LOW-DIMENSIONAL REPRESENTATIONS OF SPECIAL UNITARY GROUPS

GERHARD HISS AND GUNTER MALLE

Abstract. We prove gap results for the low-dimensional representations of unitary groups in non-defining characteristics. More precisely, we give a lower bound for the third smallest degree of a non-trivial absolutely irreducible representation of the groups mentioned above, which is almost of the order of magnitude of the square of the two smallest such representations. As a corollary we obtain the uniqueness of Weil representations in all cross characteristics for the unitary groups. Our approach uses some results on ordinary representations which can be proved using Deligne-Lusztig theory and which might be of independent interest, as well as information on decomposition matrices.

1. Introduction

In this paper we investigate the low-dimensional representations of the special unitary groups $G = \mathrm{SU}_n(q)$ in characteristics ℓ not dividing q. We prove that the non-trivial representations of the two smallest degrees are (the ℓ-modular reductions of) the Weil representations, of degree approximately q^{n-1}. We also show that the degree of any other absolutely irreducible representation of G is at least in the order of magnitude of q^{2n-4}, thus giving a gap of roughly this size between the second and third lowest dimensions. It is easy to see that there do exist irreducible representations of dimension approximately q^{2n-4} so that the bound is best possible, up to lower order terms. A similar result for the special linear groups has recently been obtained by Guralnick and Tiep [18] and further improved by Brundan and Kleshchev [2]. But while the argument in [18] for the special linear groups requires only rather elementary ingredients, we have to make use of Deligne-Lusztig theory for the unitary case.

Results of this type seem to have many applications. We have at least two immediate applications in mind. The first is to the study of irreducible alternating and symmetric squares of representations of quasi-simple groups. The second is to the matrix recognition program in computational group theory.

The proof uses three main ingredients. First, we make use of the fact that the sets $\mathcal{E}_\ell(G, s)$ are unions of ℓ-blocks, which was proved by Broué and Michel [1]. Secondly, Geck [12] has shown that the decomposition matrix of $\mathrm{GU}_n(q)$ is lower unitriangular. Finally, we are able to give upper bounds for the entries in a (small) part of the decomposition matrix of G modulo ℓ by constructing characters which are projective for all ℓ not dividing q, and which are not too big. Note that all of the above ingredients involve only assertions about ordinary characters of G.

Our general method to give lower bounds for the degrees of the irreducible modular characters of $\mathrm{SU}_n(q)$ is quite simple. Unfortunately, it only works well if n is not too small. For $n \leq 5$ we have to resort to some unpleasant ad hoc arguments (see, for examples, Lemmas 12 or 13). These complications could have been avoided if the representation theory of these small rank groups had been studied in more detail before, as was done by Geck for $\mathrm{SU}_3(q)$, or by the first author for $G_2(q)$. This shows once more that the investigation of series of groups of Lie type of small rank is of vital interest even for proving results for all groups of Lie type, very much like the "Atlas of Finite Groups" allows to deal with all kinds of degenerate situations in the theory of finite simple groups.

2. Results on ordinary characters of groups of Lie type

We first collect some general results on complex irreducible characters of finite groups of Lie type, which allow to restrict the blocks in which modular irreducible characters of small degrees may lie. Let G be a finite reductive group of characteristic p. In other words, $G = \mathbf{G}^F$ is the group of fixed points under a Frobenius morphism F of a connected reductive algebraic group $\mathbf{G}$, where F is defined with respect to some $\mathbb{F}_q$-structure with q a power of p. According to Lusztig, the set $\mathrm{Irr}(G)$ of complex irreducible characters of G may be partitioned into Lusztig series $\mathcal{E}(G,s)$, parametrized by conjugacy classes of semisimple F-stable elements s in a group $\mathbf{G}^*$, dual to $\mathbf{G}$. Moreover, there is a bijection between $\mathcal{E}(G,s)$ and the unipotent characters $\mathcal{E}(C_{G^*}(s),1)$ of the centralizer of s in $G^* := \mathbf{G}^{*F}$. If $C_{\mathbf{G}^*}(s)$ is not connected, the unipotent characters of $C_{G^*}(s)$ are, by definition, the irreducible constituents of the unipotent characters of $C_{\mathbf{G}^*}(s)^{\circ F}$, induced up to $C_{\mathbf{G}^*}(s)^F$ (see [23, Corollary 6.1] and [6]). If ψ denotes this bijection then the degrees are related by

$$\chi(1) = \left(\frac{|G|}{|C_{G^*}(s)|}\right)_{p'} \psi(\chi)(1) \tag{1}$$

(see [5, Remark 13.24]). We write $\chi_{s,\lambda}$ for $\chi \in \mathcal{E}(G,s)$ with $\psi(\chi) = \lambda$.

We now describe the above bijection in a special case, which, however, always applies in the general unitary groups. For $s \in G^*$ semisimple, put $\mathbf{G}(s) := C_{\mathbf{G}^*}(s)^*$, and $G(s) := \mathbf{G}(s)^F$. Suppose that $C_{\mathbf{G}^*}(s)$ is an F-stable Levi subgroup (not necessarily 1-split) of $\mathbf{G}^*$. Then we may identify $\mathbf{G}(s)$ with an F-stable Levi subgroup of $\mathbf{G}$, and Lusztig induction $R_{G(s)}^G$ induces a bijection

$$\mathcal{E}(G(s),1) \to \mathcal{E}(G,s), \qquad \vartheta \mapsto \varepsilon_G \varepsilon_{G(s)} R_{G(s)}^G(\hat{s}\vartheta), \tag{2}$$

where $\hat{s}$ is a suitable linear character of $G(s)$, and $\varepsilon_G = (-1)^{\mathbb{F}_q\text{-rank of } \mathbf{G}}$ (see [5, Theorem 13.25(ii) and Proposition 13.30]).

Now let ℓ be a prime not dividing q. For a semisimple ℓ'-element $s \in G^*$ define

$$\mathcal{E}_\ell(G,s) := \bigcup_t \mathcal{E}(G,st)$$

where the union runs over all ℓ-elements t in $C_{G^*}(s)$. Then, by a result of Broué and Michel, $\mathcal{E}_\ell(G,s)$ is a union of ℓ-blocks [1]. Since the degree of any ordinary character in $\mathcal{E}_\ell(G,s)$ is divisible by $(|G|/|C_{G^*}(s)|)_{p'}$ by (1), and for all ℓ-elements t centralizing s, the centralizer of st is contained in the one of s, this implies the following.

Proposition 1. *Let ρ be an absolutely irreducible ℓ-modular Brauer character of G. Assume that ρ lies in the union of ℓ-blocks $\mathcal{E}_\ell(G,s)$ for some semisimple ℓ'-element s of G^*. Then*

$$\left(\frac{|G|}{|C_{G^*}(s)|}\right)_{p'} \quad \text{divides} \quad \rho(1).$$

In particular $\chi_{s,1}$, the so-called semisimple character in $\mathcal{E}(G,s)$, is irreducible modulo ℓ.

This result already gives strong *a priori* information on the blocks which can possibly contain Brauer characters of low degree.

3. Harish-Chandra series

Keep the situation and notation of the preceding section. We have to recall some facts on the distribution of representations of a finite group of Lie type into Harish-Chandra series. For the results collected in this section see [19] and [7]. For each 1-split Levi subgroup L of G (where 1-split means that L is the Levi subgroup of a parabolic subgroup of G) we denote by ${}^*R_L^G$ and R_L^G the maps "Harish-Chandra restriction" and "Harish-Chandra induction", respectively, defined on generalized ℓ-modular characters of G, respectively L. Let ρ be an absolutely irreducible ℓ-modular character of G. There is a minimal Levi subgroup L (unique up to conjugation) such that ${}^*R_L^G(\rho)$ is non-zero. Any such minimal Levi subgroup L is called a Harish-Chandra vertex of ρ. The irreducible characters whose Harish-Chandra vertex equals G are called cuspidal. If L is a Harish-Chandra vertex of ρ, then ${}^*R_L^G(\rho)$ is a sum of irreducible, cuspidal ℓ-modular characters of L, all of which are conjugate in $N_G(L)$. Any such constituent is called a Harish-Chandra source of ρ.

We shall also need the fact that Harish-Chandra restriction commutes with restriction to the commutator subgroup.

Proposition 2. *Let $\tilde{\mathbf{G}}$ be a closed, connected, F-stable subgroup of $\mathbf{G}$ such that $\mathbf{G} = \mathbf{T}\tilde{\mathbf{G}}$ for a maximally split torus $\mathbf{T}$ of $\mathbf{G}$. Put $\tilde{G} = \tilde{\mathbf{G}}^F$ and $T = \mathbf{T}^F$.*

Let L be a Levi subgroup of G containing T. Then $\tilde{G} \cap L$ is a Levi subgroup of $\tilde{G}$ and we have for all generalized ℓ-modular characters ρ of G:

$$\mathrm{Res}^L_{\tilde{G}\cap L}({}^*R_L^G(\rho)) = {}^*R^{\tilde{G}}_{\tilde{G}\cap L}(\mathrm{Res}^G_{\tilde{G}}(\rho)).$$

A dual statement holds, of course, for Harish-Chandra induction.

