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Tensor decomposable characters of p-groups

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Contents

Preface

This work deals with the investigation of p-groups with respect to the existence of tensor decomposable characters, i.e. irreducible characters which are products of two non-linear irreducible characters.
Using the computer algebra system GAP [GAP] we made some valuable observations. On the one hand we saw that no group of order p^5 with $p \leq 17$ has a tensor decomposable character. This led us to conjecture that for any prime number p there is no group of order p^5 possessing such a character, and we finally succeeded in proving this claim.
On the other hand we noticed that there are groups of order p^6 for $p \in \{2, 3, 5, 7, 11\}$ with a tensor decomposable character. This in turn led us to conjecture that for any prime p there always is a group of order p^6 possessing such a character.
Obviously we were not looking for a trivially tensor decomposable character. By that we mean an irreducible character $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ of a group G with normal subgroup $N \trianglelefteq G$ such that there exist irreducible projective characters $\hat{\vartheta}$ of G and ε of G/N with $\hat{\vartheta}_N = \vartheta$ such that $\chi = \hat{\vartheta} \cdot \varepsilon$.

The approach to prove what was conjectured before was to explicitly construct a non-trivial example of a group of order p^6 with a tensor decomposable character for an arbitrary prime p using power commutator presentations.
In order to find an appropriate presentation GAP again happened to be rather useful. Indeed we reached the goal to find a presentation for a group with the desired properties.
As final result we worked out the generic character table of this particular group. Looking at the table we can see that not only G possesses a tensor decomposable character, but that actually all irreducible characters of degree p^2 are tensor decomposable.

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Chapter 1

Basics from Group Theory

Let throughout this chapter G be a finite group.

1.1 Semidirect products

Let us first recall some properties about the **direct product** of groups.

Let H, N be two subgroups of G.
If the three conditions

(1) $H, N \trianglelefteq G$,

(2) $HN = G$,

(3) $H \cap N = \{1\}$

hold, G is isomorphic to the **outer direct product** $H \times N$.
For the sake of simplicity we shall use the following notation for the conjugation:

Notation 1.1.1 Let $N \trianglelefteq G$ and $g \in G$. We denote the conjugation on N with g by γ_g, i.e.

$$\gamma_g \colon N \to N,\ n \mapsto g^{-1}ng.$$

It is easy to see that γ_g is an automorphism of N.

Notation 1.1.2 Let N be a group.

(1) If $\varphi \in \mathrm{Aut}(N)$ is an automorphism of N, we write n^φ for the image of n under φ, where $n \in N$.

(2) Further for two elements $x, y \in N$ the conjugation of y with x is written as $y^x = x^{-1}yx$ and ${}^x y = xyx^{-1}$ respectively.

(3) We define the commutator of $x, y \in N$ as $[x, y] = x^{-1}y^{-1}xy$.

Remark 1.1.3 Let $H, N \leq G$.

(a) If N is a normal subgroup of G, then the commutator $[h, n]$ is contained in N for all $h \in H$, $n \in N$.

(b) G is the direct product of H and N, if and only if

  (1') $hn = nh$ for all $h \in H$, $n \in N$,

  (2) $HN = G$,

  (3) $H \cap N = \{1\}$.

Proof

(a) By definition we have $[h, n] = \underbrace{h^{-1}n^{-1}h}_{\in N} n \in N$.

(b) Suppose G is the direct product of H and N. From part (a) we know that $[h, n] \in H \cap N$ for all $h \in H$, $n \in N$. Since $H \cap N = \{1\}$ it follows that $[h, n] = 1$ for all $h \in H, n \in N$, i.e. property (1') holds. Properties (2) and (3) follow immediately from the assumption.

For the other direction suppose that properties (1'), (2) and (3) hold. Let $n \in N$, $g = hm \in G$ with $h \in H$, $m \in N$. Then we have

$$g^{-1}ng = m^{-1}h^{-1}nhm \stackrel{(1')}{=} m^{-1}nm \in N, \text{ hence } N \trianglelefteq G.$$

Similarly we can show that $H \trianglelefteq G$. Thus, property (1) holds, properties (2) and (3) hold by assumption. Hence G is the direct product of H and N.

Definition 1.1.4 Let $H, N \leq G$.
G is called the (inner) **semidirect product** of H with N, if

(1) $N \trianglelefteq G$,

(2) $HN = G$,

(3) $H \cap N = \{1\}$.

In this case H is called a **complement** of N in G.

Example 1.1.5

(1) $G = S_n$ $(n \geq 2)$, $H = \langle (1,2) \rangle$, $N = A_n$.
Then G is the semidirect product of H with N.

(2) Let K be a field, G the group of monomial $(n \times n)$-matrices over K $(n \geq 2)$. Furthermore let $T \leq G$ be the group of diagonal matrices and $W \leq G$ the group of permutation matrices. Then $T \trianglelefteq G$, $W \cap T = \{1\}$, $G = WT$.

(3) Let $G = D_{2n}$, the symmetry group of a regular n-gon and let $D = \langle d \rangle \trianglelefteq G$ be the subgroup of rotations. If $S = \langle s \rangle$, where s is a reflection, then G is the semidirect product of S with D.

Remark 1.1.6 Let G be the semidirect product of H with $N \trianglelefteq G$; then:

(1) Every $g \in G$ has a unique representation as a product hn with $h \in H$, $n \in N$ (normal form).

(2) For $h, h' \in H$, $n, n' \in N$ we have $(hn)(h'n') = \underbrace{(hh')}_{\in H}\underbrace{(n^{\gamma_{h'}} n')}_{\in N}$.

Proof

(1) The existence of the decomposition is clear. Let us show uniqueness: Suppose $hn = h'n'$, with $h, h' \in H$, $n, n' \in N$. Then we obtain $(h')^{-1}h = n'n^{-1} \in H \cap N = \{1\}$, i.e. $h = h'$, $n = n'$.

(2) $(hn)(h'n') = hh' \underbrace{(h')^{-1}nh'}_{n^{\gamma_{h'}}} n' = hh'n^{\gamma_{h'}}n'$.

Definition 1.1.7 Let H and N be groups. We say that H **acts on** N **as a group of automorphisms**, if

(1) H acts on N, and

(2) the action homomorphism $\varphi : H \longrightarrow S_N$ maps H to $\mathrm{Aut}(N) \leq S_N$, i.e. $(nn')^{\varphi(h)} = n^{\varphi(h)}n'^{\varphi(h)}$ for all $h \in H,\ n, n' \in N$.

Remark 1.1.8 Let $N \trianglelefteq G$.

(1) G acts on N by conjugation, the action homomorphism is

$$\varphi\colon G \to Aut(N)\colon g \mapsto \gamma_g.$$

(2) If $H \leq G$ is a complement to N in G (i.e. $HN = G$, $H \cap N = \{1\}$), then H acts on N.

(3) If N is abelian, then G/N acts on N.
Indeed $\varphi\colon G/N \to Aut(N),\ gN \mapsto \gamma_g$ is well-defined, since N is abelian.

Definition 1.1.9 Let H, N be two groups. Suppose H acts on N as a group of automorphisms.
The **semidirect product** of H with N w.r.t. $\varphi\colon H \to Aut(N)$, written as $H \ltimes_\varphi N$ (or just as $H \ltimes N$, if φ is clear from the context), is the group with underlying set $H \times N$ and multiplication defined by:

$$(h, n) \cdot (h', n') := (hh', n^{\varphi(h')}n'), \text{ where } h, h' \in H, n, n' \in N.$$

Convention: Let H, N be groups. If we say that H acts on N, we mean an action as a group of automorphisms.

Remark 1.1.10 Let H, N be groups, $\varphi\colon H \to Aut(N)$. Then $G := H \ltimes_{\varphi} N$ is a group.
Let $\bar{H} := \{(h, 1) \in H \ltimes N \,|\, h \in H\}$ and $\bar{N} := \{(1, n) \in H \ltimes N \,|\, n \in N\}$. Then

$$\bar{H}, \bar{N} \leq G,\ \bar{N} \trianglelefteq G,\ \bar{H}\,\bar{N} = G,\ \bar{H} \cap \bar{N} = \{1\}.$$

The action of $\bar{H}$ on $\bar{N}$ by conjugation is equivalent to the given action of H on N.

1.2 Free groups and presentations

1.2.1 Free groups

A **presentation** is a way to define a group. This concept involves so-called **free groups**. In the following we shall give a brief introduction into these two topics. We shall state and prove relevant theorems we later work with.

Definition 1.2.1 Let F be a group, $X \subseteq F$. Then F is called a **free group** on X, if for all groups G and all maps $f\colon X \to G$ there is a unique homomorphism $\varphi\colon F \to G$ with $\varphi|_X = f$. In this case X is also called a **free generating set** of F.

Remark 1.2.2

(1) Let F be a free group on $X \subseteq F$. Then $\langle X \rangle = F$.

(2) Let F and G be free groups on $X \subseteq F$ and $Y \subseteq G$ respectively. If $|X| = |Y|$, then $F \cong G$.

Proof

(1) Put $G = \langle X \rangle \leq F$. Now consider the map $f\colon X \to G,\ x \mapsto x$. Since F is a free group we know that there exists a homomorphism $\varphi\colon F \to G$ such that $\varphi(x) = f(x)$ for all $x \in X$. Let $\iota\colon G \to F$ denote the inclusion; then

$$\iota \circ \varphi\colon F \to F \text{ is a group homomorphism with } \iota \circ \varphi|_X = \iota \circ f|_X, \textit{and}$$
$$\mathrm{Id}_F\colon F \to F \text{ is a group homomorphism with } \mathrm{Id}_F|_X = \iota \circ f|_X.$$

By the uniqueness of such homomorphisms we obtain $\iota \circ \varphi = \mathrm{Id}_F$, hence ι is surjective, i.e. $G = F$.

(2) Let $f\colon X \to Y \subseteq G$ and $g\colon Y \to X \subseteq F$ be maps with $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$. By definition there exist group homomorphisms $\varphi\colon F \to G$ and $\psi\colon G \to F$ with $\varphi \circ \psi|_Y = \mathrm{Id}_Y$ and $\psi \circ \varphi|_X = \mathrm{Id}_X$. By the uniqueness we conclude that $\varphi \circ \psi = \mathrm{Id}_G$ and $\psi \circ \varphi = \mathrm{Id}_F$, hence φ and ψ are isomorphisms.

Definition 1.2.3 Let X be a set.

(1) A **word** of length $n \in \mathbb{N}$ over X is a sequence $(x_1, \ldots, x_m)$ with $x_i \in X$ for all $1 \leq i \leq m$, shortly written $x_1 x_2 \cdots x_m$.
The unique word of length 0 is called the empty word and denoted by ε. The set of all words over X is denoted by X^*.

(2) For $x = x_1 x_2, \cdots x_m$, $y = y_1 y_2 \cdots y_n$ in X^* we put $xy := x_1 x_2 \cdots x_m y_1 \cdots y_n$ (the concatenation). Through

$$X^* \times X^* \colon \to X^* \colon (x, y) \mapsto xy$$

X^* becomes a monoid (with neutral element ε), called the **free monoid** over X.

Theorem 1.2.4 *Let X be a set. Then there exists a group F and an injective map $\iota\colon X \to F$ such that F is a free group on $\iota(X)$ (in this situation we usually identify X with $\iota(X) \subseteq F$ and call F the free group on X. Note that F only depends on the cardinality of X).*

Proof Put $X^{\pm} := X \times \{1, -1\}$.
For $(x, \alpha) \in X^{\pm}$ we shall write x^{α} and for $(x, 1)$ we shall also write x. Now let us define an equivalence relation $\sim$ on $(X^{\pm})^*$ by:
$u \sim v$ if and only if there is a sequence $u = w_1, w_2, \ldots, w_m = v$ such that $w_i \to w_{i+1}$ or $w_{i+1} \to w_i$ for all $1 \leq i \leq m$.
Here we write $s \to t$ for $s, t \in (X^{\pm})^*$, if t is obtained from s by an "elementary

cancellation", i.e. $t = ax^\alpha x^{-\alpha} b$, $s = ab$, with $a, b \in (X^\pm)^*$, $x \in X$, $\alpha \in \{1, -1\}$.

For $w \in (X^\pm)^*$ let $[w]$ denote the equivalence class containing w. Put $F := \{[w] \mid w \in (X^\pm)^*\}$. Then:

(1) F is a group with multiplication $[v][w] := [vw]$, $v, w \in (X^\pm)^*$.
The identity element is $[\varepsilon] \in F$ and for $x_1, \ldots, x_m \in X$, $\alpha_1, \ldots, \alpha_m \in \{1, -1\}$ we have $[x_1^{\alpha_1} \cdots x_m^{\alpha_m}]^{-1} = [x_m^{-\alpha_m} \cdots x_1^{-\alpha_1}]$.

(2) $\iota\colon X \to F\colon x \mapsto [x]$ is injective.
We have to show: If $x, y \in X$ with $x \sim y$, then $x = y$. More generally for $x \in X$ we have: If $x \sim x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ with $x_i \in X$, $\alpha_i \in \{1, -1\}$, $1 \leq i \leq m$, then $|\{i \mid 1 \leq i \leq m, x = x_i\}|$ is odd and $|\{i \mid 1 \leq i \leq m, x \neq x_i\}|$ is even.

