TWO CONJECTURES ON THE WEIL REPRESENTATIONS OF FINITE SYMPLECTIC AND UNITARY GROUPS

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Dedicated to Professor Michel Broué

Abstract. We propose two conjectures on the tensor product of Weil representations with irreducible representations in finite symplectic and unitary groups.

1. Weil representations

Let q be a power of the prime p and let n be a positive integer. We let G either denote the unitary group $\mathrm{GU}_n(q)$ or the symplectic group $\mathrm{Sp}_{2n}(q)$, in which case we also assume q to be odd. Let E be the natural vector space of G (which is a vector space over $\mathbb{F}_{q^2}$ of dimension n in case $G = \mathrm{GU}_n(q)$, and a vector space over $\mathbb{F}_q$ of dimension $2n$ if $G = \mathrm{Sp}_{2n}(q)$). Let H be a subgroup of G. We consider representations of H over the field $\mathbb{C}$ of complex numbers, and write $\mathrm{Irr}(H)$ for the set of complex irreducible characters of H.

For the definition and properties of the Weil representations of G we refer to [5]. For $G = \mathrm{GU}_n(q)$, there is a unique Weil representation, up to equivalence, splitting into $q+1$ distinct irreducible constituents. For $G = \mathrm{Sp}_{2n}(q)$ there are two equivalence classes of Weil representations, each with two distinct irreducible constituents. Let V denote a $\mathbb{C}G$-module of dimension $\sqrt{|E|} = q^n$ affording a Weil representation of G. The following property of V is well known.

Remark 1.1. The $\mathbb{C}G$-module $V \otimes_{\mathbb{C}} V^*$ is isomorphic to $\mathbb{C}E$, the permutation module of G on E.

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2. A conjecture on endomorphism rings

Keep the notation of Section 1. If $G = \mathrm{Sp}_{2n}(q)$, the space E splits into two orbits under the action of G, and if $G = \mathrm{GU}_n(q)$, the number of orbits of G on E equals $q + 1$.

Conjecture 2.1. *Let $x_1, \ldots, x_r$ be representatives for the orbits of G on E. Let Q_i denote the stabilizer in G of x_i for $i \in \{1, \ldots, r\}$. Moreover, let X be any simple $\mathbb{C}G$-module. Then*

$$\mathrm{End}_{\mathbb{C}G}(V \otimes_{\mathbb{C}} X) \cong \bigoplus_{j=1}^{r} \mathrm{End}_{\mathbb{C}Q_j}(\mathrm{Res}^G_{Q_j}(X))$$

as $\mathbb{C}$-algebras.

If $x_j = 0$, then $Q_j = G$. If $x_j \neq 0$ is isotropic, then Q_j is a normal subgroup of index $q - 1$ respectively $q^2 - 1$ in a maximal parabolic subgroup of G, the stabilizer of the line spanned by x_j. If $G = \mathrm{GU}_n(q)$, then the remaining x_j are non-isotropic and $Q_j \cong \mathrm{GU}_{n-1}(q)$.

Let us collect some evidence for Conjecture 2.1. We begin by showing that the two $\mathbb{C}$-algebras in question have the same dimension. In the considerations below, we suppress the subscript $\mathbb{C}$ from the tensor product symbol.

Proposition 2.2. *If X is a finite dimensional $\mathbb{C}G$-module, there exists an isomorphism*

$$\mathrm{End}_{\mathbb{C}G}(V \otimes X) \cong \bigoplus_{j=1}^{r} \mathrm{End}_{\mathbb{C}Q_j}(\mathrm{Res}^G_{Q_j}(X))$$

of $\mathbb{C}$-vector spaces.

Proof. We have isomorphisms

$$\begin{aligned}
\mathrm{End}_{\mathbb{C}G}(V \otimes X) &\cong \mathrm{Hom}_{\mathbb{C}G}(V \otimes X, V \otimes X) \\
&\cong \mathrm{Hom}_{\mathbb{C}G}(V \otimes V^*, X \otimes X^*) \\
&\cong \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}E, X \otimes X^*) \\
&\cong \mathrm{Hom}_{\mathbb{C}G}\Big(\bigoplus_{j=1}^{r} \mathrm{Ind}^G_{Q_j}(\mathbb{C}), X \otimes X^*\Big) \\
&\cong \bigoplus_{j=1}^{r} \mathrm{Hom}_{\mathbb{C}Q_j}(\mathbb{C}, \mathrm{Res}^G_{Q_j}(X \otimes X^*)) \\
&\cong \bigoplus_{j=1}^{r} \mathrm{End}_{\mathbb{C}Q_j}(\mathrm{Res}^G_{Q_j}(X)),
\end{aligned}$$

proving our claim. $\diamond$

Example 2.3. Consider the case that $X = \mathbb{C}$ is the trivial $\mathbb{C}G$-module. In this case for $G = \mathrm{Sp}_{2n}(q)$ the conjecture states that

$$\mathrm{End}_{\mathbb{C}G}(V) \cong \mathbb{C}^2,$$

and for $G = \mathrm{GU}_n(q)$ that

$$\mathrm{End}_{\mathbb{C}G}(V) \cong \mathbb{C}^{q+1}$$

as $\mathbb{C}$-algebras. These statements are true by the introductory remarks.

More generally, Proposition 2.2 shows that our conjecture is true for a simple $\mathbb{C}G$-module X, if $V \otimes X$ and $\mathrm{Res}^G_{Q_j}(X)$ are multiplicity free for all $1 \leq j \leq r$. Indeed, under these assumptions, $\mathrm{End}_{\mathbb{C}G}(V \otimes X)$ and $\mathrm{End}_{\mathbb{C}Q_j}(\mathrm{Res}^G_{Q_j}(X))$ are commutative, semisimple $\mathbb{C}$-algebras, and thus isomorphic to direct sums of copies of $\mathbb{C}$. Applying [6, Corollary 1.3] and [6, Theorem 1.4], we obtain the following.