Moreover,

$${}^*R^{\tilde{G}}_{\tilde{G}\cap L}({}^t\tilde{\rho}) = {}^t\left({}^*R^{\tilde{G}}_{\tilde{G}\cap L}(\tilde{\rho})\right)$$

for all generalized ℓ-modular characters $\tilde{\rho}$ of $\tilde{G}$ and all $t \in T$.

Proof. Let P be a parabolic subgroup of G with Levi decomposition $P = LU$. Then $U \subseteq \tilde{G}$ and $\tilde{G} \cap P$ is a standard parabolic subgroup of $\tilde{G}$ with Levi decomposition $\tilde{G} \cap P = (\tilde{G} \cap L)U$. (See the first paragraph of the proof of [5, Proposition 13.20] for these statements.) Moreover, U is T-invariant, since T is contained in L. The results follow easily from these facts. □

The above result will be applied to the case of $G = \mathrm{GU}_n(q)$ and $\tilde{G} = \mathrm{SU}_n(q)$. We give one example of such an application.

Corollary 3. *With the notation of Proposition 2, let ρ be an absolutely irreducible ℓ-modular character of G with Harish-Chandra vertex L. Then $\mathrm{Res}^G_{\tilde G}(\rho)$ has at most as many constituents as $\mathrm{Res}^L_{\tilde G\cap L}({}^*R^G_L(\rho))$.*

Proof. Since $\mathbf{T}\cap\tilde{\mathbf{G}}$, being a maximal torus of $\tilde{\mathbf{G}}$, is connected, we have $G = T\tilde G$. Let $\tilde\rho$ be an irreducible constituent of $\mathrm{Res}^G_{\tilde G}(\rho)$. Then $\mathrm{Res}^G_{\tilde G}(\rho) = \sum_{t\in S} {}^t\tilde\rho$ for some subset $S \subseteq T$. By Proposition 2 we have $\mathrm{Res}^L_{\tilde G\cap L}({}^*R^G_L(\rho)) = \sum_{t\in S} {}^t\left({}^*R^{\tilde G}_{\tilde G\cap L}(\tilde\rho)\right)$. This gives the desired result. □

Under additional hypotheses we obtain a slightly stronger result.

Proposition 4. *With the notation of Proposition 2, let ρ be an absolutely irreducible ℓ-modular character of G with Harish-Chandra vertex L and Harish-Chandra source σ. Suppose that $\tilde\sigma := \mathrm{Res}^L_{\tilde G\cap L}(\sigma)$ is irreducible. Suppose also that $|W_G(L,\sigma)| = |W_{\tilde G}(\tilde G\cap L,\tilde\sigma)|$, where $W_G(L,\sigma)$ denotes the ramification group of $R^G_L(\sigma)$ (see e.g.* [15, Section 3]*). Then $\mathrm{Res}^G_{\tilde G}(\rho)$ is irreducible.*

Proof. Let k be an algebraically closed field of characteristic ℓ and let Z be a kL-module with Brauer character σ. Put $\tilde L := \tilde G\cap L$ and $\tilde Z := \mathrm{Res}^L_{\tilde L}(Z)$.

It follows from the first part of Proposition 2 that $\tilde Z$ is cuspidal and also that
$$\mathrm{Res}^G_{\tilde G}(R^G_L(Z)) = R^{\tilde G}_{\tilde L}(\tilde Z).$$
Our assumption on the ramification groups now implies that $\mathrm{End}_{kG}(R^G_L(Z)) = \mathrm{End}_{k\tilde G}(R^{\tilde G}_{\tilde L}(\tilde Z)$. It follows that every indecomposable direct summand of $R^G_L(Z)$ restricts to an indecomposable direct summand of $R^{\tilde G}_{\tilde L}(\tilde Z)$. By [15, Theorem 2.4] each of these indecomposable direct summands has a simple head. Thus a simple head of an indecomposable summand of $R^G_L(Z)$ restricts to a simple module. Since ρ is the character of such a head, the result follows. □

By the comparison theorem [20, Theorem 5.9], the computation of Harish-Chandra induced unipotent characters can be reduced to the computation of induced characters in Weyl groups. In the cases we are interested in, Harish-Chandra induction of arbitrary characters can be reduced to Harish-Chandra induction of unipotent characters. Namely, let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$. Then $\mathbf{L}^*$ is an F-stable Levi subgroup of $\mathbf{G}^*$. Let s be an F-stable semisimple element of $\mathbf{L}^*$ and suppose that $C_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$. Then $C_{\mathbf{L}^*}(s) = C_{\mathbf{G}^*}(s)\cap\mathbf{L}^*$ is a Levi subgroup of $\mathbf{L}^*$. Using properties of Lusztig induction [5, 11.5, Proposition 12.6], we obtain the following commutative diagram.

$$\begin{tikzcd}
\mathcal{E}(G,s) & \mathcal{E}(G(s),1) \arrow[l, "\varepsilon_G\varepsilon_{G(s)}R^G_{G(s)}(\hat s\text{–})"'] \\
\mathcal{E}(L,s) \arrow[u, "R^G_L"] & \mathcal{E}(L(s),1) \arrow[l, "\varepsilon_L\varepsilon_{L(s)}R^L_{L(s)}(\hat s\text{–})"] \arrow[u, "R^{G(s)}_{L(s)}"']
\end{tikzcd} \qquad (3)$$

This allows to compute R_L^G from $R_{L(s)}^{G(s)}$, applied to unipotent characters. Note that $L(s)$ is a Levi subgroup of $G(s)$.

4. Approximations to decomposition matrices

We now turn to the case of the general unitary group $G = \mathrm{GU}_n(q)$, where $n \geq 3$. We will mainly be concerned with the set $\mathcal{E}(G,1)$ of unipotent characters of G. The unipotent characters are parametrized by partitions $\lambda \vdash n$ [3, 13.8]. We denote the unipotent character corresponding to a partition λ by χ_λ. Since the set of unipotent characters forms a basic set for the unipotent blocks and the decomposition matrix is triangular [12], the unipotent Brauer characters can also be labelled canonically by partitions of n. We write φ_λ for the unipotent Brauer character labelled by λ.

We say that a matrix D' approximates a decomposition matrix if the columns of D' are non-negative integral linear combinations of the columns of the decomposition matrix. In other words, the columns of D' describe the decomposition of some projective characters into ordinary characters. The same terminology is used for parts of decomposition matrices consisting of certain rows only.

The decomposition numbers of $\mathrm{GU}_3(q)$ were investigated by Geck, who showed the following.

Proposition 5. (Geck [11]) *For all q and all $\ell \nmid q$ the ℓ-modular decomposition matrix for the unipotent characters of $\mathrm{GU}_3(q)$ is approximated by*

λ			
(3)	1		
$(2,1)$	.	1	
(1^3)	1	q	1

Proposition 6. *For all $\ell \nmid q$ the ℓ-modular decomposition matrices for the unipotent characters of $\mathrm{GU}_4(q)$ and $\mathrm{GU}_5(q)$ are approximated by*

λ					
(4)	1				
$(3,1)$	1	1			
(2^2)	2	1	1		
$(2,1^2)$	1	.	1	1	
(1^4)	1	1	1	q	1

λ							
(5)	1						
$(4,1)$	.	1					
$(3,2)$	1	.	1				
$(3,1^2)$	2	q	1	1			
$(2^2,1)$	1	q	.	1	1		
$(2,1^3)$	.	1	.	.	q	1	
(1^5)	1	q	1	1	q^2	q	1

If $n = 4$ and $\ell = 2$, then all the entries in the above matrix, except q, are decomposition numbers. In case $n = 4$ and odd ℓ dividing $q+1$ one obtains the first three columns of the decomposition matrix of unipotent characters by subtracting the third column from the first.

Proof. First consider $\mathrm{GU}_4(q)$. The projective characters occurring in this and in subsequent tables come from two different sources. Firstly, we Harish-Chandra induce projective characters from proper Levi subgroups of $\mathrm{GU}_4(q)$. Thus, the first column is obtained as the permutation character on the unipotent radical of a Borel subgroup, the second is the Harish-Chandra induction of the projective cover of the Steinberg character of the Levi subgroup with semisimple part $\mathrm{GU}_2(q)$, and the third is the Harish-Chandra induction of the projective cover of the Steinberg character of the Levi subgroup with semisimple part $\mathrm{GL}_2(q^2)$. The last two projective characters are the generalized Gelfand-Graev characters corresponding to the partitions $(3,1)$ and (4) respectively (see [12]). The statements about decomposition numbers are proved in [27].

For $\mathrm{GU}_5(q)$ we just give the origins of the projective characters. The first, second and fourth are obtained by Harish-Chandra induction from the unipotent projective characters of the Levi subgroup with semisimple part $\mathrm{GU}_3(q)$ and the third by Harish-Chandra induction from the projective cover of the Steinberg character of the Levi subgroup with semisimple part $\mathrm{GL}_2(q^2)$. The last three projective characters are the generalized Gelfand-Graev characters corresponding to the partitions $(3,2)$, $(4,1)$, and (5) respectively (see [12]). ☐

For $n \geq 6$ the first seven rows of the unipotent part of the decomposition matrix can be approximated simultaneously as follows.

