# Distinguished representations of finite classical groups

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# Congratulations

Happy Birthday to You,

Professor Zalesski!

# CONTENTS

# THE FINITE CLASSICAL GROUPS

These are the classical groups defined over finite fields. i.e., linear groups preserving a form of degree 2.

## EXAMPLES

- $\mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$, $\mathrm{Sp}_{2n}(q)$, $\mathrm{SO}_{2n+1}(q)$ ... *(q a prime power)*
- *E.g.,* $\mathrm{Sp}_{2n}(q) = \{A \in GL_{2n}(q) \mid A^{tr}\tilde{J}A = \tilde{J}\}$, *where*

$$\tilde{J} = \begin{bmatrix} & J \\ -J & \end{bmatrix} \text{ with } J = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} \in \mathbb{F}_q^{n \times n}.$$

- *Related groups, e.g.,* $\mathrm{SL}_n(q)$, $\mathrm{PSL}_n(q)$, $\mathrm{CSp}(q)$ *etc. are also classical groups.*

# Distinguished Representations

1. Representations in this talk are taken mostly over the field $\mathbb{C}$ of complex numbers.

2. Distinguished representations are families of representations obtained by a uniform construction.

3. Some of the distinguished representations also exist for other groups of Lie type, or for infinite classical groups, others exist only for some of the classical groups.

# THE STEINBERG REPRESENTATION

Frobenius, 1896: $\mathrm{PSL}_2(p)$ has irreducible representation of degree $p$.

Schur, 1907: $\mathrm{SL}_2(q)$ has irreducible representation of degree $q$.

Steinberg, 1951: $\mathrm{GL}_n(q)$ has irreducible representation of degree $q^{n(n-1)/2}$.

Steinberg, 1956: Finite classical groups of characteristic $p$ have an irreducible representation whose degree equals the order of a Sylow $p$-subgroup.

Steinberg, 1957: Same result as above for groups with split $BN$-pairs of characteristic $p$.

# Construction of the Steinberg repr'n, I

Steinberg, *A geometric approach to the representations of the full linear group over a Galois field*. TAMS 1951.

Steinberg constructs all *unipotent* representations of $\mathrm{GL}_n(q)$.

These are in bijection with the partitions of $n$.

In particular, the representation

$$\Gamma(1^n) = \sum_{\nu} \varepsilon(\nu) \mathrm{Ind}_{P_\nu}^{G}(\mathbb{C}).$$

Here, $\nu$ runs through the compositions of $n$, $\varepsilon(\nu)$ is a sign, and $\mathrm{Ind}_{P_\nu}^{G}(\mathbb{C})$ is the permutation module on the flags of shape $\nu$.

Steinberg shows that $\Gamma(1^n)$ has degree $q^{n(n-1)/2}$.

# Construction of the Steinberg repr'n, II

Steinberg, *Prime power representations of finite linear groups, II*. Canad. J. Math. 1957.

Steinberg constructs the Steinberg representation for all finite groups $G$ with a split $BN$-pair of characteristic $p$:

Let $W$ be the Weyl group, $U \leq B$ a Sylow subgroup of $G$.

Put

$$e := \left(\sum_{b \in B} b\right)\left(\sum_{w \in W} (-1)^{\ell(w)} w\right).$$

Then $e\mathbb{C}G$ has basis $\{eu \mid u \in U\}$ and affords the Steinberg representation $\mathrm{St}_G$.

# CONSTRUCTION OF THE STEINBERG REPR'N, III

Charles W. Curtis, 1965: $G$ a finite group with $BN$-pair with Coxeter system $(W, S)$.

The character $\mathrm{ch}(\mathrm{St}_G)$ of the Steinberg representations equals

$$\mathrm{ch}(\mathrm{St}_G) = \sum_{J \subset S} (-1)^{|J|} \mathrm{Ind}_{P_J}^{G}(\mathbf{1}),$$

where $P_J$ is the standard parabolic subgroup corresponding to $J$, and $\mathbf{1}$ denotes the trivial character.

$\mathrm{St}_G$ is the unique constituent of $\mathrm{Ind}_B^G(\mathbb{C})$ which is not a component of $\mathrm{Ind}_{P_J}^G(\mathbb{C})$ for $J \neq \emptyset$ $[B = P_\emptyset]$.

# CONSTRUCTION OF THE STEINBERG REPR'N, IV

L. Solomon 1969, Curtis, Lehrer, Tits, 1979:

Let $G$ be a finite group with $BN$-pair of rank $r$ and let $\Delta$ be the Tits building of $G$. Then

$$H_{r-1}(\Delta) \otimes_{\mathbb{Z}} \mathbb{C}$$

affords the Steinberg representation of $G$.