Example 2.4. Let X be a $\mathbb{C}G$-module affording the Steinberg character of G. Then Conjecture 2.1 holds for X.

Let ω denote the character of V and let χ be an irreducible character of G. Then Conjecture 2.1 is equivalent to the statement that the two multisets

$$\{(\omega \cdot \chi, \psi) \mid \psi \in \mathrm{Irr}(G)\} \tag{1}$$

and

$$\bigcup_{j=1}^{r} \{(\mathrm{Res}^G_{Q_j}(\chi), \psi) \mid \psi \in \mathrm{Irr}(Q_j)\} \tag{2}$$

are equal. Here, (ϑ, μ) denotes the usual inner product on the set of complex valued class functions on G, i.e.,

$$(\vartheta, \mu) = \frac{1}{|G|} \sum_{g \in G} \vartheta(g)\mu(g^{-1}). \tag{3}$$

Using the above interpretation of Conjecture 2.1, the second author has verified this for $G = \mathrm{Sp}_2(q)$ for all odd q, for $\mathrm{Sp}_n(q)$ for all (n, q) in

$$\{(4, 3), (6, 3), (8, 3), (4, 5), (6, 5), (4, 7), (6, 7), (4, 9), (4, 11), (4, 13)\},$$

as well as for $\mathrm{GU}_n(q)$ for all (n, q) in

$$\{(2, 2), (3, 2), (4, 2), (5, 2), (2, 3), (3, 3)\}.$$

More details are given in the respective chapters of his PhD-thesis [8, Chapters 3,4,5].

We conclude this section by giving an example which shows that the assumption on X of being simple in Conjecture 2.1 is necessary.

Remark 2.5. Though Proposition 2.2 holds for any finite dimensional $\mathbb{C}G$-module X, the statement of Conjecture 2.1 may fail if X is not simple. Consider $G = \mathrm{Sp}_2(5)$. We use GAP [4] to compute with the characters of G. In particular, we use the GAP-numbering of $\mathrm{Irr}(G)$, which agrees with the numbering in the ATLAS [2]. Put $\omega := \chi_2 + \chi_7$, and $\chi := \chi_1 + \chi_9$. Then ω is a Weil character and

$$\omega \cdot \chi = 2\chi_2 + \chi_4 + \chi_5 + 2\chi_7 + \chi_8 + 2\chi_9.$$

Let Q denote the stabilizer of a non-trivial vector in the natural vector space of G. Then Q is cyclic of order 5, and we find

$$\mathrm{Res}^G_Q(\chi) = 3\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5,$$

with $\mathrm{Irr}(Q) = \{\psi_1 = 1_Q, \psi_2, \ldots, \psi_5\}$. Thus $\omega \cdot \chi$ has the multiplicities $\{1^3, 2^3\}$, whereas $\mathrm{Res}^G_G(\chi) + \mathrm{Res}^G_Q(\chi)$ has the multiplicities $\{1^6, 3\}$.

3. A conjecture on tensor products

What is the rationale behind our conjecture? Let us keep the notation of the previous sections.

Conjecture 3.1. *Let X be any simple $\mathbb{C}G$-module. Then the multiplicities of the simple constituents of $V \otimes X$ are bounded by a function in n, independently of q.*

There is a simple heuristics behind Conjecture 3.1. As before, let ω and χ denote the characters of V and X, respectively, and let $\psi \in \mathrm{Irr}(G)$. In the expression (3) for $(\omega \cdot \chi, \psi)$, the dominant term is the summand for $g = 1$, i.e.

$$\omega(1)\chi(1)\psi(1)/|G|. \tag{4}$$

To estimate the term (4), write $|G|_p$ for the p-part of $|G|$. By [7, Theorems 5.2, 5.3], there is a bound D, independent of q, such that $\psi(1) \leq D|G|_p$ for all $\psi \in \mathrm{Irr}(G)$. (One may take $D = 38(2 + \log_2(2n + 1))^{1.27}$.) For $G = \mathrm{GU}_n(q)$, we have $|G|_p = q^{n(n-1)/2}$, and thus the numerator of (4) is bounded from above by $D^2 q^{n^2}$. For $G = \mathrm{Sp}_{2n}(q)$, we have $|G|_p = q^{n^2}$, and thus the numerator of (4) is bounded from above by $D^2 q^{2n^2+n}$. From the formula for $|G|$ one concludes that $2^n|G| \geq q^{n^2}$ in the first case, and $2^n|G| \geq q^{2n^2+n}$ in the second. Thus (4) is bounded from above by $2^n D^2$.

Conjecture 3.1 holds for $G = \mathrm{Sp}_2(q)$ by the calculations in [8, Chapter 3]. It is also true for $G = \mathrm{Sp}_4(q)$, if χ is a unipotent character (see [1, Proposition 5.4]). We have also checked the conjecture for $G = \mathrm{GU}_3(q)$, using its generic character table computed by Ennola in [3].

The truth of the two conjectures would imply an upper bound, independent of q, for the multiplicities of the irreducible constituents in the restricted characters $\mathrm{Res}^G_{Q_j}(\chi)$, where $\chi \in \mathrm{Irr}(G)$, and $j = 1, \ldots, r$. A heuristics, similar to the one presented above, underpins the latter statement. If valid, this would be relevant for the modular representation theory of G in the non-defining characteristic case.

Even if Conjecture 2.1 turns out to be false, one may still hope that the multiplicities occuring in (1) are bounded in terms of those in (2) and vice versa.

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