Proposition 7. *Let $n = 2m \geq 6$ be even. Then part of the decomposition matrix for the unipotent characters of $\mathrm{GU}_n(q)$ is approximated by:*

λ							
(n)	1						
$(n-1,1)$	1	1					
$(n-2,2)$	m	1	1				
$(n-2,1^2)$	$m-1$	.	1	1			
$(n-3,3)$	m	$m-1$	1	.	1		
$(n-3,2,1)$	.	.	.	.	.	1	
$(n-3,1^3)$	$m-1$	$m-1$	1	q	1	q	1

Proof. It is sufficient to construct projective characters with the indicated decomposition for $n = 6$. Then the result for arbitrary even $n \geq 8$ follows by induction. Indeed, we may assume that the table is correct for $\mathrm{GU}_{n-2}(q)$. The projective characters listed in the table are obtained by Harish-Chandra induction from the projective characters for $\mathrm{GU}_{n-2}(q)$, as can be checked by an easy calculation inside the Weyl group B_m of $\mathrm{GU}_n(q)$.

It remains to explain the origin of the projective characters for $\mathrm{GU}_6(q)$. The first four are Harish-Chandra induced from the first four projective characters of $\mathrm{GU}_4(q)$ in Proposition 6. The fifth is the Harish-Chandra induction of the projective cover of the Steinberg character of $\mathrm{GU}_2(q) \times \mathrm{GL}_2(q^2)$, and the sixth is the generalized Gelfand-Graev character corresponding to the partition $(3,2,1)$. ☐

Proposition 8. *Let $n = 2m+1 \geq 7$ be odd. Then part of the decomposition matrix for the unipotent characters of* $\mathrm{GU}_n(q)$ *is approximated by:*

λ							
(n)	1						
$(n-1,1)$	$.$	1					
$(n-2,2)$	$m-1$	$.$	1				
$(n-2,1^2)$	m	q	1	1			
$(n-3,3)$	$.$	$m-2$	$.$	$.$	1		
$(n-3,2,1)$	1	q	$.$	1	$.$	1	
$(n-3,1^3)$	$.$	$m-1$	$.$	$.$	1	q	1

Proof. As in the previous Proposition we may use induction on m. The projective characters for $\mathrm{GU}_{2m+1}(q)$ given in the table are obtained via Harish-Chandra induction from those for $\mathrm{GU}_{2m-1}(q)$, as can be verified inside the Weyl group B_m. We now explain how to obtain the projective characters for $\mathrm{GU}_7(q)$. The first four and the sixth arise by Harish-Chandra induction from the first five projective characters of $\mathrm{GU}_5(q)$ in Proposition 6. For the fifth we consider $\mathrm{GU}_3(q) \times \mathrm{GL}_2(q^2)$. The second projective for $\mathrm{GU}_3(q)$ tensored with the projective cover of the Steinberg character of $\mathrm{GL}_2(q^2)$ is again projective. Harish-Chandra induction to $\mathrm{GU}_7(q)$ gives the fifth column in the table. $\square$

The generalized Gelfand-Graev characters used in the above proofs have been computed by Frank Lübeck. They are available in the CHEVIE-system [14], which was also used to compute the relevant scalar products.

For the convenience of the reader we collect, for later use, the degrees (multiplied by $(q^3+1)(q^2-1)(q+1)$) of the unipotent characters occurring in the above tables (see [3, 13.8]).

λ	$\chi_\lambda(1) \cdot (q^3+1)(q^2-1)(q+1)$
(n)	$(q^3+1)(q^2-1)(q+1)$
$(n-1,1)$	$q\left(q^{n-1}-(-1)^{n-1}\right)(q^3+1)(q^2-1)$
$(n-2,2)$	$q^2\left(q^n-(-1)^n\right)\left(q^{n-3}-(-1)^{n-3}\right)(q^3+1)$
$(n-2,1^2)$	$q^3\left(q^{n-1}-(-1)^{n-1}\right)\left(q^{n-2}-(-1)^{n-2}\right)(q^3+1)$
$(n-3,3)$	$q^3\left(q^n-(-1)^n\right)\left(q^{n-1}-(-1)^{n-1}\right)\left(q^{n-5}-(-1)^{n-5}\right)$
$(n-3,2,1)$	$q^4\left(q^n-(-1)^n\right)\left(q^{n-2}-(-1)^{n-2}\right)\left(q^{n-4}-(-1)^{n-4}\right)(q-1)$
$(n-3,1^3)$	$q^6\left(q^{n-1}-(-1)^{n-1}\right)\left(q^{n-2}-(-1)^{n-2}\right)\left(q^{n-3}-(-1)^{n-3}\right)$

5. A parabolic subgroup

In this section, let $G_n = \mathrm{SU}_n(q)$, $n \geq 3$, where q is a power of the prime p. To make the exposition more transparent, we first introduce a convenient matrix representation for G_n. The $(n\times n)$-identity matrix is denoted by I_n, and the $(n\times n)$-matrix with ones along the anti-diagonal and zeroes otherwise, by J_n. If a is an element of an extension field of $\mathbb{F}_q$, we put $\bar{a} := a^q$, and extend this definition to matrices by applying the bar-map to each entry of a matrix. The special unitary group $G_n = \mathrm{SU}_n(q)$ is embedded into $\mathrm{SL}_n(q^2)$ as

$$\mathrm{SU}_n(q) = \{\mathbf{a} \in \mathrm{SL}_n(q^2) \mid \mathbf{a}J_n\bar{\mathbf{a}}^t = J_n\}.$$

Let P_n be the stabilizer in G_n of an isotropic line in the natural module for G_n. Thus P_n is a maximal parabolic subgroup with unipotent radical Q_n and Levi subgroup L_n. Furthermore, Q_n is a special p-group of order $q^{1+2(n-2)}$, $|Z(Q_n)| = q$, and $Q_n/Z(Q_n)$ is isomorphic to the natural module for $\mathrm{SU}_{n-2}(q)$ (see for example [24]). In the above matrix representation, the Levi subgroup L_n may be taken to be

$$L_n = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & \mathbf{y} & 0 \\ 0 & 0 & \bar{a}^{-1} \end{bmatrix} \mid a \in \mathrm{GL}_1(q^2), \mathbf{y} \in \mathrm{GU}_{n-2}(q), a^{1-q}\det(\mathbf{y}) = 1 \right\},$$

and the unipotent radical Q_n as

$$Q_n = \left\{ \begin{bmatrix} 1 & \mathbf{u} & y \\ 0 & I_{n-2} & \mathbf{v} \\ 0 & 0 & 1 \end{bmatrix} \mid y \in \mathbb{F}_{q^2}, \mathbf{v} \in \mathbb{F}_{q^2}^{n-2}, \mathbf{u} = -\bar{\mathbf{v}}^t J_{n-2}, y + \bar{\mathbf{v}}^t J_{n-2}\mathbf{v} + \bar{y} = 0 \right\}.$$

Let $\tilde{L_n}$ and L_n' denote the subgroups of L_n consisting of those matrices with $a^{q+1} = 1$ and $a = 1$, respectively, and put $\tilde{P}_n := Q_n\tilde{L}_n$. Then $L_n' \cong \mathrm{SU}_{n-2}(q)$ is contained in $\tilde{L}_n$ and $\tilde{L}_n/L_n'$ is cyclic of order $q+1$. Moreover, L_n' is the commutator subgroup of L_n, except in the case $n = 4$, $q = 2$.

6. The Weil characters of $\mathrm{SU}_n(q)$

Assume that $n \geq 3$ and let $G = \mathrm{SU}_n(q)$. We first show the existence of certain semi-simple elements in the dual group $G^* = \mathrm{PGU}_n(q)$. These elements serve as labels for some of the Weil characters to be defined below. Let v be a non-isotropic vector in the natural module for $\mathrm{GU}_n(q)$, and let $\hat{S}$ be the subgroup of $\mathrm{GU}_n(q)$ which fixes the line $\langle v\rangle$ and induces the identity on its orthogonal complement. Then $\hat{S}$, as well as its image S in $\mathrm{PGU}_n(q)$ are cyclic groups of order $q+1$. The centralizer of each non-trivial element of $\hat{S}$ in $\mathrm{GU}_n(q)$ is a group of type $\mathrm{GU}_1(q) \times \mathrm{GU}_{n-1}(q)$. Since $n > 2$, the centralizer in $\mathrm{PGU}_n(q)$ of a non-trivial element of S is a group of type $\mathrm{GU}_{n-1}(q)$. Note that the centralizer of a non-trivial element of S in the algebraic group $\mathrm{PGL}_n(\bar{\mathbb{F}}_q)$ is connected.