(3) Universal property: Let G be a group and $f\colon \iota(X) \to G$ a map. Define $\varphi\colon F \to G$ by $\varphi([x_1^{\alpha_1} \cdots x_m^{\alpha_m}]) := f([x_1])^{\alpha_1} \cdots f([x_m])^{\alpha_m}$, $x_i \in X$, $\alpha_i \in \{1, -1\}$, $1 \leq i \leq m$. Then φ is well-defined, a group homomorphism and $f = \varphi|_{\iota(X)}$.
If $\varphi'\colon F \to G$ is a group homomorphism with $\varphi'|_{\iota(X)} = f$, then $\varphi' = \varphi$, since $\langle \iota(X) \rangle = F$ by construction.

Example 1.2.5

(1) The free group on $X = \emptyset$ is the trivial group.

(2) The free group F on $X = \{x\}$ is isomorphic to $(\mathbb{Z}, +)$, $F = \{x^z \mid z \in \mathbb{Z}\}$.

(3) If $|X| \geq 2$, the free group F on X is not abelian. In order to see that let $x \neq y \in X$. By definition of a free group there is a (surjective) homomorphism

$$F \to S_3\colon \begin{cases} x \mapsto (1,2) \\ y \mapsto (1,2,3) \\ z \mapsto 1, \text{ if } z \notin \langle x, y \rangle \end{cases}$$

Since S_3 is not abelian F neither is.

1.2.2 Presentations

Definition 1.2.6 Let G be a group, $S \subseteq G$. We define

$$\langle\langle S\rangle\rangle := \bigcap_{N \trianglelefteq G,\, S \subseteq N} N,$$

called the **normal closure** of S in G, which is the smallest normal subgroup of G containing S. Note that

$$\langle\langle S\rangle\rangle = \langle g^{-1}sg \,|\, g \in G, s \in S\rangle.$$

Definition 1.2.7 Let G be a group. A **presentation** of G (by generators and relations) is a pair (X, R), where X is a set and R is a subset of the free group F on X, such that $G \cong F/\langle\langle R\rangle\rangle =: \langle X \mid R\rangle$.
We call $\langle X \mid R\rangle$ **finite**, if X and R are finite. In this case $G \cong \langle X \mid R\rangle$ is called **finitely presented**.

Remark 1.2.8 $\langle X \mid R\rangle$ is the most general (largest) group generated by the set X whose elements satisfy the relations $r = 1$ for $r \in R$.

Remark 1.2.9 Let G be a group, $S \subseteq G$ such that $\langle S\rangle = G$. Then there exists a set X, a bijection $f\colon X \to S$ and a presentation $G \cong \langle X \mid R\rangle$. If G is finite, it is finitely presented.

Proof Let X be a set with $|X| = |S|$ and let $f\colon X \to S$ be a bijection. Let further F be the free group on X and $\varphi\colon F \to G$ be the extension of f. Then φ is surjective and $G \cong \langle X \mid R\rangle$ for $R \subseteq G$ with $\langle\langle R\rangle\rangle = \ker(\varphi)$.
Let G be finite, $\overline{G} := \{\overline{g} \,|\, g \in G\}$ a set with $|\overline{G}| = |G|$ and a bijection $f\colon \overline{G} \to G\colon \overline{g} \mapsto g$.
Let F be the free group on $\overline{G}$ and let φ be the homomorphism $F \to G$ extending f. Then $\ker(\varphi) = \langle\langle\{\overline{g}\,\overline{h}\,\overline{gh}^{-1} \,|\, g, h \in G\}\rangle\rangle$, i.e. $G \cong \left\langle \overline{G} \,|\, \overline{g}\,\overline{h}\,\overline{gh}^{-1}, g, h \in G \right\rangle$.

Theorem 1.2.10 *Let G, H be groups, $G \cong \langle X \mid R\rangle$ a presentation of G. Let F be the free group on X and $f\colon X \to H$ be a map with extension $\varphi\colon F \to H$.*

If $\varphi(r) = 1$ *for all* $r \in R$ *(in this case we say the elements* $f(x)$, $x \in X$ *satisfy the relations* R*), then there exists a homomorphism* $\overline{\varphi}\colon G \to H$ *such that the following diagram commutes*

$$\begin{tikzcd} & F \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ G \arrow[rr, "\overline{\varphi}"'] & & H \end{tikzcd}$$

Here π *is defined by*

$$\begin{tikzcd} F \arrow[r, "\nu_{\langle\langle R\rangle\rangle}"] \arrow[dr, "\pi"'] & F/\langle\langle R\rangle\rangle \arrow[d, "\alpha"] \\ & G \end{tikzcd}$$

where $\nu_{\langle\langle R\rangle\rangle}$ *denotes the canonical map* $\nu_{\langle\langle R\rangle\rangle} : F \longrightarrow F/\langle\langle R\rangle\rangle$ *and* α *denotes the isomorphism* $\alpha : F/\langle\langle R\rangle\rangle \longrightarrow G$.

In particular, if H *is generated by* $f(x)$, $x \in X$*, then* H *is isomorphic to a factor group of* G.

Proof By assumption $R \subseteq \ker(\varphi)$, hence $\langle\langle R\rangle\rangle \leq \ker(\varphi)$.
Writing $\tilde{\varphi}\colon F/\langle\langle R\rangle\rangle \to H\colon w\langle\langle R\rangle\rangle \mapsto \varphi(w)$, the homomorphism $\overline{\varphi} := \tilde{\varphi} \circ \alpha^{-1}$ satisfies the assertion.

Example 1.2.11

(1) Let C_n be the cyclic group of order n. Then $C_n \cong \langle x \mid x^n \rangle$.

(2) Let D_{2n} be the dihedral group of order $2n$. Then $D_{2n} \cong \langle x, y \mid x^2, y^2, (xy)^n \rangle$.

Proof In each of the two cases let G be the group defined by the respective presentation. We view X as a subset of G.

(1) Let $C_n = \langle a \rangle$.
Since $a^n = 1$, there is a homomorphism $\varphi\colon G \to C_n$ with $\varphi(x) = a$, which is clearly surjective. Hence $|G| \geq |C_n| = n$. Now $G = \langle x \rangle$ is cyclic, $x^n = 1$, i.e. $|G| \leq n$. Hence $|G| = n$ and φ is an isomorphism.

(2) We have that $D_{2n} = \langle s, t \rangle$ with involutions s, t such that $d := st$ has order n. Hence by 1.2.10 there exists a surjective homomorphism $G \to D_{2n}\colon x \mapsto s,\ y \mapsto t$, and so $|G| \geq |D_{2n}| = 2n$.
In G we have

$$x^{-1}(xy)x = yx = y^{-1}x^{-1} = (xy)^{-1},$$
$$y^{-1}(xy)y = y^{-1}x = y^{-1}x^{-1} = (xy)^{-1}.$$

Therefore $D := \langle xy \rangle \trianglelefteq G$ as well as $|\langle xy \rangle| \leq n$. From $G = \langle x, y \rangle = \langle xy, y \rangle$ we conclude that $G = D \cup Dy$.

Remark 1.2.12 Let X be a set, F the free group on X and let $R, S \subseteq F$. Let G be a group with presentation $\langle X \mid R \rangle$, i.e. $G \cong F/\langle\langle R \rangle\rangle$, and let $\pi\colon F \to G$ be an epimorphism with kernel $\langle\langle R \rangle\rangle$.
If $N := \langle\langle \pi(S) \rangle\rangle_G$, then $G/N \cong \langle X \mid R \cup S \rangle$.

Proof Put $H := \langle X \mid R \cup S \rangle = F/K$ with $K = \langle\langle R \cup S \rangle\rangle$ and let $f\colon X \to H\colon x \mapsto xK$ and $\varphi\colon F \to H$ its extension. Since $R \subseteq K$ we have $\varphi(r) = 1$ for all $r \in R$. By 1.2.10 there is a group homomorphism $\overline{\varphi}\colon G \to H$ such that $\overline{\varphi} \circ \pi = \varphi$. Since $H = \langle \varphi(x) \mid x \in X \rangle$ the map $\overline{\varphi}$ is surjective.
Now $\ker(\overline{\varphi}) = \pi(K) = \langle\langle \pi(R \cup S) \rangle\rangle_G = \langle\langle \pi(S) \rangle\rangle_G = N$. Since π is surjective we obtain $G/N = G/ker(\bar{\varphi}) \cong H = \langle X \mid R \cup S \rangle$.

Theorem 1.2.13 *Let $\langle X \mid R\rangle$ be a presentation for a group G. Then*

$$G/G' \cong \langle X \mid R \cup S\rangle \text{ with } S = \{[x,y] \mid x, y \in X\}.$$

Proof Let F be the free group on X, $\pi\colon F \to G = F/\langle\langle R\rangle\rangle$ the canonical homomorphism. By 1.2.12 $\langle X \mid R \cup S\rangle$ is a presentation for $G/\langle\langle \pi(S)\rangle\rangle_G$. We have to prove $\langle\langle \pi(S)\rangle\rangle_G = G'$. Let $x, y \in X$; then $\pi([x,y]) = [\pi(x), \pi(y)] \in G'$, hence $\pi(S) \subseteq G'$ and therefore $\langle\langle \pi(S)\rangle\rangle_G \subseteq G'$.
We have that G is generated by $\{\pi(x) \mid x \in X\}$.
Furthermore $[\pi(x), \pi(y)] = \pi([x,y]) \in \langle\langle \pi(S)\rangle\rangle_G$ for all $x, y \in X$. Thus $G/\langle\langle \pi(S)\rangle\rangle_G$ is abelian, i.e. $G' \subseteq \langle\langle \pi(S)\rangle\rangle_G$.

Remark 1.2.14 Let X, Y be disjoint sets, $G = \langle X \mid R\rangle$, $H = \langle Y \mid S\rangle$. Then

$$G \times H = \langle X \cup Y \mid R \cup S \cup \{[x,y] \mid x \in X, y \in Y\}\rangle.$$

Corollary 1.2.15 *Let F be the free group on a finite set $X = \{x_1, \dots, x_r\}$ with r elements. Then $F/F' \cong \langle x_1, \dots, x_r \mid [x_i, x_j],\ 1 \le i, j \le r\rangle$ and F/F' is a free abelian group of rank r.*

Proof The first statement follows from 1.2.13 (with $F = G$, $\pi = \mathrm{Id}$, $R = \emptyset$, $S = \{[x,y] \mid x, y \in X\}$). To prove the second statement we show $F/F' \cong \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ (r factors) by induction on r.
By 1.2.5(2) we have $\langle x_r \mid \emptyset\rangle \cong \mathbb{Z}$ and by induction we obtain

$$\underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{r-1\ factors} \cong \langle x_1, \dots, x_{r-1} \mid [x_i, x_j],\ 1 \le i, j \le r-1\rangle.$$

Using 1.2.14

$$\begin{aligned}
\underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{r\ factors} &\cong (\underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{r-1\ factors}) \times \mathbb{Z} \\
&\cong \langle x_1, \dots, x_{r-1} \mid [x_i, x_j],\ 1 \le i, j \le r-1\rangle \times \langle x_r \mid \emptyset\rangle \\
&\cong \langle x_1, \dots, x_r \mid [x_i, x_j],\ 1 \le i, j \le r\rangle \cong F/F'.
\end{aligned}$$

Theorem 1.2.16 *Let X and Y be sets and F_X and F_Y be the free groups on X and Y respectively. Let further $R \subseteq F_X$, $S \subseteq F_Y$ and define two groups by the presentations $H := \langle X \mid R \rangle$ and $N := \langle Y \mid S \rangle$ respectively. Moreover let $\varphi : H \longrightarrow Aut(N)$, $h \mapsto (n \mapsto n^{\varphi(h)})$ be a group homomorphism, so that we can construct the semidirect product of H and N.*
Then a presentation of the group $H \rtimes N$ is given by

$$H \rtimes N \;\cong\; \langle X \cup Y \mid R \cup S \cup \{y^x (y^{\varphi(x)})^{-1} \mid x \in X,\ y \in Y\} \rangle.$$

Let $\bar{R} := \langle\langle R \rangle\rangle \trianglelefteq F_X$, $\bar{S} := \langle\langle S \rangle\rangle \trianglelefteq F_Y$ and let $F_{X\cup Y}$ be the free group on $X \cup Y$. We shall view F_Y as a subgroup of $F_{X\cup Y}$.
For $x \in X \cup X^{-1}$ the symbol *$y^{\varphi(x)} \in F_{X\cup Y}$ denotes an arbitrarily chosen preimage of the element $(y\bar{S})^{\varphi(x\bar{R})} \in N$ in $F_Y \leq F_{X\cup Y}$. Note that $y^{\varphi(x)}\bar{S} = (y\bar{S})^{\varphi(x\bar{R})}$. Finally write $\bar{T} := \langle\langle R \cup S \cup \{y^x(y^{\varphi(x)})^{-1} \mid x \in X,\ y \in Y\}\rangle\rangle \trianglelefteq F_{X\cup Y}$.*

Proof Define

$$G \;:=\; \langle X \cup Y \mid R \cup S \cup \{y^x (y^{\varphi(x)})^{-1} \mid x \in X,\ y \in Y\} \rangle$$

as a quotient of $F_{X\cup Y}$.
We have a surjective group homomorphism

$$\begin{aligned} \phi:\ G &\longrightarrow H \rtimes_\varphi N \\ x\bar{T} &\mapsto \big(x\bar{R}, 1\big) \\ y\bar{T} &\mapsto \big(1, y\bar{S}\big) \end{aligned}$$

where $x \in X$ and $y \in Y$. This can be seen using 1.2.10. In fact ϕ is induced by a homomorphism $F_{X\cup Y}$ that maps elements $x \in X$ to $(x\bar{R}, 1)$ and $y \in Y$ to $(1, y\bar{S})$. Now check that all relations of G belong to the kernel. Elements of $R \cup S$ clearly map to $(1, 1)$. Given $y \in Y$ and $x \in X$, the element $y^x(y^{\varphi(x)})^{-1}$ is mapped to

$$\big(1, y\bar{S}\big)^{(x\bar{R},1)} (1, y^{\varphi(x)}\bar{S})^{-1} = (1, \big(y\bar{S}\big)^{\varphi\left(x\bar{R}\right)}) \left(1, \big(y\bar{S}\big)^{\varphi\left(x\bar{R}\right)}\right)^{-1} = (1,1)\,.$$

Hence ϕ actually is a group homomorphism. Clearly it is surjective since $H \rtimes_\varphi N$ is generated by $\big\{(x\bar{R},1)\,|\,x \in X\big\} \cup \big\{(1,y\bar{S})\,|\,y \in Y\big\}$. Furthermore we have a group homomorphism $N \xrightarrow{\nu} G$ that maps $y\bar{S}$ to $y\bar{T}$, where $y \in Y$. And we have a group homomorphism $H \xrightarrow{\chi} G$ that maps $x\bar{R}$ to $x\bar{T}$, where $x \in X$.

