REPRESENTATIONS OF FINITE GROUPS OF LIE TYPE

LECTURE III: REPRESENTATIONS IN NON-DEFINING CHARACTERISTICS

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RECOLLECTION

AIM

Classify all irreducible representations of all finite simple groups and related finite groups.

In the following, let $G = \mathbf{G}^F$ be a finite reductive group of characteristic p, and let k be an algebraically closed field with $\mathrm{char}(k) = \ell$.

Today we consider the case $0 \neq \ell \neq p$.

A SIMPLIFICATION: BRAUER CHARACTERS

Let V be a kG-module.

The character χ_V of V as defined in Lecture 2 does not convey all the desired information, e.g.,

$\chi_V(1)$ only gives the dimension of V modulo ℓ.

Instead one considers the Brauer character φ_V of V.

This is obtained by consistently lifting the eigenvalues of the linear transformation of $g \in G_{\ell'}$ on V to characteristic 0.
($G_{\ell'}$ is the set of ℓ-regular elements of G.)

Thus $\varphi_V : G_{\ell'} \to K$, where K is a suitable field with $\text{char}(K) = 0$, and $\varphi_V(g) =$ sum of the eigenvalues of g on V (viewed as elements of K).

In particular, $\varphi_V(1)$ equals the dimension of V.

THE BRAUER CHARACTER TABLE

If V is simple, φ_V is called an irreducible Brauer character.

Two **simple** kG-modules are isomorphic if and only if their Brauer characters are equal.

Put $\mathrm{IBr}_\ell(G) := \{\varphi_V \mid V \text{ simple } kG\text{-module}\}$.

(If $\ell \nmid |G|$, then $\mathrm{IBr}_\ell(G) = \mathrm{Irr}(G)$.)

$\mathcal{C}_{\ell'}$: set of representatives of the conjugacy classes of G contained in $G_{\ell'}$.

The **square** matrix

$$\left[\varphi(g)\right]_{\varphi \in \mathrm{IBr}_\ell(G), g \in \mathcal{C}_{\ell'}}$$

is the Brauer character table or ℓ-modular character table of G.

The 13-Modular Character Table of $SL_3(3)$

Let $G = SL_3(3)$. Then $|G| = 5\,616 = 2^4 \cdot 3^3 \cdot 13$.

Example (The 13-Modular Character Table of $SL_3(3)$)

	$1a$	$2a$	$3a$	$3b$	$4a$	$6a$	$8a$	$8b$
φ_1	1	1	1	1	1	1	1	1
φ_2	11	3	2	-1	-1	0	-1	-1
φ_3	13	-3	4	1	1	0	-1	-1
φ_4	16	0	-2	1	0	0	0	0
φ_5	26	2	-1	-1	2	-1	0	0
φ_6	26	-2	-1	-1	0	1	$\sqrt{-2}$	$-\sqrt{-2}$
φ_7	26	-2	-1	-1	0	1	$-\sqrt{-2}$	$\sqrt{-2}$
φ_8	39	-1	3	0	-1	-1	1	1

GOALS AND RESULTS

AIM

Describe all Brauer character tables of all finite simple groups and related finite groups.

In contrast to the case of ordinary character tables (i.e. $\mathrm{char}(k) = 0$, cf. Lecture 2), this is wide open:

1. For alternating groups: complete up to A_{17}
2. For groups of Lie type: only partial results
3. For sporadic groups up to McL and other "small" groups (of order $\leq 10^9$): *An Atlas of Brauer Characters*, Jansen, Lux, Parker, Wilson, 1995

   More information is available on the web site of the Modular Atlas Project: (http://www.math.rwth-aachen.de/˜MOC/)

THE DECOMPOSITION NUMBERS

For $\chi \in \mathrm{Irr}(G)$, write $\hat{\chi}$ for the restriction of χ to $G_{\ell'}$.

Then there are integers $d_{\chi\varphi} \geq 0$, $\chi \in \mathrm{Irr}(G)$, $\varphi \in \mathrm{IBr}_\ell(G)$, such that

$$\hat{\chi} = \sum_{\varphi \in \mathrm{IBr}_\ell(G)} d_{\chi\varphi}\,\varphi.$$

These integers are called the decomposition numbers of G modulo ℓ.

The matrix $D = [d_{\chi\varphi}]$ is the decomposition matrix of G.

