) Let $e = 3$ and $w = 2$. Then there are the following e-cores γ with $s_e(\gamma) < 2$: (–), (1), (2), (1, 1), (3, 1), (2, 1, 1), and (4, 2, 1, 1). The decomposition matrices (up to three unknown parameters) of the unipotent characters of these blocks have been computed by Gunter Malle and the first author.

9. Concluding Remarks

In the final section we sketch some general ideas allowing to approach Donovan's conjecture for certain classes of groups. Throughout this section we let K denote an algebraically closed field of characteristic ℓ. By a block we mean an ℓ-block of a group algebra KG for some finite group G. If B is a block, we let $d(B)$ denote its defect, and $c(B)$ the largest Cartan invariant of B.

The following two propositions were first published in [17].

Proposition 9.1. *Let $\mathcal{B}$ be a set of blocks. Suppose there exist positive integers d and c and a finite field $K_0 \subset K$ such that for all blocks $B \in \mathcal{B}$ the following conditions are satisfied.*

(a) $d(B) \leq d$,
(b) $c(B) \leq c$,
(c) $B \cong B_0 \otimes_{K_0} k$ *for some split K_0-algebra B_0.*

Then there are only finitely many Morita equivalence classes among the blocks in $\mathcal{B}$.

Proof. Let $B \in \mathcal{B}$. By the results of Brauer and Feit, B has at most $\ell^{2d}/4 + 1$ irreducible characters. By (b), the sum of the entries of $C(B)$ is at most $c(\ell^{2d}/4 + 1)^2$.

Let $P_1, \ldots, P_n$ denote a set of representatives for the isomorphism classes of the projective indecomposable modules of B_0 and let $M := P_1 \oplus \cdots \oplus P_n$. Put $A_0 := \mathrm{End}_{B_0}(M)$. Since K_0 is a splitting field for B_0 by assumption, the projective B-module $P_i \otimes_{K_0} K$ is indecomposable. Thus $P_1 \otimes_{K_0} K, \ldots, P_n \otimes_{K_0} K$ is a complete set of representatives for the projective indecomposable modules of B. It follows that $A := A_0 \otimes_{K_0} K \cong \mathrm{End}_B(M \otimes_{K_0} K)$ is a basic algebra for B. By the first paragraph of the proof, $\dim_{K_0} A_0 = \dim_K A \leq c(\ell^{2d}/4 + 1)^2$.

Since there are only finitely many isomorphism classes among the K_0-algebras of fixed, bounded dimension, the above implies that there are only finitely many isomorphism classes among the basic algebras of the blocks in $\mathcal{B}$. Since two K-algebras are Morita equivalent if and only if their basic algebras are isomorphic, the result follows. $\square$

Proposition 9.2. *Let G be a finite group and B a block of KG. Let $\varphi_1, \dots, \varphi_n$ denote the K-characters of the simple B-modules. If K_0 is the finite field containing the $\varphi_i(g)$, $i = 1, \dots, n$, $g \in G$, then there is a K_0-algebra B_0 with $B \cong B_0 \otimes_{K_0} K$. Moreover, K_0 is a splitting field for B_0.*

Proof. Let $e \in Z(KG)$ be the central idempotent with $B = KGe$. By Osima's theorem and Brauer reciprocity it follows that $e \in K_0G$ (see [26, Theorem III.2.9]). Hence $B = B_0 \otimes_{K_0} K$ with $B_0 := K_0Ge$. Also, every simple B-module is realizable over K_0 (see [19, Corollary 9.23]), and so K_0 is a splitting field for B_0. $\square$

We can now show that Donovan's conjecture is true for the class of unipotent ℓ-blocks of general linear groups.