It remains to show that ϕ is injective. Suppose that $x \in X$ and $y \in Y$.

(i) By the definition of $\bar{T}$ we have $(y\bar{T})(x\bar{T}) = (x\bar{T})\left(y^{\varphi(x)}\bar{T}\right)$.

(ii) Again by the definition of $\bar{T}$ we have $(y^{-1}\bar{T})^{x\bar{T}} = (y^x\bar{T})^{-1} = \left(y^{\varphi(x)}\bar{T}\right)^{-1}$ and thus $\left(y^{-1}\bar{T}\right)\left(x\bar{T}\right) = \left(x\bar{T}\right)\left(\left(y^{\varphi(x)}\right)^{-1}\bar{T}\right)$.

(iii) Write $y^{\varphi(x^{-1})} = y_1^{\alpha_1}\cdots y_k^{\alpha_k}$ in F_Y, where $k \geq 0$, $y_i \in Y$ and $\alpha_i \in \{-1,+1\}$. We obtain

$$\begin{aligned}
\left(y^{\varphi(x^{-1})}\bar{T}\right)^{x\bar{T}} &= ((y_1\bar{T})^{\alpha_1}\cdots(y_k\bar{T})^{\alpha_k})^{x\bar{T}} \\
&= (y_1^x\bar{T})^{\alpha_1}\cdots(y_k^x\bar{T})^{\alpha_k} \\
&\overset{\text{def. of } \bar{T}}{=} (y_1^{\varphi(x)}\bar{T})^{\alpha_1}\cdots(y_k^{\varphi(x)}\bar{T})^{\alpha_k} \\
&= ((y_1^{\varphi(x)})^{\alpha_1}\cdots(y_k^{\varphi(x)})^{\alpha_k})\bar{T}\ .
\end{aligned}$$

Now

$$\begin{aligned}
((y_1^{\varphi(x)})^{\alpha_1}\cdots(y_k^{\varphi(x)})^{\alpha_k})\bar{S} &= (y_1^{\varphi(x)}\bar{S})^{\alpha_1}\cdots(y_k^{\varphi(x)}\bar{S})^{\alpha_k} \\
&= ((y_1\bar{S})^{\varphi(x\bar{R})})^{\alpha_1}\cdots((y_k\bar{S})^{\varphi(x\bar{R})})^{\alpha_k} \\
&= ((y_1^{\alpha_1}\cdots y_k^{\alpha_k})\bar{S})^{\varphi(x\bar{R})} \\
&= (y^{\varphi(x^{-1})}\bar{S})^{\varphi(x\bar{R})} \\
&= ((y\bar{S})^{\varphi(x^{-1}\bar{R})})^{\varphi(x\bar{R})} \\
&= y\bar{S}\ .
\end{aligned}$$

An application of $N \xrightarrow{\nu} G$ yields $((y_1^{\varphi(x)})^{\alpha_1}\cdots(y_k^{\varphi(x)})^{\alpha_k})\bar{T} = y\bar{T}$. Hence $(y^{\varphi(x^{-1})}\bar{T})^{x\bar{T}} = y\bar{T}$. Therefore, $(y\bar{T})(x^{-1}\bar{T}) = (x^{-1}\bar{T})(y^{\varphi(x^{-1})}\bar{T})$.

(iv) We have $(y^{-1}\bar{T})^{x^{-1}\bar{T}} = (y^{x^{-1}}\bar{T})^{-1} \overset{(iii)}{=} (y^{\varphi(x^{-1})}\bar{T})^{-1}$ and thus $(y^{-1}\bar{T})(x^{-1}\bar{T}) =$

$(x^{-1}\bar{T})((y^{\varphi(x^{-1})})^{-1}\bar{T})$.

Let us summarise our observations. Given $x \in X$ and $y \in Y$, we have

$$\begin{aligned}
(y\bar{T})(x\bar{T}) &= (x\bar{T})(y^{\varphi(x)}\bar{T}), \\
(y^{-1}\bar{T})(x\bar{T}) &= (x\bar{T})((y^{\varphi(x)})^{-1}\bar{T}), \\
(y\bar{T})(x^{-1}\bar{T}) &= (x^{-1}\bar{T})(y^{\varphi(x^{-1})}\bar{T}), \\
(y^{-1}\bar{T})(x^{-1}\bar{T}) &= (x^{-1}\bar{T})((y^{\varphi(x^{-1})})^{-1}\bar{T}).
\end{aligned}$$

Thus any element of G may be written as a product $(\xi\bar{T})(\eta\bar{T})$, where $\xi \in F_X$ and $\eta \in F_Y$. Such an element $(\xi\bar{T})(\eta\bar{T})$ is mapped to $(\xi\bar{R}, 1)(1, \eta\bar{S}) = (\xi\bar{R}, \eta\bar{S})$ under ϕ. If it is mapped to $(1, 1)$, we have $\xi\bar{R} = 1$ and $\eta\bar{S} = 1$. Applications of ν and χ show that this implies $\xi\bar{T} = 1$ and $\eta\bar{T} = 1$. Hence $(\xi\bar{T})(\eta\bar{T}) = 1$. This proves that ϕ is injective, hence an isomorphism and the proof is complete.

Chapter 2

Basics from Character Theory

2.1 Products of characters

For two characters χ and ψ, afforded by two $\mathbb{C}[G]$-modules V and W of a finite group G it is easy to see that the product $\chi \cdot \psi$ is a class function again, i.e. a map which is constant on each conjugacy class. However it is rather not obvious that $\chi \cdot \psi$ is a character of G again. In order to see that we will define a new $\mathbb{C}[G]$-module with the property that its afforded character is exactly the product of the two characters χ and ψ. Then we can immediately deduce that the product of any two characters yields a character again. In the following we shall first define the **tensor product** of two $\mathbb{C}$-vector spaces V and W which in the first place will be a $\mathbb{C}$-vector space again. In case V and W also are finite dimensional $\mathbb{C}[G]$-modules we shall see later that the resulting tensor product also has a structure as $\mathbb{C}[G]$-module and which affords the product of the characters afforded by V and W.

Definition and Remark 2.1.1

(1) Let V and W be two finite dimensional $\mathbb{C}$-vector spaces. Then there is a $\mathbb{C}$-vector space $V \bigotimes W$ of dimension $\dim_{\mathbb{C}} V \cdot \dim_{\mathbb{C}} W$ and a $\mathbb{C}$-bilinear map $\varphi : V \times W \longrightarrow V \otimes W$ with the following property:
If $\{v_1, \ldots, v_n\}$ is a $\mathbb{C}$-basis of V and $\{w_1, \ldots, w_m\}$ a $\mathbb{C}$-basis of W, then $\{\varphi(v_i, w_j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ is a $\mathbb{C}$-basis of $V \bigotimes W$.

(2) $V \bigotimes W$ is called the **tensor product** of V and W.
For $v \in V, w \in W$ we shall use the notation $v \otimes w$ for the image $\varphi(v, w)$.

Proof Ad (1): Let $\mathscr{B} = \{v_1, ..., v_n\}$ be a $\mathbb{C}$-basis of V and $\mathscr{C} = \{w_1, ..., w_m\}$ be a $\mathbb{C}$-basis of W. Now define $T := \{f : \mathscr{B} \times \mathscr{C} \longrightarrow \mathbb{C}\}$ as the space of all mappings from $\mathscr{B} \times \mathscr{C}$ to $\mathbb{C}$. This is a $\mathbb{C}$-vector space with basis $\left\{\mathbb{1}_{\{(v_i, w_j)\}} \mid 1 \le i \le n, 1 \le j \le m\right\}$ where $\mathbb{1}_{\{(v_i, w_j)\}}(x) = \begin{cases} 0\ , \ x \neq (v_i, w_j) \\ 1\ , \ x = (v_i, w_j) \end{cases}$ is the indicator function of the set $\{(v_i, w_j)\}$.
As next step we define the map

$$\varphi : V \times W \longrightarrow T\ ,\ \left(\sum_{i=1}^{n} a_i v_i, \sum_{j=1}^{m} b_j w_j\right) \mapsto \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \mathbb{1}_{\{(v_i, w_j)\}}.$$

We see that φ is bilinear and that $\{\varphi(v_i, w_j) \mid 1 \le i \le n, 1 \le j \le m\}$ is a basis of T (†). In particular we have $\dim_{\mathbb{C}} T = n \cdot m$.
If $\mathscr{B}' = \{v_1', ..., v_n'\}$ is a basis of V and $\mathscr{C}' = \{w_1', ..., w_m'\}$ a basis of W, then we conclude from (†) together with the bilinearity of φ that
$\mathscr{M} := \left\{\varphi\left(v_i', w_j'\right) \mid 1 \le i \le n, 1 \le j \le m\right\}$ generates T. Hence $\mathscr{M}$ is a basis of T and the proof is complete.

Remark 2.1.2 From 2.1.1 it follows that the tensor product $V \bigotimes W$ is generated as a $\mathbb{C}$-vector space by $\{v \otimes w \mid v \in V, w \in W\}$. However in general it is not true that there is equality, i.e. $V \bigotimes W \neq \{v \otimes w \mid v \in V, w \in W\}$.

Remark 2.1.3 Let V and W be finite dimensional $\mathbb{C}[G]$-modules. Then the tensor product $V \bigotimes W$ also has a structure as a $\mathbb{C}[G]$-module satisfying $g(v \otimes w) = gv \otimes gw$ for all $g \in G, v \in V, w \in W$.
Further the character afforded by $V \bigotimes W$, say $\chi_{V \bigotimes W}$, is the product $\chi_V \cdot \chi_W$ of the characters afforded by V and W respectively.

Proof Define an action of G on $V \bigotimes W$ via the basis. More precisely, for $g \in G$ define $g(v_i \otimes w_j) := gv_i \otimes gw_j$, where $\mathscr{B} = \{v_1, ..., v_n\}$ is a $\mathbb{C}$-basis of V and $\mathscr{C} = \{w_1, ..., w_m\}$ is a $\mathbb{C}$-basis of W. Then, by linear extension, we obtain $g\,(v \otimes w) = gv \otimes gw$ for all $v \in V, w \in W$. Let $B = (b_{ij})$ be the matrix induced by g on V with respect to the basis $\mathscr{B}$ and $C = (c_{ij})$ be the matrix induced by g on W with respect to $\mathscr{C}$. We put the basis elements into the order $\{v_1 \otimes w_1, v_1 \otimes w_2, ..., v_1 \otimes w_m, v_2 \otimes w_1, ..., v_2 \otimes w_m, ..., v_n \otimes w_m\}$. Then the matrix of g on $V \bigotimes W$ with respect to this basis is

$$B \otimes C := \begin{pmatrix} b_{11}C & \cdot & \cdot & \cdot & \cdot & \cdot & b_{1n}C \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ b_{n1}C & \cdot & \cdot & \cdot & \cdot & \cdot & b_{nn}C \end{pmatrix}.$$

We see that the trace of this matrix, i.e., the value $\chi_{V \bigotimes W}\,(g)$, is exactly the product $\chi_V\,(g) \cdot \chi_W\,(g) = \chi_V \cdot \chi_W\,(g)$ and the proof is complete.

Definition and Remark 2.1.4 Let $G = H \times K$ be the product of two finite groups H and K. Furthermore let φ be a character of H and ϑ a character of K. Define the map $\varphi \times \vartheta$ to be the product of these two characters, i.e.

$$\varphi \times \vartheta : H \times K,\ (h, k) \mapsto \varphi\,(h) \cdot \vartheta\,(k)\,.$$

From 2.1.3 we deduce that $\varphi \times \vartheta$ is a character of G. Consider the maps

$$\hat{\varphi} : H \times K \longrightarrow \mathbb{C},\ (h, k) \mapsto \varphi\,(h)$$

and analogously

$$\hat{\vartheta} : H \times K \longrightarrow \mathbb{C},\ (h, k) \mapsto \vartheta\,(k)\,.$$

We know that for a normal subgroup of $N \trianglelefteq G$ and a character $\hat{\chi}$ of G/N we obtain a character χ of G by defining $\chi : G \longrightarrow \mathbb{C},\ g \mapsto \hat{\chi}\,(gN)$.
Using the isomorphisms $H \times K/(\{1\} \times K) \cong H$ and $H \times K/(H \times \{1\}) \cong K$ and the foregoing comment we deduce that $\hat{\varphi}$ and $\hat{\vartheta}$ are characters of $G = H \times K$.