PROPERTIES OF BRAUER CHARACTERS

$\mathrm{IBr}_\ell(G)$ is linearly independent (in $\mathrm{Maps}(G_{\ell'}, K)$) and so the decomposition numbers are uniquely determined.

The elementary divisors of D are all 1 (i.e., the decomposition map defined by $\chi \mapsto \hat{\chi}$ is surjective). Thus:

Knowing $\mathrm{Irr}(G)$ and D is equivalent to knowing $\mathrm{Irr}(G)$ and $\mathrm{IBr}_\ell(G)$.

If G is ℓ-soluble, $\mathrm{Irr}(G)$ and $\mathrm{IBr}_\ell(G)$ can be sorted such that D has shape

$$D = \left[\frac{I_n}{D'} \right],$$

where I_n is the $(n \times n)$ identity matrix (Fong-Swan theorem).

UNIPOTENT BRAUER CHARACTERS

The concept of decomposition numbers can be used to define unipotent Brauer characters of a finite reductive group.

Let $G = \mathbf{G}^F$ be a finite reductive group of characteristic p. (Recall that $\mathrm{char}(k) = \ell \neq p$.)

Recall that $\mathrm{Irr}^u(G) =$

$$\{\chi \in \mathrm{Irr}(G) \mid \chi \text{ occurs in } R_{\mathbf{T}}^{\mathbf{G}}(\mathbf{1}) \text{ for some maximal torus } \mathbf{T} \text{ of } \mathbf{G}\}.$$

This yields a definition of $\mathrm{IBr}_\ell^u(G)$.

DEFINITION (UNIPOTENT BRAUER CHARACTERS)

$\mathrm{IBr}_\ell^u(G) = \{\varphi \in \mathrm{IBr}_\ell(G) \mid d_{\chi\varphi} \neq 0$ *for some* $\chi \in \mathrm{Irr}^u(G)\}$.

The elements of $\mathrm{IBr}_\ell^u(G)$ *are called the unipotent Brauer characters of* G.

A simple kG-module is unipotent, if its Brauer character is.

JORDAN DECOMPOSITION OF BRAUER CHARACTERS

The investigations are guided by the following main conjecture.

CONJECTURE

Suppose that $Z(\mathbf{G})$ is connected. Then there is a labelling

$$\mathrm{IBr}_\ell(G) \leftrightarrow \{\varphi_{s,\mu} \mid s \in G^* \text{ semisimple}, \ell \nmid |s|, \mu \in \mathrm{IBr}_\ell^u(C_{G^*}(s))\},$$

such that $\varphi_{s,\mu}(1) = |G^* : C_{G^*}(s)|_{p'}\,\mu(1)$.

Moreover, D can be computed from the decomposition numbers of ***unipotent*** *characters of the various $C_{G^*}(s)$.*

Known to be true for $\mathrm{GL}_n(q)$ (Dipper-James, 1980s) and if $C_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$ (Bonnafé-Rouquier, 2003).

The truth of this conjecture would reduce the computation of decomposition numbers to unipotent characters.

Consequently, we will restrict to this case in the following.

THE UNIPOTENT DECOMPOSITION MATRIX

Put $D^u :=$ restriction of D to $\mathrm{Irr}^u(G) \times \mathrm{IBr}_\ell^u(G)$.

THEOREM (GECK-H., 1991; GECK, 1993)

(Some conditions apply.)

$|\mathrm{Irr}^u(G)| = |\mathrm{IBr}_\ell^u(G)|$ *and* D^u *is invertible over* $\mathbb{Z}$.

CONJECTURE (GECK, 1997)

(Some conditions apply.) With respect to suitable orderings of $\mathrm{Irr}^u(G)$ *and* $\mathrm{IBr}_\ell^u(G)$, D^u *has shape*

$$\begin{bmatrix} 1 & & & \\ \star & 1 & & \\ \vdots & \vdots & \ddots & \\ \star & \star & \star & 1 \end{bmatrix}.$$

This would give a canonical bijection $\mathrm{Irr}^u(G) \longleftrightarrow \mathrm{IBr}_\ell^u(G)$.