To define the Weil characters of G, we identify G with L_{n+2}' and put $P := P_{n+2}$, $Q := Q_{n+2}$, and $\tilde{P} := \tilde{P}_{n+2}$ (in the notation of the preceding section). Gérardin has shown that each of the $q-1$ ordinary irreducible characters of Q of degree q^n extends to QL' (see [17, Theorem 3.3]). By multiplying with a suitable linear character of L' (in case L' is not perfect), we obtain an extension which is also invariant under the elements of $\tilde{L}$. Since $\tilde{P}/QL'$ is cyclic, it follows that the extension to QL' extends further to $\tilde{P}$. Let ϕ denote the character of such an extension. The irreducible constituents of the restriction $\mathrm{Res}_G^{\tilde{P}}(\phi)$ are called *Weil characters* of G. (They are independent of the chosen extension ϕ.) Seitz [24] shows that $\mathrm{Res}_G^{\tilde{P}}(\phi)$ has $q+1$ irreducible constituents, one of degree $(q^n + (-1)^n q)/(q+1)$ and q of degree $(q^n - (-1)^n)/(q+1)$. The first of these is the unipotent character $\chi_{(n-1,1)}$, the others are the semisimple characters $\chi_{s,(n-1)}$ in Lusztig series $\mathcal{E}(G,s)$

where s runs over the non-trivial elements of the subgroup $S \leq \mathrm{PGU}_n(q)$ described above. Let x denote the smaller one of the two above-mentioned degrees, so that the other is equal to $x+1$. Landazuri and Seitz have shown [22], that except in the cases $(n,q) \in \{(3,2),(4,2),(4,3)\}$, any non-trivial irreducible character of G in characteristic ℓ not dividing q has degree at least x. In particular, the ordinary Weil characters of degree x are irreducible modulo ℓ, while those of degree $x+1$ split off at most one trivial constituent. (Using the Modular Atlas [21] or the references [11] and [27], one can see that the same statement is true for the three exceptions to the Landazuri-Seitz bound.) The following lemma clarifies when the reducible case occurs.

Proposition 9. *Let ℓ be a prime not dividing q.*

(a) *Let $n \geq 4$ be even. The Weil character $\chi_{(n-1,1)}$ of degree $x+1$ remains irreducible modulo ℓ if and only if $\ell \not| (q+1)$.*

(b) *Let $n \geq 3$ be odd. The Weil character $\chi_{s,(n-1)}$ of degree $x+1$ remains irreducible modulo ℓ if and only if the order of s is not a power of ℓ.*

Proof. The assertions are true for $(n,q) \in \{(3,2),(4,2),(4,3)\}$, by the references given above. If $n=4$, the result follows from [27].

Now assume that $n \geq 6$ is even. The Harish-Chandra restriction of $\chi_{(n-1,1)}$ to the Levi subgroup $L := L_n$ equals ${}^*R_L^G(\chi_{(n-1,1)}) = \chi_{(n-3,1)}$. Thus the result follows by induction.

In the case of odd n it follows from Proposition 1 that $\chi_{s,(n-1)}$ is irreducible if the order of s is not a power of ℓ. Otherwise, $\chi_{s,(n-1)} \in \mathcal{E}_\ell(G,1)$ and we have $\chi_{s,(n-1)}(g) = 1 + \chi_{(n-1,1)}(g)$ for all ℓ-regular elements $g \in G$. Hence the reduction modulo ℓ of $\chi_{s,(n-1)}$ contains the trivial Brauer character as claimed. $\square$

7. An induced character

Let $n \geq 3$ and let $G_n = \mathrm{SU}_n(q)$. For every $n \geq 3$ let ϕ_n denote an extension to $\tilde{P}_n$ of an ordinary irreducible character of Q_n of degree q^{n-2}. We also write $\Phi_n := \mathrm{Ind}_{\tilde{P}_n}^{P_n}(\phi_n)$, so that $\mathrm{Ind}_{\tilde{P}_n}^{G_n}(\phi_n) = \mathrm{Ind}_{P_n}^{G_n}(\Phi_n)$.

For the remainder of this section but Lemma 11 below, we omit the subscript n. Thus, in particular, $G = \mathrm{SU}_n(q)$. The next lemma shows why we are interested in the ordinary constituents of the induced character $\mathrm{Ind}_{\tilde{P}}^G(\phi)$.

Lemma 10. *Let ℓ be a prime not dividing q. Let ϑ be a non-trivial absolutely irreducible ℓ-modular character of G satisfying $\vartheta(1) < q^{n-2}(q-1)d_{n-2}$, where d_{n-2} denotes the degree of the smallest non-trivial absolutely irreducible ℓ-modular character of L'. Then ϑ is a modular constituent of $\mathrm{Ind}_{\tilde{P}}^G(\phi)$.*

Proof. Let k be an algebraically closed field of characteristic ℓ and let V be a kG-module affording ϑ. Since ℓ does not divide the order of Q, the restriction $\mathrm{Res}_Q^G(V)$ of V to Q is semi-simple. Since V is non-trivial and $Z(Q)$ is not contained in any proper normal subgroup of G, this restriction has an irreducible direct summand of dimension q^{n-2}. Then all $q-1$ irreducible kQ-modules $M_1, \ldots, M_{q-1}$ of dimension q^{n-2} are direct summands of $\mathrm{Res}_Q^G(V)$, since these are all conjugate in L.

Let $\tilde{M}_i$ denote an extension of M_i to $\tilde{P}$ (see Section 6). Then, by Clifford theory, all constituents of $\mathrm{Ind}_Q^{\tilde{P}}(M_i)$ are of the form $\tilde{M}_i \otimes_k X_i$, where X_i is an irreducible $k\tilde{L}$-module, inflated over Q. By Frobenius reciprocity, $\mathrm{Res}_{\tilde{P}}^G(V)$ has such a submodule $\tilde{M}_i \otimes_k X_i$ for each i. Our assumption on $\dim(V)$ implies that the only X_i that can occur are 1-dimensional. Hence, for each $1 \leq i \leq q-1$, there is a submodule of $\mathrm{Res}_{\tilde{P}}^{G}(V)$ extending M_i. One of these lifts to an ordinary representation with character ϕ. Thus V is a modular constituent of the induced module $\mathrm{Ind}_{\tilde{P}}^{G}(\phi)$. $\square$

For the next lemma only we resume our original notation with the subscript n.

Lemma 11. *Let $n \geq 5$. Any irreducible constituent of ${}^*R_{L_n}^{G_n}(\mathrm{Ind}_{P_n}^{G_n}(\Phi_n))$ is also a constituent of $\mathrm{Ind}_{P_{n-2}}^{L_n}(\Phi_{n-2})$. In particular, if χ_λ occurs in $\mathrm{Ind}_{P_n}^{G_n}(\Phi_n)$ for some unipotent character χ_λ then $\langle \mathrm{Ind}_{P_{n-2}}^{L_n}(\Phi_{n-2}), \chi_\mu \rangle \neq 0$ for all partitions μ that can be obtained from λ by removing a 2-hook. (Here, χ_μ denotes the unique unipotent character of L_n extending the unipotent character with label μ of $G_{n-2} = L'_n$.)*

Proof. We first consider $\mathrm{Res}_{P_n}^{G_n}(\mathrm{Ind}_{P_n}^{G_n}(\Phi_n))$. By the Mackey formula we have

$$\mathrm{Res}_{P_n}^{G_n}(\mathrm{Ind}_{P_n}^{G_n}(\Phi_n)) = \sum_{g \in P_n \backslash G_n / P_n} \mathrm{Ind}_{P_n \cap {}^gP_n}^{P_n}(\mathrm{Res}_{P_n \cap {}^gP_n}^{{}^gP_n}({}^g\Phi_n))$$

where the sum runs over double coset representatives of P_n in G_n. It is well known (see for example [3, Proposition 2.8.1]) that for parabolic subgroups the double coset representatives can be chosen to be preimages of corresponding double coset representatives in the Weyl group. The Weyl group of G_n is of type B_m, with $m = \lfloor n/2 \rfloor$, generated by fundamental reflections $s_1, \ldots, s_m$, say, where we choose notation such that $s_2, \ldots, s_m$ are conjugate, while s_1 is not conjugate to s_2. Then it is easy to see that $B_{m-1} \backslash B_m / B_{m-1} = \{1, s_m, s_m \cdots s_2 s_1 s_2 \cdots s_m\}$. For $g = 1$ we get no contribution to ${}^*R_{L_n}^{G_n}(\mathrm{Ind}_{P_n}^{G_n}(\Phi_n))$, since by construction Q_n has no fixed points on Φ_n.

Next consider $g = s_m$. Here, ${}^gQ_n \cap L_n = Q_{n-2}$ is the unipotent radical of the parabolic subgroup P_{n-2} of L'_n (this follows from [3, Proposition 2.8.9]). Also, ${}^gZ(Q_n)$ is contained in L_n and hence equals the center of Q_{n-2}. Now ${}^g\Phi_n$ restricted to ${}^gQ_n$ is the sum of the irreducible characters of ${}^gQ_n$ of degree q^{n-2}. None of these has ${}^gZ(Q_n)$ in its kernel. Hence the restriction of ${}^g\Phi_n$ to Q_{n-2} only contains characters not having $Z(Q_{n-2})$ in their kernel. It follows that if ψ is an irreducible constituent of $\mathrm{Res}_{P_n \cap {}^gP_n}^{{}^gP_n}({}^g\Phi_n)$, then $\mathrm{Res}_{\tilde{P}_{n-2}}^{P_n \cap {}^gP_n}(\psi)$ contains ϕ_{n-2}. Thus any constituent of $\mathrm{Ind}_{P_n \cap {}^gP_n}^{P_n}(\mathrm{Res}_{P_n \cap {}^gP_n}^{{}^gP_n}({}^g\Phi_n))$ is also contained in $\mathrm{Ind}_{\tilde{P}_{n-2}}^{P_n}(\phi_{n-2})$. Taking Q_n-fixed points of the latter just gives $\mathrm{Ind}_{P_{n-2}}^{L_n}(\Phi_{n-2})$.