Hence, by 2.1.3, the product $\hat{\varphi} \cdot \hat{\vartheta}$ is a character of G as well. This is exactly the map $\varphi \times \vartheta$ and we conclude that $\varphi \times \vartheta$ is a character of G.

We will now see how the irreducible characters of a direct product $H \times K$ look like. We shall see that in this case the most intuitive approach is correct.

Theorem 2.1.5 *The irreducible characters of the direct product of two groups H and K can be describes as follows:*

$$\mathrm{Irr}_{\mathbb{C}}(H \times K) = \{\varphi \times \vartheta \mid \varphi \in \mathrm{Irr}_{\mathbb{C}}(H), \vartheta \in \mathrm{Irr}_{\mathbb{C}}(K)\}.$$

Proof [I, p.50, (4.21)].

2.2 Induced characters

Given a subgroup $H \leq G$ of a finite group G and a character χ of G, it is easy to see that its restriction to H yields a character of H. Hence it is natural to rise the question whether it is possible to obtain a character of G from a given character of H. In the following section we will present the concept of **induced characters** and further give some results from the theory around these characters. The most powerful theorem in this section is **Mackey's Tensor Product Theorem**. This will in particular be useful in the next chapter when it comes to investigate certain products of characters.

Let for this section again G be a finite group and $H \leq G$ be a subgroup of G.
Given a character χ of G we use the notation χ_H for the restricted character

$$\chi_H : H \longrightarrow \mathbb{C}, h \mapsto \chi(h).$$

Definition 2.2.1 Let φ be a class function of H, i.e. a function $\varphi : H \longrightarrow \mathbb{C}$ which is constant on each conjugacy class of H (confer [I, p.16

f.]). Define the function $\dot{\varphi} : G \longrightarrow \mathbb{C},\ g \mapsto \begin{cases} \varphi(g)\ , \text{ if } g \in H, \\ 0\ , \text{ otherwise.} \end{cases}$

Then φ^G, the **induced class function** on G, is defined as

$$\varphi^G(g) = \frac{1}{|H|} \sum_{x \in G} \dot{\varphi}\left(xgx^{-1}\right).$$

Remark 2.2.2 Note that for a set R of right coset representatives for G/H we can also write $\varphi^G(g) = \sum_{x \in R} \dot{\varphi}\left(xgx^{-1}\right).$

We now state a rather useful lemma which is known as **Frobenius Reciprocity**.

Lemma 2.2.3 (Frobenius Reciprocity)
Let φ be a class function on H and ϑ a class function of G. Then for the inner product we have the equality

$$(\varphi, \vartheta_H)_H = \left(\varphi^G, \vartheta\right)_G .$$

Proof [I, p.62, (5.2)].

Corollary 2.2.4 *If φ is a character of H, then φ^G is a character of G.*

Proof [I, p.63, (5.3)].

Remark 2.2.5 If φ is an irreducible character of H, then φ^G is **not** necessarily irreducible again.

We shall now state and prove two handy properties of induced character. The proofs will be rather technical, however they are a good exercise to get a better understanding of induced characters.

Lemma 2.2.6 *Let G be a finite group.*

(1) Let $U \leq H \leq G$ be a chain of subgroups of G, $\varphi \in \mathrm{Irr}_{\mathbb{C}}(U)$ a character of U. Then $(\varphi^H)^G = \varphi^G$.

(2) Let $H \leq G$ be a subgroup of G, φ be a character of H, ψ a character of G and ψ_H the restriction of ψ to H. Then $(\psi_H \cdot \varphi)^G = \psi \cdot (\varphi^G)$

Proof

(1) Let R be a set of coset representatives for G/H. Then

$$\begin{aligned}
\left(\varphi^H\right)^G(g) &= \sum_{x \in R} \dot{\varphi^H}\left(xgx^{-1}\right) \\
&= \sum_{\substack{x \in R,\\ xgx^{-1} \in H}} \varphi^H\left(xgx^{-1}\right) \\
&= \sum_{\substack{x \in R,\\ xgx^{-1} \in H}} \frac{1}{|U|} \sum_{y \in H} \dot{\varphi}\left(yxgx^{-1}y^{-1}\right) \\
&= \frac{1}{|U|} \sum_{\substack{x \in R, y \in H\\ xgx^{-1} \in H}} \dot{\varphi}\left((xy)g(xy)^{-1}\right) \\
&\overset{(*)}{=} \frac{1}{|U|} \sum_{\substack{z \in G,\\ zgz^{-1} \in H}} \dot{\varphi}\left(zgz^{-1}\right) \\
&= \frac{1}{|U|} \sum_{z \in G} \dot{\varphi}\left(zgz^{-1}\right) \\
&= \varphi^G(g).
\end{aligned}$$

Ad($*$): Since $H = yHy^{-1}$ for any $y \in H$ we have that $\{x \in R, y \in H \mid xgx^{-1} \in H\} = \{x \in R, y \in H \mid yxgx^{-1}y^{-1} \in H\}$. Now R is a set of coset representatives for G/H, hence $\{xy \mid x \in R, y \in H\} = G$ and the set above yields $\{z \in G \mid zgz^{-1} \in H\}$.

(2) Again we just use the definition of an induced character. We obtain for $g \in G$:

$$(\varphi \cdot \psi_H)(g) = \sum_{\substack{x \in G,\\ xgx^{-1} \in H}} (\varphi \cdot \psi)\left(xgx^{-1}\right) = \sum_{\substack{x \in G,\\ xgx^{-1} \in H}} \varphi\left(xgx^{-1}\right) \psi\left(xgx^{-1}\right).$$

However, ψ is a class function of G, hence constant on conjugacy classes. Therefore $\psi\left(xgx^{-1}\right) = \psi\left(g\right)$ for all $x \in G$ and we get

$$\sum_{\substack{x\in G,\\ xgx^{-1}\in H}} \varphi\left(xgx^{-1}\right)\psi\left(xgx^{-1}\right) = \psi\left(g\right)\cdot \underbrace{\sum_{\substack{x\in G,\\ xgx^{-1}\in H}} \varphi\left(xgx^{-1}\right)}_{\varphi^{G}(g)}.$$

We see that the second expression is just $\varphi^{G}\left(g\right)$, hence $\left(\varphi\cdot\psi_{H}\right)^{G}\left(g\right) = \left(\psi\cdot\varphi^{G}\right)\left(g\right)$ and the proof is complete.

The following note is a nice lemma which describes how to obtain the **kernel of an induced character**.

Lemma 2.2.7 *Let ϑ be a character of H, where $H \leq G$ is a subgroup of G. Then for the kernel of the induced character ϑ^{G} we have*

$$\ker\left(\vartheta^{G}\right) = \bigcap_{x\in G}\left(\ker\vartheta\right)^{x},$$

where S^{x} denotes the conjugate of a set S by the element x, i.e. $S^{x} = x^{-1}Sx = \{x^{-1}sx \mid s\in S\}$.

Proof Define $\chi := \vartheta^{G}$. By definition of the kernel of a character an element $g\in G$ is in the kernel of χ if and only if $\chi(g) = \chi(1)$, i.e.

$$\frac{1}{|H|}\sum_{x\in G}\dot{\vartheta}\left(xgx^{-1}\right) = \frac{1}{|H|}\sum_{x\in G}\vartheta\left(1\right).$$

Hence we have to investigate the condition $|\sum_{x\in G}\dot{\vartheta}\left(xgx^{-1}\right)| = \sum_{x\in G}\vartheta\left(1\right)$. However we know that that $\left|\dot{\vartheta}\left(xgx^{-1}\right)\right| \leq \vartheta\left(1\right)$ and we obtain $|\sum_{x\in G}\dot{\vartheta}\left(xgx^{-1}\right)| \leq \sum_{x\in G}|\dot{\vartheta}\left(xgx^{-1}\right)| \leq \sum_{x\in G}\vartheta\left(1\right)$. Therefore it follows that $g\in\ker\left(\chi\right)$ if and only if $|\dot{\vartheta}\left(xgx^{-1}\right)| = \vartheta\left(1\right)$ for all $x\in G$. This is the case if and only if $g\in\ker\left(\vartheta^{x}\right)$ for all $x\in G$ and the proof is complete.

Let us now assume that we have two subgroups $U, V \leq G$ and characters ϑ of U and ψ of V. We wonder what the two induced characters ϑ^{G}

and ψ^G have in common. Therefore we take a look at their inner product $\left(\psi^G, \vartheta^G\right)$. Using Frobenius reciprocity we obtain $\left(\psi^G, \vartheta^G\right)_G = \left(\left(\psi^G\right)_U, \vartheta\right)_U$. The following theorem, known as **Mackey's subgroup theorem**, is concerned with the expression $\left(\psi^G\right)_U$.
Before we state and prove the theorem we shall give some preliminary definitions and remarks.

Definition and Remark 2.2.8 Let $x \in G$ and $V \leq G$. We shall use the notation
${}^xV := xVx^{-1}$ and $V^x := x^{-1}Vx$. Now let ψ be a class function (or a character) of V and define $\psi^x : V^x \longrightarrow \mathbb{C},\ x^{-1}vx \mapsto \psi(v)$. It is easy to see that ψ^x is a class function (or a character) of V^x.

Theorem 2.2.9 (Mackey's Subgroup Theorem)
Let $U, V \leq G$ be subgroups of G and ψ be a class function of V. Let further T be a set of (V,U)-double coset representatives so that $G = \bigcup_{t \in T} VtU$ is a disjoint union. Then

$$\left(\psi^G\right)_U = \sum_{t \in T} \left(\left(\psi^t\right)_{V^t \cap U}\right).$$

Proof For each $t \in T$ choose a set R_t of left cosets representatives for ${}^tU \cap V$ in V, i.e. $V = \bigcup_{r \in R_t} r\left({}^tU \cap V\right)$ is a disjoint union. We have $|R_t| = |V|/|{}^tU \cap V| = |V|/|U \cap V^t|$.
Then we obtain $\{r \cdot t \mid r \in R_t, t \in T\} \subseteq G$ as a set of representatives for the left cosets of U in G, hence $G = \bigcup_{t \in T} \bigcup_{r \in R_t} rtU$, a disjoint union. This is because we have
$G = \bigcup_{t \in T} VtU$ and for a fixed $t \in T$ we have $VtU = \bigcup_{r \in R_t} rtU$.
In particular for every $x \in G$ there are uniquely determined elements $w \in U$, $t \in T$, $r \in R_t$ such that $x = rtw$.
Now define $\dot{\psi} : G \longrightarrow \mathbb{C},\ \dot{\psi}(x) = \begin{cases} \psi(x) & \text{, if } x \in V, \\ 0 & \text{, otherwise.} \end{cases}$

Let $u \in U$. Then we have

$$\begin{aligned}
\psi^G(u) &= \frac{1}{|V|}\sum_{x\in G}\dot{\psi}\left({}^x u\right) \\
&= \frac{1}{|V|}\sum_{t\in T}\sum_{r\in R_t}\sum_{w\in U}\dot{\psi}\left({}^{rtw}u\right) \\
&= \frac{1}{|V|}\sum_{t\in T}\sum_{r\in R_t}\sum_{w\in U}\dot{\psi}\left({}^{tw}u\right) \quad \left({}^{rtw}u \in V \text{ if and only if } {}^{tw}u \in V\right) \\
&= \frac{1}{|V|}\sum_{t\in T}|R_t|\sum_{w\in U}\dot{\psi}\left({}^{tw}u\right) \\
&= \sum_{t\in T}\frac{1}{|V^t\cap U|}\sum_{\substack{w\in U,\\ {}^{tw}u\in V}}\psi\left({}^{tw}u\right) \\
&= \sum_{t\in T}\frac{1}{|V^t\cap U|}\sum_{\substack{w\in U,\\ {}^{w}u\in V^t\cap U}}\psi\left({}^{tw}u\right) \\
&= \sum_{t\in T}\frac{1}{|V^t\cap U|}\sum_{\substack{w\in U,\\ {}^{w}u\in V^t\cap U}}\psi^t\left({}^{w}u\right) \\
&= \sum_{t\in T}\left({\psi^t}_{V^t\cap U}\right)^U(u)
\end{aligned}$$

and the proof is complete.

In this context there is another interesting theorem by Mackey which is known as **Mackey's tensor product theorem**. This theorem will also become important later and we will state it now.