# THE GELFAND-GRAEV REPRESENTATIONS

Let $G$ be a finite group of Lie type, $U$ its unipotent subgroup.

$U = \prod_{\alpha \in \Phi^+} U_\alpha$, where $\Phi_+$ is the set of positive roots and $U_\alpha$ a root subgroup.

$\lambda : U \to \mathbb{C}^*$ is *in general position*, if $\lambda_{\downarrow U_\alpha} \neq \mathbf{1}$ if $\alpha$ is simple, and $\lambda_{\downarrow U_\alpha} = \mathbf{1}$, otherwise.

$$\Gamma := \mathrm{Ind}_U^G(\lambda),$$

where $\lambda$ is a linear representation of $U$ in general position, is called a *Gelfand-Graev representation* of $G$.

In particular, the degree of $\Gamma$ equals $|G|/|U| = |G|/\deg(\mathrm{St}_G)$.

For some groups, e.g., $\mathrm{Sp}_{2n}(q)$, $q$ odd, there is more than one Gelfand-Graev representation.

# SOME PROPERTIES OF THE GELFAND-GRAEV REPR'N

**THEOREM (GELFAND-GRAEV '62, YOKONUMA '67, STEINBERG '68)**

*A Gelfand-Graev representation $\Gamma$ is multiplicity free.*

Idea of prove: $\mathrm{End}_{\mathbb{C}G}(\Gamma)$ is abelian.

The characters of the irreducible constituents of $\Gamma$ are called *regular characters* of $G$. E.g., $\mathrm{ch}(\mathrm{St}_G)$ is regular.

**Note**:

$$\mathrm{St}_G \otimes_{\mathbb{C}} \Gamma_G$$

equals the regular representation of $G$.

# HISTORY OF THE WEIL REPRESENTATION

1961: Bolt, Room & Wall define Weil representations for finite conformal symplectic groups.

1964: Weil constructs Weil representations associated to symplectic vector spaces over local fields.

1972: Ward defines Weil representations for finite symplectic groups.

Howe (1973), Isaacs (1973), Lehrer (1974), Seitz (1975) & Gérardin (1975) contribute to different aspects of the

- construction of the Weil representations for finite general linear and unitary groups,
- determination of the characters of the Weil representations,
- determination of the irreducible constituents of the Weil representations.

# Weil character for general unitary groups

$\varepsilon = \pm 1$, $G = \mathrm{GL}_n(\varepsilon q)$ acting on $V$ [$\mathrm{GL}_n(-q) := \mathrm{GU}_n(q)$]

$\omega$ character of Weil representation of $G$

$\omega(1) = q^n$

$\omega(g) = \varepsilon^n (\varepsilon q)^{N(V,g)}$, $g \in G$ [$N(V,g) = \dim(\ker(1-g))$]

For $G = \mathrm{GL}_n(q)$, $\omega$ is the permutation character on $V$.

$\omega$ is multiplicative, i.e.,

if $V = V_1 \oplus V_2$, then $\omega_{\downarrow G(V_1) \times G(V_2)} = \omega_{G(V_1)} \boxtimes \omega_{G(V_2)}$.

# Construction of Weil representation, I

Suppose $G = \mathrm{GU}_n(q)$ acting on $V = \mathbb{F}_{q^2}^n$, preserving the Hermitian form $\beta$.

For $v \in V$, let $v^*$ denote the linear form determined by $\beta$ and $v$.

The Heisenberg group of $\beta$ is the group

$$H = \left\{ \begin{bmatrix} 1 & v^* & z \\ 0 & I_n & v \\ 0 & 0 & 1 \end{bmatrix} \mid v \in V, z \in \mathbb{F}_{q^2}, z + v^*(v) + z^q = 0 \right\}.$$

This is a special $p$-group of order $q^{2n+1}$.

The centre of $H$ is elementary abelian of order $q$.

# Construction of Weil representation, II

Let $\zeta$ be an irreducible representation of $H$ with $Z(H)$ not in kernel. ($H$ has $q-1$ distinct such representations.)

$G$ acts on $H$ in a natural way, fixing $Z(H)$ element-wise.

Consider the semidirect product $P := HG$.
Then $\zeta$ is invariant in $P$.

**Fact**: $\zeta$ extends to an irreducible representation $\hat{\zeta}$ of $P$.

$\Omega_n := \mathrm{Res}_G^P(\hat{\zeta})$ is the Weil representation of $G = \mathrm{GU}_n(q)$.

Each such $\zeta$ yields the same $\Omega_n$.

An analogous construction works for the symplectic groups.