ABOUT GECK'S CONJECTURE

Geck's conjecture on D^u is known to hold for

- $\mathrm{GL}_n(q)$ (Dipper-James, 1980s)
- $\mathrm{GU}_n(q)$ (Geck, 1991)
- G a classical group and ℓ "linear" (Gruber-H., 1997)
- $\mathrm{Sp}_4(q)$ (White, 1988 – 1995)
- $\mathrm{Sp}_6(q)$ (An-H., 2006)
- $G_2(q)$ (Shamash-H., 1989 – 1992)
- $F_4(q)$ (Köhler, 2006)
- $E_6(q)$ (Geck-H., 1997; Miyachi, 2008)
- Steinberg triality groups ${}^3D_4(q)$ (Geck, 1991)
- Suzuki groups (for general reasons)
- Ree groups (Himstedt-Huang, 2009)

LINEAR PRIMES, I

Suppose $G = \mathbf{G}^F$ with $F(a_{ij}) = (a_{ij}^q)$ for some power q of p.

Put $e := \min\{i \mid \ell \text{ divides } q^i - 1\}$, the order of q in $\mathbb{F}_\ell^*$.

If G is classical ($\neq \mathrm{GL}_n(q)$) and e is odd, ℓ is linear for G.

EXAMPLE

$G = \mathrm{SO}_{2m+1}(q)$, $|G| = q^{m^2}(q^2-1)(q^4-1)\cdots(q^{2m}-1)$.
If $\ell \mid |G|$ and $\ell \nmid q$, then $\ell \mid q^{2d}-1$ for some minimal d.
Thus $\ell \mid q^d - 1$ (ℓ linear and $e = d$) or $\ell \mid q^d + 1$ ($e = 2d$).

Now $\mathrm{Irr}^u(G)$ is a union of Harish-Chandra series $\mathcal{E}_1, \ldots, \mathcal{E}_r$.

THEOREM (FONG-SRINIVASAN, 1982, 1989)

Suppose that $G \neq \mathrm{GL}_n(q)$ is classical and that ℓ is linear.
Then $D^u = \mathrm{diag}[\Delta_1, \ldots, \Delta_r]$ with square matrices Δ_i corresponding to $\mathcal{E}_i$.

Linear primes, II

Let $\Delta := \Delta_i$ be one of the decomposition matrices from above. Then the rows and columns of Δ are labelled by bipartitions of a for some integer a. (Harish-Chandra theory.)

Theorem (Gruber-H., 1997)

In general,

$$\Delta = \begin{bmatrix} \Lambda_0 \otimes \Lambda_a & & & & \\ & \ddots & & & \\ & & \Lambda_i \otimes \Lambda_{a-i} & & \\ & & & \ddots & \\ & & & & \Lambda_a \otimes \Lambda_0 \end{bmatrix}$$

Here $\Lambda_i \otimes \Lambda_{a-i}$ is the Kronecker product of matrices, and Λ_i is the ℓ-modular unipotent decomposition matrix of $\mathrm{GL}_i(q)$.

THE v-SCHUR ALGEBRA

Let v be an indeterminate an put $A := \mathbb{Z}[v, v^{-1}]$.

Dipper and James (1989) have defined a remarkable A-algebra $\mathcal{S}_{A,v}(S_n)$, called the generic v-Schur algebra, such that:

1. $\mathcal{S}_{A,v}(S_n)$ is free and of finite rank over A.
2. $\mathcal{S}_{A,v}(S_n)$ is constructed from the generic Iwahori-Hecke algebra $\mathcal{H}_{A,v}(S_n)$, which is contained in $\mathcal{S}_{A,v}(S_n)$ as a subalgebra (with a different unit).
3. $\mathbb{Q}(v) \otimes_A \mathcal{S}_{A,v}(S_n)$ is a quotient of the quantum group $\mathcal{U}_v(\mathfrak{gl}_n)$.

THE q-SCHUR ALGEBRA

Let $G = \mathrm{GL}_n(q)$.

Then $D^u = (d_{\lambda,\mu})$, with $\lambda, \mu \in \mathcal{P}_n$.

Let $\mathcal{S}_{A,v}(S_n)$ be the v-Schur algebra, and let $\mathcal{S} := \mathcal{S}_{k,q}(S_n)$ be the k-algebra obtained by specializing v to the image of $q \in k$.