Theorem 2.2.10 (Mackey's Tensor Product Theorem)
Let $H_1, H_2 \leq G$ two subgroups of G, ψ_1 a character of H_1 and ψ_2 be a character of H_2. Then the character $\psi_1^G \cdot \psi_2^G$ of G is given by

$$\psi_1^G \cdot \psi_2^G = \sum_{x^{-1}y\in D}\left[\left({\psi_1}^x\,{\psi_2}^y\right)_{{H_1}^x\cap {H_2}^y}\right]^G ,$$

where the sum is taken over the (H_1, H_2)-double cosets D in G. There is one summand for each D; namely we choose a pair $(x, y) \in G \times G$

with $x^{-1}y \in D$ and take the indicated summand. If also $u^{-1}v \in D$, then $\left[({\psi_1}^x\, {\psi_2}^y)_{{H_1}^x \cap {H_2}^y} \right]^G = \left[({\psi_1}^u\, {\psi_2}^v)_{{H_1}^u \cap {H_2}^v} \right]^G$

Proof [Curtis Reiner, p.242, (10.19)]

2.3 Normal subgroups

In this work we shall use some theorems whose origin lies in **Clifford theory**. Let us assume for this section that G is as usual a finite group and $N \trianglelefteq G$ a normal subgroup of G. We will now investigate the irreducible characters of N with respect to the irreducible characters of G. This will help us later to explicitly construct an irreducible tensor decomposable character of a group of order p^6.

Definition 2.3.1 Let $\varphi \in \mathrm{Irr}_{\mathbb{C}}(N)$ and take an element $x \in G$. Consider the map

$$\varphi^x : N \longrightarrow \mathbb{C}, \;\; n \mapsto \varphi(xnx^{-1})$$

We say φ^x is **conjugate** to φ in G.

Remark 2.3.2 It is easy to see that $\varphi^x \in \mathrm{Irr}_{\mathbb{C}}(N)$, if $\varphi \in \mathrm{Irr}_{\mathbb{C}}(N)$.

We will now state the theorem of Clifford.

Theorem 2.3.3 (Clifford) *Let $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ and $\vartheta \in \mathrm{Irr}_{\mathbb{C}}(N)$ and suppose that $(\chi_N, \vartheta) =: e \neq 0$, i.e. ϑ occurs as a constituent of the restricted character χ_N. Let further $\vartheta = \vartheta_1,\ \vartheta_2,\ \ldots\ ,\ \vartheta_t$ be all the distinct characters which are conjugate to ϑ in G. Then we have*

$$\chi_N := e \sum_{i=1}^{t} \vartheta_i$$

Proof [I, (6.2), p.79].

We have an action of G on $\mathrm{Irr}_{\mathbb{C}}(N)$ by conjugation. The stabilizer of an irreducible character $\vartheta \in \mathrm{Irr}_{\mathbb{C}}(N)$ under G is called the **inertia group** of ϑ in G.
More precisely this means the following:

Definition 2.3.4 Let $\vartheta \in \mathrm{Irr}_{\mathbb{C}}(N)$. The **inertia group** of ϑ in G is defined as $I_G(\vartheta) := \{g \in G \mid \vartheta^g = \vartheta\}$.

Lemma 2.3.5 *Let $\vartheta \in \mathrm{Irr}_{\mathbb{C}}(N)$. We then have $\vartheta^G \in \mathrm{Irr}_{\mathbb{C}}(G)$ if and only if $I_G(\vartheta) = N$.*

Proof Let us first assume that $\vartheta^G \in \mathrm{Irr}_{\mathbb{C}}(G)$. We have to show that $I_G(\vartheta) = N$. We have $\left(\vartheta^G, \vartheta^G\right) = 1$ since ϑ^G is irreducible. Applying Frobenius Reciprocity we obtain $\left(\vartheta, \left(\vartheta^G\right)_N\right) = \left(\vartheta^G, \vartheta^G\right) = 1$. We further have $\left(\vartheta^G\right)_N(1) = \vartheta^G(1) = [G : N]\,\vartheta(1)$. Applying Cliffords theorem with $e = 1$ we obtain $\left(\vartheta^G\right)_N(1) = \sum_{i=1}^{t} \vartheta_i(1) = t\vartheta(1)$, where $\vartheta_1, ..., \vartheta_t$ are all distinct conjugates of ϑ in G.
Hence $t = [G : N]$. On the other hand t, the length of the orbit of ϑ, equals the index of the stabilizer of ϑ in G, which is the inertia group $I_G(\vartheta)$. Thus we obtain $I_G(\vartheta) = N$.
For the other direction we assume $I_G(\vartheta) = N$. As above we have $t = [G : N]$. Let $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ with $(\vartheta^G, \chi) = e > 0$. By Frobenius Reciprocity $(\vartheta, \chi_N) = e$. By Cliffords theorem $\chi_N = e \sum i = 1^t \vartheta_i$, where $\vartheta_1, ..., \vartheta_t$ are the conjugates of ϑ in G. Hence $\vartheta(1)|G\colon N| \geq \chi(1) = et\vartheta(1) \geq |G\colon N|\vartheta(1)$. Thus $\chi(1) = |G\colon N|\vartheta(1)$, i.e. $\chi = \vartheta^G$.

Let us state another rather useful theorem.

Theorem 2.3.6 *Let $\vartheta \in \mathrm{Irr}_{\mathbb{C}}(N)$ and $T := I_G(\vartheta)$. Define*

$$\mathrm{Irr}_{\mathbb{C}}(G \mid \vartheta) := \{\chi \in \mathrm{Irr}_{\mathbb{C}}(G) \mid (\chi_N, \vartheta) \neq 0\}.$$

Then:

1. *If* $\psi \in \mathrm{Irr}_{\mathbb{C}}(T \mid \vartheta)$*, then* $\psi^G \in \mathrm{Irr}_{\mathbb{C}}(G)$*.*

2. *The map* $\mathrm{Irr}_{\mathbb{C}}(T \mid \vartheta) \longrightarrow \mathrm{Irr}_{\mathbb{C}}(G \mid \vartheta),\ \psi \mapsto \psi^G$ *is bijective.*

3. *If* $\psi \in \mathrm{Irr}_{\mathbb{C}}(T \mid \vartheta),\ \chi = \psi^G$*, then* $(\chi_N, \vartheta) = (\psi_N, \vartheta)$*.*

Proof [I, (6.11), p.82].

Later, in the context of tensor decomposable characters of p-groups, we shall use the notion of an M-group. We will now define what is meant by that and subsequently state some properties of M-groups.

Definition 2.3.7 A finite group G is said to be an **M-group**, if every character $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ is monomial, i.e. if there is a subgroup $H \leq G$ and a linear character $\lambda \in \mathrm{Irr}_{\mathbb{C}}(H)$ such that $\chi = \lambda^G$.

Corollary 2.3.8 *Every nilpotent group is an M-group.*

Proof [I, (6.14), p.83].

Corollary 2.3.9 *Let G be a p-group. Then G is a M-Group.*

Proof Since p-groups are nilpotent the claim follows from 2.3.8.

Another nice application of Clifford theory is Ito's theorem which we shall use later on and therefore state it here.

Theorem 2.3.10 (Ito)
Let A *be an abelian normal subgroup of* G*. Then* $\chi(1)$ *divides* $[G : A]$ *for all* $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$*.*

Proof [I, (6.15), p.84].

A rather useful lemma when it later comes to determine the character table of a certain group is the following:

Lemma 2.3.11 (Brauer's permutation lemma)
Let A be a group which acts on $\mathrm{Irr}_{\mathbb{C}}(G)$ and on the set of conjugacy classes of G. Assume that $\chi(g) = \chi^a(g^a)$ for all $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$, $a \in A$ and $g \in G$; where g^a is an element of $Cl(g)^a$. Then for each $a \in A$ the number of fixed irreducible characters of G is equal to the number of fixed classes.

Proof [I, (6.32), p.93].

2.4 Extendability of characters

Again we assume that G is a finite group and let $N \trianglelefteq G$ be a normal subgroup. There is a lot of theory about how to extend characters of normal subgroups to characters of the group itself under certain conditions. We will now state and prove some theorems regarding this topic which will be of great value later on.

Theorem 2.4.1 *Let $H \leq G$ be a subgroup of G such that G is the semi-direct product of H and N, i.e. $G = HN$ and $H \cap N = \{1\}$. Let further $\lambda : N \longrightarrow \mathbb{C}$ be a homomorphism such that $\lambda^g = \lambda$ for all $g \in G$, i.e. λ is a 1-dimensional G-invariant representation of N.*
Then λ is extendable to a representation of G, i.e. there is a 1-dimensional representation $\hat{\lambda} : G \longrightarrow \mathbb{C}$ such that $\hat{\lambda}_N = \lambda$.

Proof Let $g \in G$. Since G is the semidirect product of H and N there exist unique elements $h_g \in H$ and $n_g \in N$ such that $g = h_g n_g$. Now define a map

$$\hat{\lambda} : G \longrightarrow \mathbb{C}, \quad g \mapsto \lambda(n_g).$$

We only have to prove that $\hat{\lambda}$ is a group homomorphism. Let $x, y \in G$, $x = h_x n_x$, $y = h_y n_y$ with $h_x, h_y \in H$ and $n_x, n_y \in N$. Using the definition of $\hat{\lambda}$ and the fact that λ is invariant in G we obtain:

$$\hat{\lambda}(xy) = \hat{\lambda}(h_x n_x h_y n_y) = \hat{\lambda}\left(\underbrace{h_x h_y}_{\in H}\underbrace{{h_y}^{-1} n_x h_y n_y}_{\in N}\right) = \lambda({h_y}^{-1} n_x h_y n_y)$$
$$= \lambda({h_y}^{-1} n_x h_y)\lambda(n_y) = \lambda^{{h_y}^{-1}}(n_x)\lambda(n_y) = \lambda(n_x)\lambda(n_y) = \hat{\lambda}(n_x)\hat{\lambda}(n_y).$$

We have shown that $\hat{\lambda}$ is a homomorphism and the proof is complete.

Theorem 2.4.2 *Let $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ such that $\chi_N = \vartheta \in \mathrm{Irr}_{\mathbb{C}}(N)$. Then the characters $\beta \cdot \chi$ for $\beta \in \mathrm{Irr}_{\mathbb{C}}(G/N)$ are irreducible, distinct and are all constituents of ϑ^G.*

Proof See [I, p.85, (6.17)].

Theorem 2.4.3 *Let $[G : N]$ be a prime number and let $\vartheta \in \mathrm{Irr}_{\mathbb{C}}(N)$ be invariant in G. Then ϑ is extendable to G. Moreover there are exactly $[G : N]$ extensions of ϑ to an irreducible character of G, namely $\{\beta \cdot \chi \mid \beta \in \mathrm{Irr}_{\mathbb{C}}(G/N)\}$, where $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ is an arbitrary extension of ϑ.*

Proof The proof of the first assertion is given in [I, p.86, (6.20)]. Any extension $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ of ϑ is a constituent of ϑ^G. This can be easily seen by applying Frobenius reciprocity: $(\vartheta^G, \chi) = (\vartheta, \chi_N) = (\vartheta, \vartheta) = 1$. Hence the second assertion immediately follows from 2.4.2.

Chapter 3

Tensor decomposable Characters in p-Groups

In this chapter we will investigate p-groups with respect to the existence of tensor decomposable characters. That means we wonder which p-groups may possess irreducible characters which can be written as a product of two non-linear characters.
Let us first give a proper definition of a tensor decomposable character.

Definition 3.0.4 Let G be a finite group.
$\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ is called **tensor decomposable**, if there are characters ϕ, $\psi \in \mathrm{Irr}_{\mathbb{C}}(G)$, $\phi(1) > 1$, $\psi(1) > 1$, such that $\chi = \phi \cdot \psi$.

In the subsequent section we will investigate p-groups of order p^5.

3.1 Considering p-groups of order $\leq p^5$

Fix a prime p. Considering the fundamental formula

$$|G| = \sum_{\chi \in \mathrm{Irr}_{\mathbb{C}}(G)} \chi(1)^2 \tag{3.1}$$

it is obvious that no p-group of order less than or equal than p^4 can have a tensor decomposable character. Yet how is the situation for groups of order

p^5? In this section we shall see that these groups do not have such a character either. However the argument is rather not as easy as the one before.