# THE WEIL-STEINBERG REPRESENTATION

$G = G_n(q) \in \{\mathrm{GL}_n(q), \mathrm{Sp}_{2n}(q) (q \text{ odd}), \mathrm{GU}_{2n}(q), \mathrm{GU}_{2n+1}(q)\}$

$\mathrm{St}_n$: Steinberg representation of $G$

$\Omega_n$: Weil representation of $G$

$\Gamma_n$: Gelfand-Graev representation of $G$

$\mathrm{WSt}_n := \Omega_n \otimes \mathrm{St}_n$: *Weil-Steinberg representation of* $G$
[Zalesski, H., 2008]

# Weil-Steinberg representation, I

$G = \mathrm{GL}_n(q)$, $V$ natural vector space for $G$

$P_m$: stabilizer of $m$-dim. subspace of $V$ $(0 \leq m \leq n)$

$L_m = \mathrm{GL}_m(q) \times \mathrm{GL}_{n-m}(q)$ Levi subgroup of $P_m$

**Theorem (Zalesski, H., '08)**

$$\Omega_n \otimes \mathrm{St}_n = \sum_{m=0}^{n} \mathrm{Ind}_{P_m}^{G} \left( \mathrm{Infl}_{L_m}^{P_m} (\mathrm{St}_m \boxtimes \Gamma_{n-m}) \right).$$

Brundan, Dipper, Kleshchev (2001), Memoirs, Section 5:
Contains related and more general results.

# Weil-Steinberg representation, II

$G$ as above, $G \neq \mathrm{GL}_n(q)$, $V$ natural vector space for $G$

$P_m$: stabilizer of $m$-dim. isotropic subspace of $V$ ($0 \leq m \leq n$)

$L_m = \mathrm{GL}_m(q') \times G_{n-m}(q)$ Levi subgroup of $P_m$
[$q' = q^2$ if $G$ is unitary, $q' = q$, otherwise]

**Theorem (Zalesski, H., '08)**

$$\Omega_n \otimes \mathrm{St}_n = \sum_{m=0}^{n} \mathrm{Ind}_{P_m}^{G} \left( \mathrm{Infl}_{L_m}^{P_m} \left( \mathrm{St}_m \boxtimes \Gamma'_{n-m} \right) \right).$$

$\Gamma'_{n-m}$: *truncated* Gelfand-Graev representation of $G$, defined via Deligne-Lusztig theory

# The truncated Gelfand-Graev representation

Suppose $G = \mathrm{GU}_n(q)$.

$$\mathrm{Irr}(G) = \bigcup_{s \in \mathcal{S}} \mathcal{E}(G, s),$$

a disjoint union of *Lusztig series* $\mathcal{E}(G, s)$.
Here, $\mathcal{S}$ is a set of representatives of the conjugacy classes of semisimple elements of $G$.
Every $\mathcal{E}(G, s)$ contains exactly one regular character, $\chi_s$, and

$$\mathrm{ch}(\Gamma_G) = \sum_{s \in \mathcal{S}} \chi_s.$$

Now $\mathrm{ch}(\Gamma'_s)$ is the sum of those $\chi_s$ such that $s$ has no eigenvalue $-1$ on $V$. [1 if $q$ is even.]
A similar definition applies for the symplectic groups.

## Some steps in the proof

Suppose $G \neq \mathrm{GL}_n(q)$.

Write $\omega$ and st for the characters of $\Omega_n$ and $\mathrm{St}_n$, respectively.

1. $$\omega \cdot \mathrm{st} = \sum_{(T,\theta)} \frac{\varepsilon_g \varepsilon_T (\omega \cdot \mathrm{st}, R_{T,\theta})}{|W(T,\theta)|} R_{T,\theta}$$
2. $(\omega \cdot \mathrm{st}, R_{T,\theta}) = (\omega, \mathrm{st} \cdot R_{T,\theta}) = (\omega, \mathrm{Ind}_T^G(\theta)) = (\mathrm{Res}_T^G(\omega), \theta)$
3. Use multiplicativity of $\omega$ to compute $\mathrm{Res}_T^G(\omega)$.
4. If $(T, \theta)$ corresponds to a semisimple element without eigenvalue $-1$ on $V^*$, then $(\mathrm{Res}_T^G(\omega), \theta) = 1$.
5. If $(\mathrm{Res}_T^G(\omega), \theta) \neq 0$, then $T$ stabilizes an isotropic subspace of $V$.

# CONSEQUENCES, I

**Corollary 1**: $G \neq \mathrm{GL}_n(q)$. Then $\Omega_n \otimes \mathrm{St}_n$ is multiplicity free.

This is not the case for $G = \mathrm{GL}_n(q)$.