Theorem 9.3. *For non-negative integers n and d, let $\mathcal{B}_{n,d}$ denote the set of unipotent blocks of $KGL_n(q)$ of defect d, where q runs through the prime powers not divisible by ℓ. Then there are only finitely many Morita equivalence classes among the blocks in*

$$\bigcup_{n\geq 1} \mathcal{B}_{n,d}.$$

Proof. Fix a prime power q not divisible by ℓ. The results of Jost [21, Theorem 6.2] imply that there is a bound N, depending only on d and ℓ, but not on q, such that every unipotent block of defect d of $KGL_m(q)$ for some $0 \neq m \in \mathbb{N}$ is Morita equivalent to a unipotent block of the same defect of $KGL_n(q)$ for some $n \leq N$. Thus every block in $\cup_{n\geq 1}\mathcal{B}_{n,d}$ is Morita equivalent to one in $\cup_{n=1}^{N}\mathcal{B}_{n,d}$.

Assumption (a) of Proposition 9.1 is trivially satisfied for $\mathcal{B}_{n,d}$. By results of Dipper and James [9], the decomposition numbers of the blocks in $\mathcal{B}_{n,d}$ are bounded independently of q. Thus (b) is also satisfied. All unipotent characters of $GL_n(q)$ are rational valued. The same is true for the irreducible Brauer characters of the unipotent blocks. By Proposition 9.2, Assumption (c) of Proposition 9.1 is satisfied with $K_0 = \mathbb{F}_\ell$.

Thus each $\mathcal{B}_{n,d}$ contains only finitely many Morita equivalence classes and we are done. $\square$

A similar argument works for the class of unipotent ℓ-blocks of classical groups of fixed dimension, for which ℓ is a linear prime.

Theorem 9.4. *For non-negative integers n and d, let $\mathcal{B}_{n,d}^{l}$ denote the set of unipotent blocks of $KG_n(q)$ of defect d, where q runs through the prime powers not divisible by ℓ, and $G_n(q)$ is one of the classical groups introduced in 4.1 such that ℓ is linear for $G_n(q)$. Then there are only finitely many Morita equivalence classes among the blocks in $\mathcal{B}_{n,d}^{l}$.*

Proof. We have to check that the assumptions of Proposition 9.1 are satisfied. Again, the decomposition numbers of the blocks in $\mathcal{B}_{n,d}^{l}$ are bounded independently of q (see [16, Corollary 8.10]). Finally, the unipotent characters of all the classical groups are rational valued. This follows from the facts that, by [30, 4.23], the unipotent characters of the classical groups are uniquely determined by their multiplicities in the Deligne-Lusztig generalized characters $R_T^G(1)$, and that the latter are rational valued. (We thank Meinolf Geck for this argument.) $\square$

To obtain the finiteness of the Morita equivalence classes in $\cup_{n\geq 1}\mathcal{B}^l_{n,d}$, we would need a result like Theorem 8.3 for the linear primes as well. At the moment we can only prove such a result for q-Schur algebras, and thus for certain quotients of the blocks.

On the other hand, for a prime power q not divisible by ℓ and a non-negative integer d, let $\mathcal{B}^u_{q,d}$ denote the set of unipotent blocks of $KG_n(q)$ of defect d, where n runs through the positive integers, and $G_n(q)$ is one of the classical groups introduced in 4.1 such that ℓ is unitary for $G_n(q)$. Then Theorem 8.3 gives the finiteness of Morita equivalence classes in $\mathcal{B}^u_{q,d}$. To obtain the finiteness for $\cup_q\mathcal{B}^u_{q,d}$, where q runs through the prime powers, we would need a result bounding the decomposition numbers of the blocks in $\mathcal{B}^u_{n,d}$ independently of q (where the latter symbol has a meaning analogous to $\mathcal{B}^l_{n,d}$ with "linear" replaced by "unitary"). Such a result has been proved by Okuyama and Waki for the 4-dimensional symplectic groups $Sp_4(q)$ [31].

There is a reduction theorem to Donovan's conjecture, due to Burkhard Külshammer [25], which allows to restrict attention to blocks of groups which are generated by their defect groups. In his PhD-thesis, Olav Düvel has reduced the answer to Brauer's Problem 22 to the case of quasi-simple groups. It is not unlikely that a similar reduction theorem for Donovan's or perhaps even for Puig's conjecture exists. The proof of Donovan's conjecture could then be completed with the steps suggested here.

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