Theorem 3.1.1 *Let G be a group of order p^5. Then G does not have a tensor decomposable character.*

Proof Suppose that G has a tensor decomposable character $\chi = \phi \cdot \psi$ with $\phi(1),\ \psi(1) > 1$. Let us first have a look at possible degrees of these characters. The order of G is p^5 and considering again formula (3.1) it follows that $\chi(1) \leq p^2$. We obtain $\chi(1)^2 = p^2$ and hence $\phi(1) = \psi(1) = p$.
Now we will use the fact that p-groups are M-groups (confer 2.3.9). By definition of an M-group there are subgroups $U, V \leq G$ and linear characters $\lambda \in \mathrm{Irr}_{\mathbb{C}}(U)$, $\mu \in \mathrm{Irr}_{\mathbb{C}}(V)$ such that $\phi = \lambda^G$ and $\psi = \mu^G$. We have $p = \lambda^G(1) = [G\colon U]\,\lambda(1) = [G\colon U]$ as well as $p = [G\colon V]$. Hence $|U| = |V| = p^5$. Using 2.2.6 we obtain $\phi \cdot \psi = \lambda^G \cdot \mu^G = \left\{ \begin{array}{c} \left(\left(\lambda^G\right)_V \cdot \mu\right)^G \\ \left(\lambda \cdot \left(\mu^G\right)_U\right)^G \end{array} \right\}$ $\in \mathrm{Irr}_{\mathbb{C}}(G)$. It follows that $\left(\lambda^G\right)_V \in \mathrm{Irr}_{\mathbb{C}}(V)$ and $\left(\mu^G\right)_U \in \mathrm{Irr}_{\mathbb{C}}(U)$. Since $\lambda^G(1) = \mu^G(1) = p$ we conclude that U and V are not abelian because all irreducible characters of abelian groups are of degree 1. This implies that the derived subgroups $U' \leq U$ and $V' \leq V$ are not trivial.
This fact will be useful in order to show that ϕ and ψ are not faithful, i.e. $\ker(\phi), \ker(\psi) \neq \{1\}$.
Since λ and μ are linear characters we obtain $U' \subseteq \ker(\lambda)$ and $V' \subseteq \ker(\mu)$. As U is a maximal subgroup of the p-group G we conclude that U is normal in G. Together with the fact that U' is characteristic in U it follows that U' is normal in G as well. By 2.2.7 we get $\ker(\phi) = \bigcap_{g \in G} \underbrace{(\ker \lambda)^g}_{\supseteq U'}$ and we conclude that $\ker(\phi) \supseteq U' \supsetneq \{1\}$. Analogously we conclude that $\ker(\psi) \supsetneq \{1\}$ $(*)$.
Furthermore we have that $\ker(\phi) \cap \ker(\psi) = \{1\}$ $(**)$, as otherwise $G/\left(\ker \phi \cap \ker \psi\right)$ would also have a tensor decomposable character. Yet as we already argued earlier a group of order $\leq p^4$ cannot have such a character.
As a last step in our preparation for the final contradicting argument we will analyse the center of G.
From [I, (2.28), p. 27] and [I, (2.30), p. 28] we obtain $p^4 = \chi(1)^2 \leq$

$[G : Z(\chi)] \leq [G : Z(G)]$. Since the center of a p-group always is non-trivial it follows that $Z(G) \neq \{1\}$ and we conclude that $|Z(G)| = p$.
Now we put everything together:
For all non-trivial normal subgroups N of a p-group G it holds that $N \cap Z(G) \neq \{1\}$. Hence, using $(*)$, we deduce that $\ker(\phi) \cap Z(G) \gneqq \{1\}$ and $\ker(\psi) \cap Z(G) \gneqq \{1\}$. Since $|Z(G)| = p$ we finally obtain $Z(G) \leq \ker(\phi)$ as well as $Z(G) \leq \ker(\psi)$ and therefore $Z(G) \leq \ker(\phi) \cap \ker(\psi)$, which by $(**)$ is the trivial group.
Hence we derive a contradiction and the proof is complete.

Remark 3.1.2 Considering 3.1.1 and in particular considering the proof we can already make some remarks about groups of order p^6 which possess a tensor decomposable character $\chi = \phi \cdot \psi$ with $\phi(1), \psi(1) > 1$. Reasoning just as above but for groups of order p^6 we see that $\ker(\phi) \gneqq \{1\}$, $\ker(\psi) \gneqq \{1\}$ and $\ker(\phi) \cap \ker(\psi) = \{1\}$. Since $\{1\} \lneqq Z(G) \cap \ker(\phi)$ and $\{1\} \lneqq Z(G) \cap \ker(\psi)$ we conclude that $Z(G)$ has two nontrivial subgroups which have trivial intersection. By [I, (2.30), p. 28] we obtain $|Z(G)| \leq p^2$ and hence $Z(G) \cong C_p \times C_p$. These facts will be of great value in the following section when it comes to investigate groups of order p^6 with respect to the existence of tensor decomposable characters.

3.2 Groups of order p^6

After experiments using the computer algebra system GAP (confer Chapter 5) we saw that for any prime number $p \leq 11$ there are groups of order p^6 which possess a tensor decomposable character. The final aim for this section is to prove that in general for any prime number p there is such a group. This will be done by giving a precise construction of an suitable group together with a subsequent construction of a tensor decomposable character of this group. Further experiments with GAP happened to be rather useful in this context. In this section we will first analyse the structure of groups of order p^6 which have a tensor decomposable character. The intention is to obtain as much structural information as possible in order to be able to construct a 'nice' group with the required properties, i.e. a group of order p^6 possessing a tensor decomposable character.

Obviously we are looking for a non-trivial example of a group of order p^6 with a tensor decomposable character. What we mean by a trivial example we will explain in the following definition and the subsequent remark and corollary.

Definition 3.2.1 We call a character $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ **trivially tensor decomposable**, if there exists a normal subgroup $N \trianglelefteq G$ such that $\chi_N = e\vartheta$ for some $\vartheta \in \mathrm{Irr}_{\mathbb{C}}(N)$, $e \in \mathbb{N}$ with $\vartheta(1) > 1$ and $e > 1$.

At first sight we probably do not see in which way the above definition is related to tensor decomposable characters. However this will become clear with the following remark and corollary.

Remark 3.2.2 If $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ is trivially tensor decomposable and N, e, ϑ are as in 3.2.1, then there exist irreducible projective characters (i.e. characters of an irreducible projective representation in the sense of [I, p.174, (11.1)]) $\hat{\vartheta}$ of G and ε of G/N with $\hat{\vartheta}_N = \vartheta$ and $\chi = \hat{\vartheta} \cdot \varepsilon$.

Proof [H, (21.2)]

There are two special cases of trivially tensor decomposable characters $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ which we will discuss now.

Corollary 3.2.3 *(i) If $\chi = \hat{\vartheta} \cdot \varepsilon$ is an irreducible character of G with $\hat{\vartheta}, \varepsilon \in \mathrm{Irr}_{\mathbb{C}}(G)$, where $\hat{\vartheta}$ is an extension of an irreducible G-invariant character $\vartheta \in \mathrm{Irr}_{\mathbb{C}}(N)$ of a normal subgroup $N \trianglelefteq G$ and ε is considered as the inflation of an irreducible character $\varepsilon \in \mathrm{Irr}_{\mathbb{C}}(G/N)$, then χ is trivially tensor decomposable.*

(ii) If G is the direct product of two non-abelian groups, then G has a trivially tensor decomposable character.

Proof (i) An irreducible character of G is in particular a projective character of G. The claim then immediately follows from 3.2.2.

(ii) This case is again a special case of (i). Suppose $G = H \times L$, with H, L non-abelian and set $N = H$; then $G/N \cong L$. Now consider irreducible characters ϑ of H and ε of L which both have degree larger than 1 (these exist since H and L are non-abelian). Further let $\hat{\vartheta}$ be the lift of ϑ to G, consider ε as the inflation of ε to G and define $\chi = \hat{\vartheta} \cdot \varepsilon$. We easily see that $\hat{\vartheta}$ and ε satisfy the assumptions from case (i). From 2.1.5 it now immediately follows that χ is an irreducible character of G.

From the previous section plus additional general theory we can already gather a lot of information which will be of great value in order to reach our aim. We obtain the following theorem:

Theorem 3.2.4 *Let G be a group of order p^6 which has a tensor decomposable character. With $Z(G)$ we denote the center of G, with G' its derived subgroup.*
Then the following properties hold:

(1) There exist subgroups U, $V \leq G$, $|U| = |V| = p^5$ and irreducible linear characters $\lambda \in \mathrm{Irr}_{\mathbb{C}}(U)$, $\mu \in \mathrm{Irr}_{\mathbb{C}}(V)$ such that $\lambda^G = \phi$, $\mu^G = \psi$.

(2) $U \cdot V = G$.

(3) U and V are not abelian.

(4) $U' \cap V' = \{1\}$.

(5) $U \cap V$ is abelian.

(6) $G/U \cap V$ is elementary abelian of order p^2.

(7) $G' \subset U \cap V$.

(8) G' is abelian.

(9) $Z(G)$ is elementary abelian of order p^2.

(10) $Z(G) \subseteq U \cap V$.

Proof Let $\chi = \psi \cdot \phi$ be a tensor decomposable character of G with $\phi(1), \psi(1) > 1$. Since $|G| = p^6$ it follows from 3.1 that $\chi(1) = p^2$ and $\phi(1) = \psi(1) = p$.

(1) By 2.3.9 we have that G is an M-group and hence ϕ and ψ are induced from linear characters of subgroups of G, say ϕ is induced from a subgroup $U \leq G$ and ψ from a subgroup $V \leq G$. Since $\phi(1) = \psi(1) = p$ the orders of U and V must be p^5.

(2) Let T be a set of representatives for the (U, V)-double cosets in G so that $G = \bigcup_{t \in T} UtV$ is a disjoint union. Using 2.2.10 we obtain $\chi = \phi \cdot \psi = \lambda^G \cdot \mu^G = \sum_{t \in T} (\lambda^t_{U^t \cap V} \cdot \ \mu_{U^t \cap V})^G \in \mathrm{Irr}_{\mathbb{C}}(G)$. Thus $U \cdot V = G$, because otherwise $|T| \gneq 1$ and χ would be a sum of at least two characters of G. Since χ is irreducible this is not possible.

(3) By 2.2.6 we have $\phi \cdot \psi = \lambda^G \cdot \mu^G = \left\{ \begin{array}{l} ((\lambda^G)_V \cdot \mu)^G \\ (\lambda \cdot (\mu^G)_U)^G \end{array} \right\} \in \mathrm{Irr}_{\mathbb{C}}(G)$. Hence $(\lambda^G)_V \in \mathrm{Irr}_{\mathbb{C}}(V)$ and $(\mu^G)_U \in \mathrm{Irr}_{\mathbb{C}}(U)$. Since $\lambda^G(1) = \mu^G(1) = p$ we conclude that U and V are not abelian.

(4) Considering 3.1.2 together with 2.2.7 we obtain $\{1\} = \ker(\phi) \cap \ker(\psi) = \ker(\lambda^G) \cap \ker(\mu^G) \supseteq U' \cap V'$. Therefore $U' \cap V' = \{1\}$.

(5) We have $(U \cap V)' \subseteq U' \cap V' \overset{(4)}{=} \{1\}$, i.e. $U \cap V$ is abelian.

(6) Define $W := U \cap V$. Using the second isomorphism theorem we conclude that $|W| = p^4$. As maximal subgroups of a p-group, U and V are normal in G. Therefore W is also normal in G and we can consider G/W. This is a group of order p^2 and hence abelian. Therefore G/W is elementary abelian or cyclic. Suppose that G/W is cyclic. Then G/W contains exactly one subgroup of order p and G contains exactly one

subgroup of order p^5 containing W. Yet by (2) we have that $U \cdot V = G$, hence $U \neq V$. Further it holds that $W \leq U$ and $W \leq V$ which contradicts the fact that G has only one subgroup of order p^5. Thus G/W cannot be cyclic and hence must be elementary abelian.

(7) Follows from (6) since G' is the smallest normal subgroup N of G such that G/N is abelian.

(8) Follows from (5) and (7).

(9) Confer 3.1.2 .

(10) Suppose that $Z(G) \nsubseteq U \cap V$. Then $H := \langle U \cap V, Z(G) \rangle \gneqq U \cap V$ is an abelian normal subgroup of G. We know that G is not abelian, hence $H \neq G$ and thus $U \cap V \lneqq H \lneqq G$. Since $|U \cap V| = p^4$ and $|G| = p^6$ we conclude that $|H| = p^5$. Applying 2.3.10, Itos theorem, every irreducible character of G would have degree $\leq p$. Yet this contradicts the assumption that χ is an irreducible tensor decomposable character of G and the claim follows.

Our next aim is to use all the information gathered here to construct a group of order p^6 which possesses a tensor decomposable character. We shall see the result in the following section.

3.3 Existence of indecomposable groups of order p^6 possessing tensor decomposable characters

As we see in Chapter 5 experiments with the computer algebra system GAP showed that there actually are groups of order p^6 possessing tensor decomposable characters for all prime numbers p we worked with (which were all prime numbers ≤ 11). This led us to conjecture that for an arbitrary prime number p there is a group with a tensor decomposable character which is not trivially tensor decomposable in the meaning of 3.2.1. But the question now is how to find a general group with the required properties? One possible approach is to work with so called power commutator presentations.

In order to construct a group of order p^6 which has a tensor decomposable character we know from 3.2.4 that we need an elementary abelian center of order p^2 and an abelian derived subgroup. Further experiments with GAP showed that for $p \in \{5, 7, 11\}$ all groups G of order p^6 with $Z(G) \cong C_p \times C_p$ and derived subgroup $G' \cong C_p \times C_p \times C_p \times C_p$ have such a character. This led us to conjecture that all groups with these properties have a tensor decomposable character. Yet solving this problem seemed to be anything but easy. However we wondered whether it might be possible to find a presentation for a group of order p^6 possessing a tensor decomposable character. The construction was aimed at providing the group with all the properties mentioned before. One nice side effect of choosing G' to be elementary abelian was to then obtain an action of G/G' on G'. Hence we obtain a homomorphism $G/G' \longrightarrow Aut(G') \cong Gl_4(\mathbb{F}_p)$. All these facts helped a lot to find a group we were looking for.

The subsequent presentation I finally found turned out to be a nice one which in addition satisfies all our demands. We will now present this group in order to see a generic example of a group of order p^6 possessing a tensor decomposable character.

Theorem 3.3.1 *Let $p \in \mathbb{P}$ be a prime, $p \neq 2, 3$. Define*

$$G := \langle a, b, c, d, x, y \mid a^p, b^p, c^p, d^p, x^p, y^p, [a,b], [a,c], [a,d], [b,c], [b,d], [c,d], \\ [a,x]\, b^{-1}, [a,y]\, d^{-1}, [b,x]\, c^{-1}, [b,y], [c,x], [c,y], [d,x], [d,y], [x,y]\, a^{-1} \rangle.$$

Then G is a group of order p^6 and has a tensor decomposable character, which is not a trivially tensor decomposable character in the meaning of 3.2.1. The derived subgroup G' is elementary abelian of order p^4.