This is called the q-Schur algebra, and satisfies:

1. $\mathcal{S}$ has a set of (finite-dimensional) standard modules $\mathbf{S}^\lambda$, indexed by $\mathcal{P}_n$.
2. The simple $\mathcal{S}$-modules $\mathbf{D}^\lambda$ are also labelled by $\mathcal{P}_n$.
3. If $[\mathbf{S}^\lambda : \mathbf{D}^\mu]$ denotes the multiplicity of $\mathbf{D}^\mu$ as a composition factor in $\mathbf{S}^\lambda$, then $[\mathbf{S}^\lambda : \mathbf{D}^\mu] = d_{\lambda,\mu}$.

As a consequence, the $d_{\lambda,\mu}$ are bounded independently of q and of ℓ.

CONNECTIONS TO DEFINING CHARACTERISTICS, I

Let $\mathcal{S}_{k,q}(S_n)$ be the q-Schur algebra introduced above.

Suppose that $\ell \mid q-1$ so that $q \equiv 1 (\mathrm{mod}\ \ell)$.

Then $\mathcal{S}_{k,q}(S_n) \cong \mathcal{S}_k(S_n)$, where $\mathcal{S}_k(S_n)$ is the Schur algebra studied by J. A. Green (1980).

A partition λ of n may be viewed as a dominant weight of $\mathrm{GL}_n(k)$ [$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \leftrightarrow \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_m\varepsilon_m$].

Thus there are corresponding $k\mathrm{GL}_n(k)$-modules $V(\lambda)$ and $L(\lambda)$.

If λ and μ are partitions of n, we have

$$[V(\lambda) : L(\mu)] = [\mathbf{S}^\lambda : \mathbf{D}^\mu] = d_{\lambda,\mu}.$$

The first equality comes from the significance of the Schur algebra, the second from that of the q-Schur algebra.

CONNECTIONS TO DEFINING CHARACTERISTICS, II

Thus the ℓ-modular decomposition numbers of $\mathrm{GL}_n(q)$ for prime powers q with $\ell \mid q-1$, determine the composition multiplicities of **certain** simple modules $L(\mu)$ in **certain** Weyl modules $V(\lambda)$ of $\mathrm{GL}_n(k)$, namely if λ and μ are partitions of n.

FACTS (SCHUR, GREEN)

Let λ and μ be partitions with at most n parts.

1. $[V(\lambda) : L(\mu)] = 0$*, if λ and μ are partitions of different numbers.*
2. *If λ and μ are partitions of $r \geq n$, then the composition multiplicity $[V(\lambda) : L(\mu)]$ is the same in $\mathrm{GL}_n(k)$ and $\mathrm{GL}_r(k)$.*

Hence the ℓ-modular decomposition numbers of **all** $\mathrm{GL}_r(q)$, $r \geq 1$, $\ell \mid q-1$ determine the composition multiplicities of **all** Weyl modules $V(\lambda)$ of $\mathrm{GL}_n(k)$. (Thank you Jens!)

CONNECTIONS TO SYMMETRIC GROUP REPR'S

As for the Schur algebra, there are standard kS_n- modules S^λ, called Specht modules, labelled by the partitions λ of n.

The simple kS_n-modules D^μ are labelled by the ℓ-regular partitions μ of n (no part of μ is repeated ℓ or more times).

The ℓ-modular decomposition numbers of S_n are the $[S^\lambda : D^\mu]$.

THEOREM (JAMES, 1980)

$[S^\lambda : D^\mu] = [V(\lambda) : L(\mu)]$, *if* μ *is* ℓ*-regular.*

THEOREM (ERDMANN, 1995)

For partitions λ, μ *of* n, *there are* ℓ*-regular partition* $t(\lambda)$, $t(\mu)$ *of* $\ell n + (\ell - 1)n(n-1)/2$ *such that*

$$[V(\lambda) : L(\mu)] = [S^{t(\lambda)} : D^{t(\mu)}].$$

AMAZING CONCLUSION

Recall that ℓ is a fixed prime and k an algebraically closed field of characteristic ℓ.

Each of the following three families of numbers can be determined from any one of the others:

1. The ℓ-modular decomposition numbers of S_n for all n.
2. The ℓ-modular decomposition numbers of the unipotent characters of $\mathrm{GL}_n(q)$ for all primes powers q with $\ell \mid q-1$ and all n.
3. The composition multiplicities $[V(\lambda) : L(\mu)]$ of $k\mathrm{GL}_n(k)$-modules for all n and all dominant weights λ, μ.