For $g = s_m \cdots s_2 s_1 s_2 \cdots s_m$ we have $P_n \cap {}^gP_n = L_n$. By the results of Seitz cited in Section 6, the restriction of ${}^g\phi_n$ to L'_n contains exactly the Weil characters of L'_n as constituents. But clearly induction from L_n to P_n and then taking Q_n-fixed points just gives back the same character, so we get all Weil characters as contribution to the Q_n-fixed points of $\mathrm{Ind}_{L_n}^{P_n}(\mathrm{Res}_{L_n}^{{}^gP_n}({}^g\Phi_n))$, restricted to L'_n.

By Lemma 10 it follows that all Weil characters of L'_n occur in $\mathrm{Ind}_{P_{n-2}}^{L'_n}(\Phi_{n-2})$, thus the preceding discussion shows that all constituents of ${}^*R_{L_n}^{G_n}(\mathrm{Ind}_{P_n}^{G_n}(\Phi_n))$ occur in $\mathrm{Ind}_{P_{n-2}}^{L_n}(\Phi_{n-2})$, which is the first part of the assertion. For the second note that the constituents of ${}^*R_{L_n}^{G_n}(\chi_\lambda)$ are just those χ_μ where μ can be obtained from λ by removing a 2-hook. $\square$

In the next three Lemmas we investigate the ordinary constituents of $\mathrm{Ind}_{\tilde{P}}^{G}(\phi)$. The first result rules out certain unipotent characters as constituents if n is even. We will use it later on only in the case $n = 4$.

Lemma 12. *Let $n \geq 4$ be even. Let $\lambda = (\lambda_1, \lambda_2) \vdash n$ be a 2-part partition different from $(n-1, 1)$. Then $\langle \mathrm{Ind}_{\tilde{P}}^G(\phi), \chi_\lambda \rangle = 0$.*

Proof. Let R denote the stabilizer in G of a maximal totally isotropic subspace containing the line stabilized by P. Thus R is a maximal parabolic subgroup such that the derived subgroup of its Levi factor is isomorphic to $\mathrm{SL}_m(q^2)$, where $n = 2m$. Note that the constituents of the permutation character $\mathrm{Ind}_R^G(\mathbf{1})$ are precisely the unipotent characters χ_λ for 2-part partitions λ (see for example [16, 6.1.9]). We will prove the assertion by evaluating $l := \langle \mathrm{Ind}_P^G(\Phi), \mathrm{Ind}_R^G(\mathbf{1}) \rangle$. By the Mackey formula this equals

$$l = \sum_{g \in P \backslash G / R} \langle \mathrm{Res}^P_{P \cap {}^g R}(\Phi), \mathbf{1} \rangle,$$

where the sum ranges over a set of double coset representatives of P, R in G. In the Weyl group of G we have $\langle s_1, \ldots, s_{m-1} \rangle \backslash B_m / \langle s_2, \ldots, s_m \rangle = \{1, s_m \cdots s_1\}$ (in the notation of the proof of Lemma 11).

Clearly the intersection $P \cap R$ of two standard parabolic subgroups is again a parabolic subgroup (of type A_{m-2}). Its unipotent radical contains the unipotent radical of P which (by the definition of ϕ) has no fixed points on ϕ. Hence the trivial character is not a constituent of $\mathrm{Res}^P_{P \cap R}(\Phi)$ and thus the trivial double coset does not give a contribution to l.

Now consider $g = s_m \cdots s_1$. Let $T \leq P$ be the maximally split maximal torus of G consisting of the diagonal matrices in G. Then clearly $P = \tilde{P}T$. Moreover T is contained in the intersection $X := P \cap {}^g R$. A short calculation using root subgroups shows that X has a normal unipotent subgroup Q_1 which is an abelian subgroup of Q of order q^{n-2} intersecting $Z(Q)$ trivially. Since $\mathrm{Res}^P_Q(\Phi)$ vanishes outside $Z(Q)$ it follows that $\mathrm{Res}^{\tilde{P}}_{Q_1}(\phi)$ is the regular character of Q_1.

Since $Q_1 \leq \tilde{P}$ we may write $\tilde{P} \cap Q_1 T = Q_1 \tilde{T}$ for some torus $\tilde{T} \leq T$. We have $\Phi = \mathrm{Ind}_{\tilde{P}}^P(\phi)$, hence

$$\mathrm{Res}^P_{Q_1 T}(\Phi) = \mathrm{Res}^P_{Q_1 T}(\mathrm{Ind}^P_{\tilde{P}}(\phi)) = \mathrm{Ind}^{Q_1 T}_{Q_1 \tilde{T}}(\mathrm{Res}^{\tilde{P}}_{Q_1 \tilde{T}}(\phi))$$

by the Mackey formula, since $P = \tilde{P}T$. But this implies that

$$\langle \mathrm{Res}^P_X(\Phi), \mathbf{1} \rangle \leq \langle \mathrm{Res}^P_{Q_1 T}(\Phi), \mathbf{1} \rangle = \langle \mathrm{Ind}^{Q_1 T}_{Q_1 \tilde{T}}(\mathrm{Res}^{\tilde{P}}_{Q_1 \tilde{T}}(\phi)), \mathbf{1} \rangle = \langle \mathrm{Res}^{\tilde{P}}_{Q_1 \tilde{T}}(\phi), \mathbf{1} \rangle$$

by Frobenius reciprocity. But the latter is certainly not larger than $\langle \mathrm{Res}^{\tilde{P}}_{Q_1}(\phi), \mathbf{1} \rangle$, which equals 1 since we saw above that $\mathrm{Res}^{\tilde{P}}_{Q_1}(\phi)$ is the regular character of Q_1.

Thus $l \leq 1$. On the other hand, since $\chi_{(n-1,1)}(1) = (q^n + (-1)^n q)/(q+1)$ is smaller than $q^{n-2} d_{n-2}$, Lemma 10 shows that $\chi_{(n-1,1)}$ is a constituent of $\mathrm{Ind}_P^G(\Phi)$. Hence no other χ_λ occurs in $\mathrm{Ind}_P^G(\Phi)$. □

Lemma 13. *Let $n = 5$ and denote by St the ordinary Steinberg character of $G = \mathrm{SU}_5(q)$, i.e., $St = \chi_{(1^5)}$. Then $\langle \mathrm{Ind}_{\tilde{P}}^G(\phi), St \rangle = 0$.*

Proof. By Frobenius reciprocity it suffices to show that $\langle \phi, \mathrm{Res}^G_{\tilde{P}}(St) \rangle = 0$. Since ϕ is the extension of an irreducible character of $P' = QL'$ to $\tilde{P}$, this is the case if $l := \langle \mathrm{Res}^{\tilde{P}}_{P'}(\phi), \mathrm{Res}^G_{P'}(St) \rangle = 0$. The values of the Steinberg character are well-known (see for example [3, 6.5]); in particular St vanishes on all non-semisimple elements. Hence in order to evaluate l we have to determine the fusion of semisimple elements from $L' = \mathrm{SU}_3(q)$ into G, and to find the values of ϕ on these elements.

By the result of Seitz the restriction of ϕ to L' is known, hence its values can be computed from the known character table of $\mathrm{SU}_3(q)$ (see [25] or [14]). The fusion of semisimple elements of L' into G is easily calculated from the explicit matrix representation given in Section 5. This allows the computation of l, which turns out to be equal to 0 in all cases (the values and the fusion depend on the congruence of $q \pmod 6$). □

We are now ready to prove the crucial reduction on the possible characteristic 0 constituents of the induced character $\mathrm{Ind}_P^G(\phi)$.

Lemma 14. *Let $n \geq 6$. If $\langle \mathrm{Ind}_P^G(\phi), \chi_\lambda \rangle \neq 0$ then*

$$\lambda \in \{\, (n-1,1),\ (n-3,3),\ (n-3,2,1),\ (n-3,1^3)\,\}\,.$$

Let s be a semisimple element of $\mathrm{PGU}_n(q)$ with centralizer $\mathrm{GU}_{n-1}(q)$ (see Section 6). If $\langle \mathrm{Ind}_P^G(\phi), \chi_{s,\lambda} \rangle \neq 0$ then

$$\lambda \in \{\, (n-1), (n-2,1),\ (n-3,2),\ (n-3,1^2)\,\}$$

for $n \geq 10$. For $6 \leq n \leq 9$, moreover $\lambda \in \{(n-4,3),\ (n-4,2,1)\}$ may occur, as well as $\lambda = (2,1^3)$ for $n = 6$.

Proof. First note that $\mathrm{Ind}_P^G(\phi)$ clearly does not contain the trivial character if $n > 2$. Thus by Lemma 11 it does neither contain $\chi_{(n-2,2)}$ nor $\chi_{(n-2,1^2)}$ for $n > 4$. By Lemma 12 it does not contain $\chi_{(2^2)}$ for $n = 4$.