Proof First of all we have to prove that the given presentation actually yields a group of order p^6. After being finished with this we show that G has a tensor decomposable character. This will be done by explicit contruction of such a character.
In order to prove the claim we shall pursue the following steps:

(1) Define a group $\widetilde{G}$ using the semidirect product and show that this group is isomorphic to G. We then shall easily see that G is a group of order p^6.

(2) Consider two particular linear characters $\lambda, \mu \in \mathrm{Irr}_{\mathbb{C}}(G')$ which have pairwise different inertia groups in G of order p^5. Denote these subgroups by U and V respectively.

(3) Extend λ to a character $\hat{\lambda} \in \mathrm{Irr}_{\mathbb{C}}(U)$ and analogously μ to $\hat{\mu} \in \mathrm{Irr}_{\mathbb{C}}(V)$.

(4) Show that $\hat{\lambda}^G \cdot \hat{\mu}^G \in \mathrm{Irr}_{\mathbb{C}}(G)$. This will be a tensor decomposable character of G.

Ad (1) Let $\langle \tilde{x} \rangle$ and $\langle \tilde{y} \rangle$ be cyclic groups of order p. Furthermore let $\langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \rangle$ be an elementary abelian group of order p^4, i.e. $\left\{ \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \right\}$ is a generating set of elements of order p which commute pairwise. Now using the semidirect product define a group $\widetilde{G}$ as follows:

$$\widetilde{G} := \langle \tilde{y} \rangle \ltimes_{\psi} \langle \tilde{x} \rangle \ltimes_{\phi} \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \rangle,$$

where ϕ and ψ are the homomorphims defined as follows:

$$\phi : \langle \tilde{x} \rangle \longrightarrow Aut\left(\langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \rangle\right) (\cong GL_4(\mathbb{F}_p))$$

$$\tilde{x} \longmapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ i.e. } \begin{pmatrix} \tilde{a} & \mapsto & \tilde{a}\tilde{b} \\ \tilde{b} & \mapsto & \tilde{b}\tilde{c} \\ \tilde{c} & \mapsto & \tilde{c} \\ \tilde{d} & \mapsto & \tilde{d} \end{pmatrix}$$

and

$$\psi : \langle \tilde{y} \rangle \longrightarrow Aut\left(\langle \tilde{x} \rangle \ltimes_\phi \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \rangle\right).$$

$$\tilde{y} \longmapsto \begin{pmatrix} \tilde{a} & \mapsto & \tilde{a}\tilde{d} \\ \tilde{b} & \mapsto & \tilde{b} \\ \tilde{c} & \mapsto & \tilde{c} \\ \tilde{d} & \mapsto & \tilde{d} \\ \tilde{x} & \mapsto & \tilde{x}\tilde{a} \end{pmatrix}$$

Before proving the isomorphism $G \cong \widetilde{G}$ it first remains to show that $\widetilde{G}$ is well defined. In order to do so we have to show that both ϕ and ψ are well defined group homomorphisms.

(Ad ϕ) The matrix $M_x := \phi(\tilde{x}) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ has determinant 1, hence $M_x \in GL_4(\mathbb{F}_p)$ and $\phi : \langle \tilde{x} \rangle \longrightarrow GL_4(\mathbb{F}_p)$ is a well defined map.
To prove that ϕ is a group homomorphism we use 1.2.10.
But $\langle \tilde{x} \rangle \cong \langle g \mid g^p \rangle$. The claim is proven if $\phi(\tilde{x})^p = M_x^p = E_4$.
By induction (see end of the proof for explicit details of the induc-

tion) it is easy to show that $M_x^i = \begin{pmatrix} 1 & i & \frac{i(i-1)}{2} & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ for all $i \in \mathbb{N}$.
We assumed that $p \neq 2, 3$, hence $2 \mid p-1$ and $\frac{p(p-1)}{2} \in p\mathbb{N}$. We obtain ${M_x}^p = \begin{pmatrix} 1 & p & \frac{p(p-1)}{2} & 0 \\ 0 & 1 & p & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_4$, and the claim follows.

(Ad ψ) Showing that ψ is a well defined group homomorphism is a little more complicated than to show this for ϕ in the previous case. However the idea remains very similar.
Define $\widetilde{U} := \langle \tilde{x} \rangle \ltimes_\phi \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \rangle$. By 1.2.16 we obtain the following presentation of $\widetilde{U}$:

$$\begin{aligned} \widetilde{U} \;\cong\; \widehat{U} := \langle\, & \hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{x} \mid \hat{a}^p, \hat{b}^p, \hat{c}^p, \hat{d}^p, \hat{x}^p,\ [\hat{a}, \hat{b}],\ [\hat{a}, \hat{c}],\ [\hat{a}, \hat{d}],\ [\hat{b}, \hat{c}], \\ & [\hat{b}, \hat{d}],\ [\hat{c}, \hat{d}],\ [\hat{x}, \hat{a}^{-1}]\hat{b}^{-1},\ [\hat{x}, \hat{b}^{-1}]\hat{c}^{-1},\ [\hat{x}, \hat{c}^{-1}],\ [\hat{x}, \hat{d}^{-1}] \,\rangle. \end{aligned}$$

In order to see that ψ is well defined, i.e. $\psi(\tilde{y}) \in Aut(\widetilde{U})$, we again use 1.2.10. Let F be the free group on the generating set $\left\{\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{x}\right\}$ and define a homomorphism $\tau : F \longrightarrow \widetilde{U}$ which maps $\hat{a}$ to $\tilde{a}\tilde{d}$, $\hat{b}$ to $\tilde{b}$, $\hat{c}$ to $\tilde{c}$, $\hat{d}$ to $\tilde{d}$ and $\hat{x}$ to $\tilde{x}\tilde{a}$. We now check if all relations of $\widehat{U}$ belong to the kernel of τ so that we obtain a homomorphism $\bar{\tau} : \widehat{U} \longrightarrow \widetilde{U}$ as in 1.2.10. After that we shall prove that $\bar{\tau} : \widehat{U} \longrightarrow \widetilde{U}$ is bijective and hence an isomorphism. Thus we obtain a diagram of isomorphisms

$$\begin{tikzcd} \widetilde{U} \arrow[r, "\cong"] \arrow[dr, "\psi(\tilde{y})"'] & \widehat{U} \arrow[d, "\bar{\tau}"] \\ & \widetilde{U} \end{tikzcd}$$

Clearly it then follows that $\psi(\tilde{y}) \in Aut(\widetilde{U})$.

We have $\tau(\hat{a}^p) = (\tilde{a}\tilde{d})^p = \tilde{a}^p\tilde{d}^p = 1$. Analogoulsy we have $\tau(\hat{b}^p) = \tau(\hat{c}^p) = \tau(\hat{d}^p) = 1$.
Furthermore $\tau([\hat{a}, \hat{b}]) = [\tilde{a}\tilde{d}, \tilde{b}] = 1$, as well as $\tau([\hat{a}, \hat{c}]) = \tau([\hat{a}, \hat{d}]) = \tau([\hat{b}, \hat{c}]) = \tau([\hat{b}, \hat{d}]) = \tau([\hat{c}, \hat{d}]) = 1$. Next we have $\tau([\hat{x}, \hat{a}^{-1}]\hat{b}^{-1}) = (\tilde{x}\tilde{a})^{-1}(\tilde{a}\tilde{d})(\tilde{x}\tilde{a})(\tilde{a}\tilde{d})^{-1}\tilde{b}^{-1} = \tilde{a}^{-1}\tilde{x}^{-1}\tilde{a}\tilde{x}\tilde{b}^{-1} = [\tilde{a}, \tilde{x}]\tilde{b}^{-1} = 1$.
And we have $\tau([\hat{x}, \hat{b}^{-1}]\hat{c}^{-1}) = (\tilde{x}\tilde{a})^{-1}\tilde{b}(\tilde{x}\tilde{a})\tilde{b}^{-1}\tilde{c}^{-1} =$
$\tilde{a}^{-1}\underbrace{\tilde{x}^{-1}\tilde{b}\tilde{x}}_{\tilde{b}\tilde{c}}\tilde{b}^{-1}\tilde{a}\tilde{c}^{-1} = 1$.
Analogously we obtain $\tau([\hat{c}, \hat{x}]) = \tau([\hat{d}, \hat{x}]) = 1$.
It remains to show that $\tau(\hat{x}^p) = 1$. Induction (see end of the proof for explicit details of the induction) yields
$\tau(\hat{x}^i) = (\tilde{x}\tilde{a})^i = \tilde{x}^i\tilde{a}^i\tilde{b}^{\frac{i(i-1)}{2}}\tilde{c}^{\frac{i(i-1)(i-2)}{6}}$ for all i $\in \mathbb{N}, i \geq 2$. Hence $\tau(\hat{x}^p) = (\tilde{x}\tilde{a})^p = \tilde{x}^p\tilde{a}^p\tilde{b}^{\frac{p(p-1)}{2}}\tilde{c}^{\frac{p(p-1)(p-2)}{6}} = 1$, since $p \notin 2, 3$ and therefore $\frac{p-1}{2}, \frac{(p-1)(p-2)}{6} \in \mathbb{N}$.
Hence we have shown that all relations of $\widehat{U}$ are in $\ker(\tau)$ and we obtain a homomorphism $\bar{\tau} : \widehat{U} \longrightarrow \widetilde{U}$.
Since $\widetilde{U}$ is generated by $\left\{\tau(\hat{a}), \tau(\hat{b}), \tau(\hat{c}), \tau(\hat{d}), \tau(\hat{x})\right\}$ we conclude $\bar{\tau}$ is surjective. Both $\widehat{U}$ and $\widetilde{U}$ are groups of order p^5. Hence $\bar{\tau}$ must be bijective.
All in all we now have shown that $\bar{\tau}$ is an isomorphism from $\widehat{U}$ to $\widetilde{U}$ and we conclude that $\psi(\tilde{y}) \in Aut(\widetilde{U})$.
It remains to prove that ψ is a homomorphism. Again we will use 1.2.10 together with the presentation $\langle x \mid x^p \rangle$ for C_p. Thus it suffices to show that $\psi(\tilde{y})^p = id_{\widetilde{U}}$.
Clearly $\psi(\tilde{y})^p(\tilde{b}) = \tilde{b}$, $\psi(\tilde{y})^p(\tilde{c}) = \tilde{c}$ and $\psi(\tilde{y})^p(\tilde{d}) = \tilde{d}$.
Induction (see end of the proof for explicit details of the induction) leads to the following results:
$\psi(\tilde{y})^i(\tilde{a}) = \tilde{a}\tilde{d}^i$ for all $i \in \mathbb{N}, i \geq 2$ and $\psi(\tilde{y})^i(\tilde{x}) = \tilde{x}\tilde{a}^i\tilde{d}^{\frac{i(i-1)}{2}}$ for all $i \in \mathbb{N}$. Hence $\psi(\tilde{y})^p(\tilde{a}) = \tilde{a}\tilde{d}^p = \tilde{a}$ and $\psi(\tilde{y})^p(\tilde{x}) = \tilde{x}\tilde{a}^p\tilde{d}^{\frac{p(p-1)}{2}} = \tilde{x}$.
This means that $\psi(\tilde{y})^p$ is the identity on a generating set of $\widetilde{U}$ and therefore the identity map on $\widetilde{U}$, i.e., $\psi(\tilde{y})^p = id_{\widetilde{U}}$ and we conclude that ψ is a homomorphism.

Summarizing all, we now know that $\widetilde{G}$ is a well defined group. Since for a semidirect product we have $|H \ltimes N| = |H| \cdot |N|$ we easily conclude that $\left|\widetilde{G}\right| = p^6$.

Finally we now come to prove that $G \cong \widetilde{G}$. Again we shall use 1.2.10. Consider the map $f : \{a, b, c, d, x, y\} \longrightarrow \widetilde{G}$ where $f(a) = \tilde{a}, f(b) = \tilde{b}, f(c) = \tilde{c}, f(d) = \tilde{d}, f(x) = \tilde{x}$ and $f(y) = \tilde{y}$. Now extend f to a homomorphism ρ on the free group $F = F_{\{a,b,c,d,x,y\}}$, i.e. $\rho : F \longrightarrow \widetilde{G}$. The question now is whether we can turn ρ into a homomorphism on $G \cong F/\langle\langle a^p, b^p, c^p, d^p, x^p, y^p,\ [a, b]\,,\ [a, c]\,,\ [a, d]\,,\ [b, c]\,,\ [b, d]\,,\ [c, d]\,,$
$[a, x]\, b^{-1},\ [a, y]\, d^{-1},\ [b, x]\, c^{-1},\ [b, y]\,, [c, x]\,, [c, y]\,, [d, x]\,, [d, y]\,, [x, y]\, a^{-1}\rangle\rangle$.
This means we again have to check whether $\rho(r) = 1$ for all relations r of G. However this is just a very easy check which does not require any tricks and will therefore be omitted here.
Since $\widetilde{G}$ is generated by $\rho(\{a, b, c, d, x, y\})$ we conclude that there exists a surjective homomorphism $\hat{\rho} : G \longrightarrow \widetilde{G}$ where $\hat{\rho}(a) = \tilde{a}$, ... and $\hat{\rho}(y) = \tilde{y}$.
In order to prove that $\hat{\rho}$ is bijective we take a look at the order of G. We have the equality $|G| = [G : G'] \cdot |G'|$ and will now investigate the order of G and the order of G/G'.
We claim that $G' = \langle a, b, c, d\rangle$.
Considering the relations of G it is obvious that $\langle a, b, c, d\rangle \subseteq G'$. To see the other inclusion we observe that $\langle a, b, c, d\rangle \trianglelefteq G$ and that the factor group $G/\langle a, b, c, d\rangle$ is abelian since the generators of $G/\langle a, b, c, d\rangle$ commute.
Hence our claim is proven and we easily conclude that $|G'| \leqslant p^4$. We cannot say yet whether this actually is an equality since it still is possible that one of a, b, c or d or a product of these is equal to 1. However the above inequality will be sufficient in order to show that $\hat{\rho}$ is bijective. Now how does G/G' look like? Using 1.2.13 and simplifying the relations we obtain $G/G' \cong \langle x, y \mid x^p = y^p = 1, [x, y] = 1\rangle$ which, by 1.2.15, is isomorphic to $C_p \times C_p$.
Thus $|G/G'| = p^2$.