Thus all these problems are really hard.

JAMES' CONJECTURE

Let $G = \mathrm{GL}_n(q)$. Recall that $e = \min\{i \mid \ell \text{ divides } q^i - 1\}$.

James has computed all matrices D^u for $n \leq 10$.

CONJECTURE (JAMES, 1990)

If $e\ell > n$, then D^u only depends on e (neither on ℓ nor q).

THEOREM

(1) The conjecture is true for $n \leq 10$ (James, 1990).

(2) If $\ell >> 0$, D^u only depends on e (Geck, 1992).

LARGE PRIMES

In fact, Geck proved the following factorisation property.

THEOREM (GECK, 1992)

Let D^u be the ℓ-modular decomposition matrix of the q-Schur algebra $\mathcal{S}_{k,q}(S_n)$. Then $D^u = D_e D_\ell$ for two square matrices D_e and D_ℓ, where D_e only depends on e and D_ℓ only on ℓ. Moreover, $D_\ell = I$ for $\ell >> 0$.

There is an algorithm to compute the matrices D_e.

THEOREM (LASCOUX-LECLERC-THIBON; ARIKI; VARAGNOLO-VASSEROT (1996 – 99))

The matrix D_e can be computed from the canonical basis of a certain highest weight module of the quantum group $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$.

A unipotent decomposition matrix for $\mathrm{GL}_5(q)$

Let $G = \mathrm{GL}_5(q)$, $e = 2$ (i.e., $\ell \mid q + 1$ but $\ell \nmid q - 1$), and assume $\ell > 2$. Then $D^u = D_e$ equals

(5)	1						
$(4, 1)$		1					
$(3, 2)$			1				
$(3, 1^2)$	1		1	1			
$(2^2, 1)$			1	1	1		
$(2, 1^3)$		1				1	
(1^5)	1			1			1

The triangular shape defines φ_λ, $\lambda \in \mathcal{P}_5$.

On the degree polynomials

The degrees of the φ_λ are "polynomials in q".

λ	$\varphi_\lambda(1)$
(5)	1
$(4,1)$	$q(q+1)(q^2+1)$
$(3,2)$	$q^2(q^4+q^3+q^2+q+1)$
$(3,1^2)$	$(q^2+1)(q^5-1)$
$(2^2,1)$	$(q^3-1)(q^5-1)$
$(2,1^3)$	$q(q+1)(q^2+1)(q^5-1)$
(1^5)	$q^2(q^3-1)(q^5-1)$

Theorem (Brundan-Dipper-Kleshchev, 2001)

The degrees of $\chi_\lambda(1)$ and of $\varphi_\lambda(1)$ as polynomials in q are the same.

AN ANALOGUE OF JAMES' CONJECTURE?

Let $\{G(q) \mid q$ a prime power with $\ell \nmid q\}$ be a series of finite groups of Lie type, e.g. $\{\mathrm{GU}_n(q)\}$ or $\{\mathrm{SO}_n(q)\}$ (n fixed).

QUESTION

Is an analogue of James' conjecture true for $\{G(q)\}$?

More precisely, is there a factorisation $D^u = D_e D_\ell$ such that D_e and D_ℓ only depend on e, respectively ℓ, and such that $D_\ell = I$ for large enough ℓ?

If **yes**, only finitely many matrices D^u to compute:

- e takes only finitely many values since the rank of $G(q)$ is fixed;
- there are only finitely many "small" ℓ's.

OTHER q-SCHUR ALGEBRAS?

The following is a weaker form of the above question.

CONJECTURE

The entries of D^u are bounded independently of q and ℓ.

This conjecture is known to be true for

- $\mathrm{GL}_n(q)$ (Dipper-James),
- G classical and ℓ linear (Gruber-H., 1997),
- $\mathrm{GU}_3(q)$, $\mathrm{Sp}_4(q)$ (Okuyama-Waki, 1998, 2002),
- Suzuki groups (cyclic defect) and Ree groups ${}^2G_2(q)$ (Landrock-Michler, 1980).

QUESTION

Is there a q-Schur algebra for $\{G(q)\}$, whose ℓ-modular decomposition matrix equals D^u?

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