From the definition of ϕ it follows that

$$\mathrm{Ind}_P^G(\phi)(1) = q^{n-2}\frac{(q^n-(-1)^n)(q^{n-1}+(-1)^n)}{q+1}\,.$$

Thus any unipotent constituent χ_λ of the induced character can have degree at most that large. If χ_λ is cuspidal, then either $n = 6$ and $\lambda = (3,2,1)$, or $n \geq 10$ and λ is a triangular partition. But then $\chi_\lambda(1)$ exceeds $\mathrm{Ind}_P^G(\phi)(1)$. We may hence assume that ${}^*R_L^G(\chi_\lambda) \neq 0$ so that Lemma 11 applies. It is straightforward to check (for example with MAPLE [4]) from the degree formula for $\chi_\lambda(1)$, that $\chi_\lambda(1) \leq \mathrm{Ind}_P^G(\phi)(1)$ implies that

$$\lambda \in \{(6),(5,1),(4,2),(4,1^2),(3^2),(3,2,1),(3,1^3),(2^3),(2^2,1^2)\}$$

for $n = 6$. We just argued that the third and fourth of these cannot occur in $\mathrm{Ind}_P^G(\phi)$. Furthermore, (3^2), (2^3) and $(2^2,1^2)$ can be obtained from (2^2) by adding a 2-hook, thus the corresponding characters do not occur in $\mathrm{Ind}_P^G(\phi)$ by Lemma 11. This proves the assertion for $n = 6$. For $n = 7$, the degree formula shows that only

$$\lambda \in \{(7),(6,1),(5,2),(5,1^2),(4,3),(4,2,1),(4,1^3),(3^2,1),(3,2^2),(3,2,1^2)\}$$

are possible. Again the third and fourth are out. The partitions $(3^2,1)$, $(3,2^2)$ and $(3,2,1^2)$ can be obtained from either $(3,2)$ or $(3,1^2)$ by adding a 2-hook, but the two latter do not correspond to constituents for $n = 5$. Thus again by Lemma 11 we are done.

Now let $n \geq 8$ and assume the result is true for $n-2$. If χ_λ is a constituent of $\mathrm{Ind}_P^G(\phi)(1)$, Lemma 11 shows that λ can be obtained from an admissible partition of $n-2$ by adding a 2-hook. It is easy to write down all such partitions: the ones which do not occur in the statement are precisely those of the form $(n-5,\mu)$ with $\mu \vdash 5$, or (4^2). Except for

$$\{(6,5),(4,3,2),(4,3,1^2),(5,4,1),(3,2,1^3),(3,2^2,1)\}$$

all of these possess another 2-hook whose removal leads to a non-admissible partition of $n-2$. For the six partitions above, the corresponding character degree is larger than $\mathrm{Ind}_{\tilde{P}}^{G}(\phi)(1)$. Using Lemma 11 we are done by induction for the unipotent characters.

Now let's consider $\chi_{s,\lambda}$. By (1) its degree is given by $(q^n-(-1)^n)/(q+1)\cdot\chi_\lambda(1)$. If $n=10$ or $n=11$ then comparison with the degree of $\mathrm{Ind}_{\tilde{P}}^{G}(\phi)$ shows that only the λ in the statement can occur. For $n\geq 12$ we may then proceed by induction as in the unipotent case above, using Lemma 11. The cases $6\leq n\leq 9$ can be checked from the degree formula. □

Remark 15. It can be verified by explicit computations that χ_λ for $\lambda=(2,1^3)$ (respectively $\lambda=(3,1^3)$) is indeed a constituent of $\mathrm{Ind}_{\tilde{P}}^{G}(\phi)$ for $G=\mathrm{SU}_5(2)$ (respectively $G=\mathrm{SU}_6(2)$). This shows that at least in these cases the unipotent character of largest degree among those listed in the preceding lemma does in fact occur.

On the other hand, if n is even, $\lambda=(n-3,3)$ cannot occur by Lemma 12.

8. Lower bounds for dimensions of modules

We are now ready to prove the main result of our paper. In order to be able to state the theorem we introduce the quantity $d(n,q,\ell)$ (for an integer $n\geq 3$, a prime power q and a prime ℓ not dividing q) as follows:

$$d(3,q,\ell):=\begin{cases}\frac{1}{3}(q-1)(q^2-q+1), & \text{if } 3\mid(q+1),\ \ell\neq 3,\\ \frac{1}{6}(q-1)(q^2+3q+2), & \text{if } 3\mid(q+1),\ \ell=3,\\ \frac{1}{3}(2q^3-q^2+2q-3), & \text{if } 3\nmid(q+1),\ q>3,\\ 14, & \text{if } q=3,\ \ell=2,\end{cases}$$

$$d(4,q,\ell):=\begin{cases}\frac{1}{2}(q^2+1)(q^2-q+1), & \text{if } q,\ell \text{ odd},\\ \frac{1}{2}(q^2+1)(q^2-q+1)-1, & \text{if } q \text{ odd},\ \ell=2,\\ (q^2+1)(q^2-q+1)-1, & \text{if } q \text{ even},\end{cases}$$

$$d(n,q,\ell):=\begin{cases}q^{n-2}(q-1)(q^{n-2}-q)/(q+1), & \text{if } n\geq 5 \text{ odd},\\ q^{n-2}(q-1)(q^{n-2}-1)/(q+1), & \text{if } n\geq 6 \text{ even},\end{cases}$$

Theorem 16. *Let $G=\mathrm{SU}_n(q)$, $n\geq 3$, $(n,q)\neq(3,2),(4,2)$. Let k be an algebraically closed field of characteristic ℓ not dividing q, and let V be a non-trivial irreducible kG-module of dimension $\dim(V)<d(n,q,\ell)$. Then*

$$\dim(V)\in\{\frac{q^n-(-1)^n}{q+1},\ \frac{q^n+q(-1)^n}{q+1}\}.$$

In particular, V is the ℓ-modular reduction of an ordinary Weil module of G.

If n is odd, there exists one such module of dimension $(q^n-q)/(q+1)$ and $(q+1)_{\ell'}-1$ such modules of dimension $(q^n+1)/(q+1)$, while if n is even, there exist $(q+1)_{\ell'}-\epsilon(q,\ell)$ inequivalent modules of dimension $(q^n-1)/(q+1)$ and $\epsilon(q,\ell)$ modules of dimension $(q^n+q)/(q+1)$. Here, $\epsilon(q,\ell)=0$ if $\ell\mid(q+1)$ and $\epsilon(q,\ell)=1$ otherwise.

Proof. Let V be as in the statement and let ρ denote the Brauer character of V. First we determine the union of ℓ-blocks $\mathcal{E}(G,s)$ which possibly can contain ρ. If ρ is not unipotent, it can also lie in $\mathcal{E}(G,s)$ for a semisimple s with centralizer $\mathrm{GU}_{n-1}(q)$, $s^{q+1}=1$. Since $\rho(1) < d(n,q,\ell)$ it follows from Proposition 1 and [26, Table V] that these are the only possibilities if $n \geq 5$. There are further possibilities if $n=3$ or 4. To find these, we used the explicit character table of $\mathrm{SU}_3(q)$ determined by Simpson and Frame [25], as well as the list of the semisimple class types of $\mathrm{PGU}_4(q)$ determined by Frank Lübeck with the help of CHEVIE.

For $n=3$ and odd ℓ the result now follows from the tables of decomposition matrices computed by Geck in [11] (see also Proposition 5). For $n=3$, $\ell=2$ and $q \equiv 1 \pmod 4$, Erdmann has given the decomposition numbers of the principal block of $\mathrm{SU}_3(q)$ in [9, Lemma 11.4] (see also [8]). The decomposition matrices for the other relevant blocks are easy to determine. The result follows from these tables. For $n=3$, $\ell=2$ and $q \equiv 3 \pmod 4$, we have mimicked the argument of Geck ([11, Proof of Theorem 4.3(a)]) to bound the unknown decomposition number in the principal block. We obtain a bound similar to Geck's and thus the result in this case as well. So from now on we may assume that $n \geq 4$.

Step 1: Unipotent Constituents. Suppose first that ρ lies in a unipotent block. Then it must be a constituent of a unipotent character χ_λ, where λ is as in Lemma 14. Each of these χ_λs has a unique extension to a unipotent character of $\mathrm{GU}_n(q)$, denoted by the same symbol. By the triangular shape of the decomposition matrix of $\mathrm{GU}_n(q)$, this implies that ρ is a constituent of the restriction to $\mathrm{SU}_n(q)$ of φ_λ.

By Clifford's theorem, $\rho(1) \geq \varphi_\lambda(1)/(q+1)$, if ρ is contained in the restriction of φ_λ. Of course, $\varphi_{(n)}$ and $\varphi_{(n-1,1)}$ restrict irreducibly to $\mathrm{SU}_n(q)$, the latter by [24]. If ℓ does not divide $\gcd(n, q+1)$, then all unipotent Brauer characters of $\mathrm{GU}_n(q)$ restrict irreducibly to $\mathrm{SU}_n(q)$. This follows from [13] which asserts that the restrictions of the ordinary unipotent characters of $\mathrm{GU}_n(q)$ to $\mathrm{SU}_n(q)$ form a basic set for the unipotent Brauer characters.

1a. If $n=4$, we use the decomposition numbers of the unipotent blocks of $\mathrm{GU}_4(q)$ determined by Wings (see [27] and also Proposition 6). Since the restriction of φ_λ to $\mathrm{SU}_4(q)$ is irreducible if ℓ is odd, and has at most 4 irreducible constituents otherwise, the result follows except for $\ell=2$ and ρ a constituent of the restriction of $\varphi_{(2^2)}$. In the latter case the Harish-Chandra vertex of $\varphi_{(2^2)}$ is the Levi subgroup isomorphic to $\mathrm{GL}_2(q^2)$ and the Harish-Chandra source is the modular Steinberg character. Its restriction to $\mathrm{SL}_2(q^2)$ has at most two irreducible constituents, giving the result with the help of Corollary 3.