**Corollary 2** (Schröer): If $G = \mathrm{GL}_n(q)$ then

$$\Omega_n \otimes \mathrm{St}_n = \sum_{m=0}^{n} \sum_{s' \in \mathcal{S}'_m} \sum_{k=0}^{\lfloor (n-m)/2 \rfloor} (n - m - 2k + 1)\chi_{(n-m-k,k)';s'}.$$

Here, $\mathcal{S}'_m$ is a set of representatives of the conjugacy classes of semisimple elements of $\mathrm{GL}_m(q)$ without eigenvalue 1, and

$\chi_{(n-m-k,k)';s'} \in \mathrm{Irr}(G)$ has Jordan decomposition $(s, \psi)$ with

$s = 1_{n-m} \times s'$, $C_G(s) = \mathrm{GL}_{n-m}(q) \times C_{\mathrm{GL}_m(q)}(s')$,

$\psi = \chi_{(n-m-k,k)'} \boxtimes \mathrm{St} \in \mathrm{Irr}_u(C_G(s))$.

# Consequences, II

$G = G_n(q) \in \{\mathrm{GL}_n(q), \mathrm{Sp}_{2n}(q)(q \text{ odd}), \mathrm{GU}_{2n}(q), \mathrm{GU}_{2n+1}(q)\}$

$V$ natural vector space for $G$, $v \in V$

**Corollary (An, Brunat, Zalesski, H., '06 – '08)**

*Let $H$ be the stabilizer of $v$ in $G$. Then*

$$\mathrm{Res}_H^G(\mathrm{St}_G) \text{ is multiplicity free.}$$

**Question**

*Is this also true for $G = \mathrm{SO}_{2n+1}(q)$, $\mathrm{SO}_{2n}^{\pm}(q)$?*

# The symplectic group in even characteristic, I

$G = \mathrm{Sp}_{2n}(q)$, $q$ even.

No Weil representation is defined for this group.

Replace $\Omega_n$ by generalized spinor representation $\Sigma_n$ with highest weight $(0, \ldots, 0, q-1)$, $\deg(\Sigma_n) = q^n$.

View $\mathrm{St}_n$ as a projective, irreducible $\bar{\mathbb{F}}_q$-representation of $G$. Then $\Sigma_n \otimes \mathrm{St}_n$ is a projective $\bar{\mathbb{F}}_q$-representation of $G$.

By general theory, $\Sigma_n \otimes \mathrm{St}_n$ lifts to an ordinary representation, which we call the Weil-Steinberg representation $\mathrm{WSt}_n$ of $G$.

# The symplectic group in even characteristic, II

The same formula as for odd $q$ holds:

**Theorem (Zalesski, H., '08)**

$$\mathrm{WSt}_n = \sum_{m=0}^{n} \mathrm{Ind}_{P_m}^{G}\left(\mathrm{Infl}_{L_m}^{P_m}\left(\mathrm{St}_m \boxtimes \Gamma'_{n-m}\right)\right).$$

We also find the decomposition of $\mathrm{WSt}_n$ into PIMs.

**Theorem (Zalesski, H., '08)**

$$\mathrm{WSt}_n = \sum_{j=0}^{q-1} \Phi_{\nu_j},$$

*where $\Phi_{\nu_j}$ is the (lift of) the projective cover of the simple representation $\nu_j$ with highest weight $(q-1, \ldots, q-1, j)$.*

# MODULAR REPRESENTATIONS

Let $\mathbf{G}$ be a connected reductive group defined over $\mathbb{F}_p$. Let $q$ be a power of $p$. Then $G := \mathbf{G}(q)$ is a finite group of Lie type.
Let $M$ be a simple $\bar{\mathbb{F}}_q\mathbf{G}$-module.
Call $M$ $G$-regular, if for every rational maximal torus $\mathbf{T}$ of $\mathbf{G}$, distinct $\mathbf{T}$-weight spaces of $M$ are non-isomorphic $\mathbf{T}(q)$-mod's.

**THEOREM (ZALESSKI, H., '09)**

*Suppose that $M$ is $G$-regular and $d = \dim(M)$. Let $\varphi$ denote the Brauer character of $\mathrm{Res}^{\mathbf{G}}_{G}(M)$.*
*Then $\varphi \otimes \mathrm{ch}(\mathrm{St}_G)$ is a sum of $d$ distinct regular characters of $G$.*

**THEOREM (ZALESSKI, H., '09)**

*Given $\mathbf{G}$ and $M$, there are only finitely many $q$ such that $M$ is not $\mathbf{G}(q)$-regular.*

Thank you for your attention!