1b. Assume that $n \geq 5$. Then, by [24], $d(n,q,\ell) \leq q^{n-2}(q-1)d_{n-2}$, where d_{n-2} denotes the degree of the smallest non-trivial irreducible Brauer character of $\mathrm{SU}_{n-2}(q)$. By Lemma 10, ρ is a modular constituent of $\mathrm{Ind}_P^G(\phi)$ (in the notation of that Lemma). This is no restriction if $n=5$. In this case we use the approximation to the decomposition matrix of the unipotent characters of $\mathrm{GU}_5(q)$ given in Proposition 6. As in the case for $n=4$ this gives the desired result except for $\lambda=(1^5)$. But $\chi_{(1^5)}$ is not contained in $\mathrm{Ind}_P^G(\phi)$ by Lemma 13, so we are done in this case.

If $n>5$, then ρ is a constituent of the restriction to $\mathrm{SU}_n(q)$ of φ_λ where λ is larger, in the lexicographic ordering of partitions, than $(n-3,1^3)$. From the approximations to the decomposition matrices in Propositions 7 and 8 and the formulas for the degrees $\chi_\lambda(1)$ given in Section 4 we obtain lower bounds for the degrees of the Brauer characters φ_λ of $\mathrm{GU}_n(q)$.

Step 2: Restriction to $\mathrm{SU}_n(q)$. We next investigate the possible splittings of these φ_λs on restriction to $\mathrm{SU}_n(q)$. It is easy to check, using MAPLE, that $\varphi_\lambda(1) > d(n,q,\ell)$ for all n and q, if λ is not one of (n) or $(n-1,1)$. Thus the result follows if ℓ does not divide $q+1$.

Suppose now that $\ell \mid q+1$. Then $\varphi_\lambda(1) > d(n,q,\ell)(q+1)$, if $\lambda \in \{(n-3,1^3),(n-3,2,1)\}$, or if $n \geq 6$ is even and $\lambda = (n-3,3)$. We illustrate the proof of this claim with the example of $\varphi_{(n-3,3)}$ and $n \geq 6$ even. First we have $\varphi_{(n-3,3)}(1) \geq \chi_{(n-3,3)}(1) - \chi_{(n-2,2)}(1) - (m-1)\chi_{(n-1,1)}(1) - m$, by Proposition 7. We then find, using MAPLE once again:

$$\begin{aligned}
&\left[\varphi_{(n-3,3)}(1) - (q+1)d(n,q,\ell)\right](q^3+1)(q^2-1)(q+1) \geq \\
&q^{3n-3} - q^{2n+3} + 2\,q^{2n+1} - q^{2\,n} - 2\,q^{2n-1} + 3\,q^{2n-2} - q^{2n-3} - q^{2n-4} + \\
&q^{n+5} - 2\,q^{n+3} + q^{n+2} + q^{n+1} - 3\,q^n + q^{n-1} + q^6 + q^5 - q^4 + q^2 - q + \\
&m\left[-q^{n+5} + q^{n+3} - q^{n+2} + q^n - 2\,q^6 - q^5 + 2\,q^4 - q^3 - q^2 + 2\,q + 1\right] \geq \\
&\qquad\qquad q^{3n-3} - q^{2n+3} + q^{2n+1} - q^{2n-1} + \\
&\qquad\qquad q^{2n+1} - q^{2\,n} - q^{2n-1} - q^{2n-3} - q^{2n-4} + \\
&\qquad\qquad 3q^{2n-2} - mq^{n+5} \\
&\qquad\qquad > 0,
\end{aligned}$$

for $n \geq 8$ (for $n > 8$ use $m \leq n/2 < q^{n/2}$). For $n = 6$ the claim can be proved directly using the first of the above estimates.

It remains to consider the three characters $\varphi_{(n-2,2)}$, $\varphi_{(n-2,1^2)}$ and $\varphi_{(n-3,3)}$, the latter only for odd $n \geq 7$. By our construction of projective characters in Section 4, it follows that the projective cover of $\varphi_{(n-2,2)}$ is a summand of a projective character Harish-Chandra induced from the Levi subgroup $M \cong \mathrm{GL}_2(q^2) \times T$, where T is the maximally split torus of $\mathrm{GU}_{n-4}(q)$. By Frobenius reciprocity (for Harish-Chandra induction), the Harish-Chandra vertex of $\varphi_{(n-2,2)}$ is contained in M. Moreover, if M is the Harish-Chandra vertex of $\varphi_{(n-2,2)}$, then $\vartheta := {}^*R_M^G(\varphi_{(n-2,2)})$ is irreducible and equals the Harish-Chandra source of $\varphi_{(n-2,2)}$, a fact which can easily be checked by counting multiplicities. Since $\varphi_{(n-2,2)}$ is unipotent, T is in the kernel K of ϑ. Also, $T(M \cap \mathrm{SU}_n(q)) = M$ since $n \geq 5$. Hence $K(M \cap \mathrm{SU}_n(q)) = M$, and so the restriction of ϑ to $M \cap \mathrm{SU}_n(q)$ is irreducible. It follows from Corollary 3 that the restriction of $\varphi_{(n-2,2)}$ to $\mathrm{SU}_n(q)$ is irreducible. On the other hand, if the Harish-Chandra vertex of $\varphi_{(n-2,2)}$ is properly contained in M we are done by Proposition 4.

Now suppose that n is even and consider $\varphi_{(n-2,1^2)}$. Here we find that its Harish-Chandra vertex is contained in $\mathrm{GU}_4(q) \times T$, where T is the maximally split torus of $\mathrm{GL}_{(n/2-2)}(q^2)$. Moreover, if the Harish-Chandra vertex of $\varphi_{(n-2,1^2)}$ is not equal to the maximally split torus of $\mathrm{GU}_n(q)$, then its Harish-Chandra source is irreducible. As above one shows that the restriction of $\varphi_{(n-2,1^2)}$ to $\mathrm{SU}_n(q)$ is irreducible.

Finally, let n be odd. The Harish-Chandra vertex of $\varphi_{(n-2,1^2)}$ is contained in $M_1 \cong \mathrm{GU}_3(q) \times T_1$ and that of $\varphi_{(n-3,3)}$ in $M_2 \cong (\mathrm{GU}_3(q) \times \mathrm{GL}_2(q^2)) \times T_2$ where T_1 and T_2 are appropriate tori (and $T_2 = \{1\}$ in case $n = 7$). We may assume that the Harish-Chandra vertices are larger than the maximally split torus of $\mathrm{GU}_n(q)$. Thus M_1 is the Harish-Chandra vertex of $\varphi_{(n-2,1^2)}$ and its Harish-Chandra source is the modular Steinberg character of $\mathrm{GU}_3(q)$, inflated to M. Moreover, ${}^*R_{M_1}^G(\varphi_{(n-2,1^2)})$ is irreducible and restricts irreducibly to $M_1 \cap \mathrm{SU}_n(q)$. We are done in this case.

Similarly we can prove the desired result if the Harish-Chandra vertex of $\varphi_{(n-3,3)}$ equals M_2 (with some extra care if $n = 7$). It remains to consider the case that the Harish-Chandra vertex of $\varphi_{(n-3,3)}$ equals M_1. In this case its Harish-Chandra source equals $\varphi_{(2,1)}$, which restricts irreducibly to $\mathrm{SU}_n(q) \cap M_1$. Proposition 4 implies that $\varphi_{(n-3,3)}$ restricts irreducibly to $\mathrm{SU}_n(q)$.

Step 3: Non-unipotent Constituents. If ρ is not unipotent, then we have to study the decomposition matrix for $\mathcal{E}(G, s)$ with s as above. If $n \geq 6$ we may apply Lemma 14. It shows that if $\chi_{s,\lambda}$ is a constituent of $\mathrm{Ind}_P^G(\phi)$, for $\lambda \vdash (n-1)$, then the first part of λ is at least $n-4$, but $\lambda \neq (n-4, 1^3)$ (except possibly for $n = 6$, where $\lambda = (2, 1^3)$ may also occur).

Let $\hat{G} = \mathrm{GU}_n(q)$. We may and will identify $\hat{G}$ with its dual group, so that we have $s \in \hat{G}$. Every column in one of the tables of Propositions 6, 7 and 8 which was obtained by Harish-Chandra induction, also gives a column in an approximation to the decomposition matrix of $\mathcal{E}(\hat{G}, s)$. This follows from the fact that s may be chosen to lie in a maximally split torus, if n is odd, and in the Levi subgroup with simple component $\mathrm{GU}_2(q)$, if n is even. Therefore, every Levi subgroup of $C_{\hat{G}}(s)$ is of the form $M \cap C_{\hat{G}}(s)$ where M is a Levi subgroup of $\hat{G}$ containing s. Now apply the commutative diagram (3) to induce the projective characters of M contained in $\mathcal{E}(M, s)$ to $\hat{G}$.

Thus the first five columns of the matrix in Proposition 7 approximate the first five columns of the decomposition matrix for $\mathcal{E}(\hat{G}, s)$, for odd $n = 2m + 1 \geq 7$. Similarly, the first five columns of the matrix in Proposition 8 approximate the first five columns of the decomposition matrix for $\mathcal{E}(\hat{G}, s)$, for even $n = 2(m + 1) \geq 8$.

For the cases $n = 4, 5$ and 6 (the latter only for $\lambda = (2, 1^3)$), we need approximations to the other decomposition numbers of $\mathcal{E}(\hat{G}, s)$ as well. We obtain these by computing the restrictions to the unipotent elements of the characters in $\mathcal{E}(\hat{G}, s)$ (for $4 \leq n \leq 6$), and tensoring them with the generalized Gelfand Graev characters. Again this was done with the help of CHEVIE [14], which contains tables of Green functions for the relevant groups. In all cases we obtain approximations as in Proposition 5 (for $n = 4$) and Proposition 6 (for $n = 5$ and 6).

Similar, but easier calculations and considerations as in the case of unipotent characters now give the desired bounds on the degrees of the characters. $\square$

Remarks 17. (1) Under the restrictions on n and q made in the theorem, the group $\mathrm{SU}_n(q)$ is the Schur covering group of the simple group $\mathrm{U}_n(q)$, except when $(n, q) \in \{(4, 3), (6, 2)\}$. Thus our result can also be viewed as a result on dimensions of projective representations (in the sense of Schur) of $\mathrm{U}_n(q)$. The three smallest nontrivial degrees of absolutely irreducible projective representations in the exceptional cases may be found in the Modular Atlas [21] or in GAP [10]. We collect the result in Table 1.

(2) The bounds given in the theorem are asymptotically correct. For example, if $n \geq 5$ and $\ell > 3$, then $\varphi_{(n-2,2)}$ remains irreducible on restriction to $\mathrm{SU}_n(q)$, and $\varphi_{(n-2,2)}(1)$ has leading term q^{2n-4} (as polynomial in q).

TABLE 1. First three modular degrees for some small groups

G	d_1	d_2	d_3	char.
$U_4(2)$	4	5	6	$\ell \neq 2, 3$
	4	5	10	$\ell = 3$
$U_4(3)$	6	15	20	$\ell \neq 3$
$U_6(2)$	21	22	56	$\ell \neq 2, 3$
	21	56	120	$\ell = 3$

For odd q and $\ell = 2$ we give, without proof, the degree of $\varphi_{(n-2,2)}$. Write $n = 2m+1-\varepsilon$ with $\varepsilon \in \{0, 1\}$ and $m \geq 2$, and let ε_m denote the parity of m. Then

$$\varphi_{(n-2,2)}(1) = \frac{q^2(q^n - (-1)^n)(q^{n-3} - (-1)^{n-3})}{(q^2-1)(q+1)} - \varepsilon \frac{q(q^{n-1} - (-1)^{n-1})}{q+1} - (1 - \varepsilon_m).$$

(3) The bounds for $n = 4$ are best possible. For odd q and ℓ there exist ordinary irreducible characters of degree $d(4, q, \ell)$ (see [26, Table V]), which are irreducible modulo ℓ by Proposition 1. If q is odd and $\ell = 2$, then $\varphi_{(2^2)}$ has degree $(q^2 + 1)(q^2 - q + 1) - 2$, and the restriction of $\varphi_{(2^2)}$ to $\mathrm{SU}_4(q)$ splits into the sum of two irreducibles. Finally, if q is even and ℓ divides $q + 1$, then $\varphi_{(2^2)}$ has degree $(q^2 + 1)(q^2 - q + 1) - 1$. These statements can be derived from the decomposition matrices given in [27].

(4) The bound for $n = 3$, $\ell \neq 3$ and 3 dividing $q + 1$ is best possible. The reason is the same as for $n = 4$ and q, ℓ odd.

Geck [11] has determined the decomposition numbers for odd primes ℓ of $\mathrm{SU}_3(q)$ up to one unknown parameter and conjectured that this parameter should take the smallest possible value. An analogous conjecture can be made for $\ell = 2$. If these conjectures are true, then the smallest degree of an irreducible representation of $\mathrm{SU}_3(q)$ (with $q > 2$) larger than $q^2 - q + 1$ equals $(q - 1)(q^2 - q + 1)/\gcd(3, q + 1)$, exactly as in characteristic 0. Okuyama and Waki have announced a proof of Geck's conjecture.

Acknowledgment: We thank Frank Lübeck for the computation of the generalized Gelfand-Graev characters as well as for the enumeration of the semisimple class types of $\mathrm{PGU}_4(q)$.

REFERENCES

[1] M. BROUÉ AND J. MICHEL, Blocs et séries de Lusztig dans un groupe réductif fini, *J. reine angew. Math.* **395** (1989), 56–67.

[2] J. BRUNDAN AND A. KLESHCHEV, Lower bounds for degrees of irreducible Brauer characters of finite general linear groups, *J. Algebra* **223** (2000), 615–629.

[3] R.W. CARTER, Finite groups of Lie type: Conjugacy classes and complex characters, Wiley, Chichester, 1985.

[4] B. W. CHAR, K. O. GEDDES, G. H. GONNET, B. L. LEONG, M. B. MONAGAN, AND S. M. WATT, Maple V, language reference manual, Springer-Verlag, 1991.

[5] F. Digne and J. Michel, Representations of finite groups of Lie type, London Math. Soc. Students Texts 21, Cambridge University Press, 1991.

[6] F. DIGNE AND J. MICHEL, Groupes réductifs non connexes, *Ann. Scient. Éc. Norm. Sup., IV. Ser. 27* **3** (1994), 345–406.

[7] R. DIPPER AND J. DU, Harish-Chandra vertices, *J. reine angew. Math.* **437** (1993), 101–130.

[8] K. Erdmann, On 2-blocks with semidihedral defect groups, *Trans. Amer. Math. Soc.* **256** (1979), 267–287.

[9] K. Erdmann, Algebras and semidihedral defect groups II, *Proc. London Math. Soc.* **60** (1990), 123–165.

[10] The GAP Group, GAP – Groups, algorithms, and programming, Version 4, Aachen, St Andrews, 1998, (http://www-gap.dcs.st-and.ac.uk/~gap).

[11] M. Geck, Irreducible Brauer characters of the 3-dimensional special unitary groups in non-defining characteristic, *Comm. Algebra* **18** (1990), 563–584.

[12] M. Geck, On the decomposition numbers of the finite unitary groups in non-defining characteristic, *Math. Z.* **207** (1991), 83–89.

[13] M. Geck, Basic sets of Brauer characters of finite groups of Lie type, II, *J. London Math. Soc.* **47** (1993), 255–268.

[14] M. Geck, G. Hiss, F. Lübeck, G. Malle, and G. Pfeiffer, CHEVIE—A system for computing and processing generic character tables, *AAECC* **7** (1996), 175–210.

[15] M. Geck, G. Hiss, and G. Malle, Towards a classification of the irreducible representations in non-defining characteristic of a finite group of Lie type, *Math. Z.* **221** (1996), 353–386.

[16] M. Geck and G. Pfeiffer, Characters of finite groups and generic Iwahori-Hecke algebras, Oxford University Press, Oxford, 2000 (to appear).

[17] P. Gérardin, Weil representations associated to finite fields, *J. Algebra* **46** (1977), 54–101.

[18] R. M. Guralnick and P. H. Tiep, Low-dimensional representations of special linear groups in cross characteristics, *Proc. London Math. Soc.* **78** (1999), 116–138.

[19] G. Hiss, Harish-Chandra series of Brauer characters in a finite group with a split BN-pair, *J. London Math. Soc.* **48** (1993), 219–228.

[20] R. B. Howlett and G. I. Lehrer, Representations of generic algebras and finite groups of Lie type, *Trans. Amer. Math. Soc.* **280** (1983), 753–779.

[21] C. Jansen, K. Lux, R. A. Parker, and R. A. Wilson, An atlas of Brauer characters, Clarendon Press, Oxford, 1995.

[22] V. Landazuri and G. M. Seitz, On the minimal degrees of projective representations of the finite Chevalley groups, *J. Algebra* **32** (1974), 418–443.

[23] G. Lusztig, On the representations of reductive groups with disconnected centre, *in:* Orbites unipotentes et représentations, I. Groupes finis et algèbres de Hecke, Astérisque 168, 1988, pp. 157–166.

[24] G. M. Seitz, Some representations of classical groups, *J. London Math. Soc.* **10** (1975), 115–120.

[25] W. A. Simpson and J. S. Frame, The character tables for $SL(3,q)$, $SU(3,q^2)$, $PSL(3,q)$, $PSU(3,q^2)$, *Can. J. Math.* **25** (1973), 486–494.

[26] P. H. Tiep and A. E. Zalesskii, Minimal characters of the finite classical groups, *Comm. Algebra* **24** (1996), 2093–2167.

[27] E. Wings, Die Zerlegungszahlen der Hauptblöcke der unitären Gruppen $\mathrm{GU}_4(q^2)$ in nicht-definierender Charakteristik, Diplomarbeit, RWTH Aachen, Germany, 1990.

G.H.: Lehrstuhl D für Mathematik, RWTH Aachen, Templergraben 64, D-52062 Aachen, Germany

G.M.: FB Mathematik/Informatik, Universität Kassel, Heinrich-Plett-Strasse 40, D-34132 Kassel, Germany

E-mail address: G.H.: Gerhard.Hiss@Math.RWTH-Aachen.DE

E-mail address: G.M.: Malle@Mathematik.Uni-Kassel.DE