Now we put the information above together and obtain
$|G| = [G : G'] \cdot |G'| \leq p^2 \cdot p^4 = p^6$.
Since $\hat{\rho}$ is surjective and $|\widetilde{G}| = p^6$ it follows that $|G/\ker(\hat{\rho})| = p^6$. Hence $|G| \geqslant p^6$.

Together we conclude that $|G| = p^6$ and $\ker(\hat{\rho}) = \{1\}$. Therefore $\hat{\rho}$ is an isomorphism and claim (1) is proven.

Ad (2) In order to obtain a tensor decomposable character of G we first take a look at two linear characters of $G' = \langle a, b, c, d \rangle$.
Let $\zeta \in \mathbb{C}$ be a root of unity of order p and define two linear characters λ and μ on G' by defining them on the generators a, b, c and d:

$$\begin{array}{llll} \lambda(a) = 1 & \lambda(b) = 1 & \lambda(c) = 1 & \lambda(d) = \zeta, \\ \mu(a) = 1 & \mu(b) = 1 & \mu(c) = \zeta & \mu(d) = 1. \end{array}$$

We know that G' is elementary abelian and by 1.2.10 λ and μ clearly are homomorphisms from G to $\mathbb{C}$, i.e. linear characters of G.
Now we wonder about the inertia groups $T_G(\lambda)$ and $T_G(\mu)$ of λ and μ. We claim the following:

(a) $U := \langle G', x \rangle = T_G(\lambda)$.

(b) $V := \langle G', y \rangle = T_G(\mu)$.

We shall prove this claim now.

(a) We have $\lambda^x = \lambda$, since
$\lambda^{x^{-1}}(a) = \lambda(x^{-1}ax) = \lambda(ab) = \lambda(a)\lambda(b) = 1 = \lambda(a)$
$\lambda^{x^{-1}}(b) = \lambda(x^{-1}bx) = \lambda(bc) = \lambda(b)\lambda(c) = 1 = \lambda(b)$
$\lambda^{x^{-1}}(c) = \lambda(x^{-1}cx) = \lambda(c)$
$\lambda^{x^{-1}}(d) = \lambda(x^{-1}dx) = \lambda(d)$,
i.e. $\lambda^{x^{-1}} = \lambda$ on a generating set of G'. Thus $\lambda^{x^{-1}} = \lambda$ for all elements of G' and we conclude that $x^{-1} \in T_G(\lambda)$ and hence $x \in T_G(\lambda)$. Therefore $G' \lneq \langle G', x \rangle \leq T_G(\lambda)$. We further notice

that $\lambda^{y^{-1}} \neq \lambda$ because for instance $\lambda^{y^{-1}}(a) = \lambda(y^{-1}ay) = \lambda(ad) = \zeta \neq 1 = \lambda(a)$. This means $y^{-1} \notin T_G(\lambda)$ and we obtain $G' \lneq T_G(\lambda) \lneq G$. Since $[G : G'] = p^2$ we conclude that $[G : T_G(\lambda)] = p$ and $[T_G(\lambda) : G'] = p$. Hence $T_G(\lambda) = \langle G', x\rangle = U$.

(b) This proof is analogous to case (a).
We have $\mu^{y^{-1}} = \mu$, since
$\mu^{y^{-1}}(a) = \mu(y^{-1}ay) = \mu(ab) = \mu(a)\mu(b) = 1 = \mu(a)$
$\mu^{y^{-1}}(b) = \mu(y^{-1}by) = \mu(b)$
$\mu^{y^{-1}}(c) = \mu(y^{-1}cy) = \mu(c)$
$\mu^{y^{-1}}(d) = \mu(y^{-1}dy) = \mu(d)$,
i.e. $\mu^{y^{-1}} = \mu$ on a generating set of G'. Thus $\mu^{y^{-1}} = \mu$ for all elements of G' and we conclude that $y^{-1} \in T_G(\lambda)$ and hence $y \in T_G(\lambda)$. Therefore $G' \lneq \langle G', y\rangle \leq T_G(\mu)$. We further notice $\mu^{x^{-1}} \neq \mu$, because for instance $\mu^{x^{-1}}(b) = \mu(x^{-1}bx) = \mu(bc) = \mu(b)\mu(c) = \zeta \neq 1 = \mu(b)$. This means $x^{-1} \notin T_G(\mu)$ and we get $G \lneq T_G(\mu) \lneq G$. Hence we conclude that $T_G(\mu) = \langle G', y\rangle = V$.

From the foregoing proofs it is easy to see that both inertia groups U and V have order p^5.

Ad (3) We have $[U : G'] = [V : G'] = p$ and furthermore we know that λ is invariant in U and as well μ is invariant in V. Now from part (1) we deduce that U is the semidirect product of G' with $\langle x\rangle$ as well as V is the semidirect product of G' with $\langle y\rangle$. From 2.4.1 we now conclude that we can extend λ and μ to a linears character on U and V respectively. Let us denote these extensions by $\hat{\lambda}$ and $\hat{\mu}$ respectively.
Obviously $\hat{\lambda} \in \mathrm{Irr}_{\mathbb{C}}(U)$ and $\hat{\mu} \in \mathrm{Irr}_{\mathbb{C}}(V)$ respectively since these are still linear characters.

Ad (4) Finally it remains to show that $\hat{\lambda}^G \cdot \hat{\mu}^G \in \mathrm{Irr}_{\mathbb{C}}(G)$.
Therefore we first observe that $T_G(\hat{\lambda}) = U$. This is since $\lambda^y \neq \lambda$ (confer part (a) from above) and therefore the inertia group $T_G(\hat{\lambda}) \lneq G$ and must hence be U.
Analogously we have $T_G(\hat{\mu}) = V$.

From 2.3.5 it follows that $\hat{\lambda}^G$ and $\hat{\mu}^G$ are irreducible characters of G. Applying 2.2.6 we conclude $\hat{\lambda}^G \cdot \hat{\mu}^G = \left((\hat{\lambda}^G)_V \cdot \hat{\mu} \right)^G$. Using 2.3.5 we now obtain the equivalence $\hat{\lambda}^G \cdot \hat{\mu}^G \in \mathrm{Irr}_{\mathbb{C}}(G)$ if and only if $T_G\left((\hat{\lambda}^G)_V \cdot \hat{\mu} \right) = V$. Therefore it is reasonable to investigate the inertia group $T := T_G\left((\hat{\lambda}^G)_V \cdot \hat{\mu} \right)$.

We have $(\hat{\lambda}^G)_V \in \mathrm{Irr}_{\mathbb{C}}(V)$ because using 2.2.10, Mackeys theorem, together with the fact that $U \cdot V = G$ we obtain $(\hat{\lambda}^G)_V = \left(\hat{\lambda}_{U \cap V} \right)^V = \lambda^V$. Let us again use that $\lambda^y \neq \lambda$, hence $T_V(\lambda) = G'$ and therefore $\lambda^V \in \mathrm{Irr}_{\mathbb{C}}(V)$.

Obviously $V \subseteq T$. However we are able to show that $T \neq G$ because x^{-1} does not stabilize $(\hat{\lambda}^G)_V \cdot \hat{\mu}$ as we see in the following:

$$\left((\hat{\lambda}^G)_V \cdot \hat{\mu} \right)^{x^{-1}} (b) = \hat{\lambda}^G(x^{-1}bx) \cdot \mu(x^{-1}bx)$$
$$= \hat{\lambda}^G(b) \cdot \mu(bc) = \hat{\lambda}^G(b) \cdot \zeta \overset{(*)}{=} p \cdot \zeta \neq p = \left((\hat{\lambda}^G)_V \cdot \hat{\mu} \right)(b).$$

Ad($*$): Let R be a set of coset representatives for G/U, say $R = \{1, y, y^2, ..., y^{p-1}\}$. Then $\hat{\lambda}^G(b) \overset{2.2.2}{=} \sum_{g \in R} \hat{\lambda}(g^{-1}bg) = \sum_{i=0}^{p-1} \hat{\lambda}(y^{-i}by^i) \overset{(y^{-1}by=b)}{=} p \cdot \hat{\lambda}(b) = p \cdot 1 = p$.
Hence we now know that $V \subseteq T \neq G$.

The index $[G : V] = p$ and it follows that $T = V$, i.e. $T_G\left((\hat{\lambda}^G)_V \cdot \hat{\mu} \right) = V$ which is equivalent to $\left((\hat{\lambda}^G)_V \cdot \hat{\mu} \right)^G = \hat{\lambda}^G \cdot \hat{\mu}^G \in \mathrm{Irr}_{\mathbb{C}}(G)$.

Hence we proved claim 4 and the whole proof is finished.

Inductions

We have to prove the following claims:

(1) $M_x^i = \begin{pmatrix} 1 & i & \frac{i(i-1)}{2} & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ for all $i \in \mathbb{N}$, (where $M_x = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$).

(2) $\tau(\hat{x})^i = \tilde{x}^i \tilde{a}^i \tilde{b}^{\frac{i(i-1)}{2}} \tilde{c}^{\frac{i(i-1)(i-2)}{6}}$ for all $i \in \mathbb{N}, i \geq 2$.

(3) $\psi(\tilde{y})^i(\tilde{a}) = \tilde{a}\tilde{d}^i$ for all $i \in \mathbb{N}$.

(4) $\psi(\tilde{y})^i(\tilde{x}) = \tilde{x}\tilde{a}^i \tilde{d}^{\frac{i(i-1)}{2}}$ for all $i \in \mathbb{N}, i \geq 2$.

Ad(1) We immediately see from the definition of M_x that the claim is true for $i = 1$.

Let us assume that the claim is correct for some $i \in \mathbb{N}$ $(*)$. Now do the step $\underline{i \mapsto i+1}$:

We have

$$\begin{aligned} {M_x}^{i+1} = {M_x}^i \cdot M_x \overset{(*)}{=} & \begin{pmatrix} 1 & i & \frac{i(i-1)}{2} & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ = & \begin{pmatrix} 1 & i+1 & \frac{i(i+1)}{2} & 0 \\ 0 & 1 & i+1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence the claim is correct for $i+1$ and therefore for all $i \in \mathbb{N}$.

Ad(2) For $i = 2$ we have $\tau(\hat{x})^2 = (\tilde{x}\tilde{a})^2 = \tilde{x}\tilde{a}\tilde{x}\tilde{a} = \tilde{x}\tilde{x} \underbrace{\tilde{x}^{-1}\tilde{a}\tilde{x}}_{=\phi(\hat{x})(a)=\tilde{a}\tilde{b}} \tilde{a} = \tilde{x}^2\tilde{a}^2\tilde{b}$.

Hence the claim is correct for $i = 2$.

Let us assume that the claim is correct for some $i \in \mathbb{N}, i > 2$ $(**)$.

Now do the step $\underline{i \mapsto i+1}$:

We have

$$\begin{aligned}
\tau\left(\hat{x}\right)^{i+1} &= \left(\tilde{x}\tilde{a}\right)^{i+1} = \left(\tilde{x}\tilde{a}\right)^{i}\cdot\left(\tilde{x}\tilde{a}\right) \overset{(**)}{=} \tilde{x}^{i}\tilde{a}^{i}\tilde{b}^{\frac{i(i-1)}{2}}\tilde{c}^{\frac{i(i-1)(i-2)}{6}}\cdot\tilde{x}\tilde{a} \\
&= \tilde{x}^{i+1}\left(\tilde{x}^{-1}\tilde{a}^{i}\tilde{x}\right)\left(\tilde{x}^{-1}\tilde{b}^{\frac{i(i-1)}{2}}\tilde{x}\right)\tilde{c}^{\frac{i(i-1)(i-2)}{6}}\tilde{a} \\
&= \tilde{x}^{i+1}\phi\left(\tilde{x}\right)\left(\tilde{a}\right)^{i}\phi\left(\tilde{x}\right)\left(\tilde{b}\right)^{\frac{i(i-1)}{2}}\tilde{c}^{\frac{i(i-1)(i-2)}{6}}\tilde{a} \\
&= \tilde{x}^{i+1}\tilde{a}^{i}\tilde{b}^{i}\tilde{b}^{\frac{i(i-1)}{2}}\tilde{c}^{\frac{i(i-1)}{2}}\tilde{c}^{\frac{i(i-1)(i-2)}{6}}\tilde{a} = \tilde{x}^{i+1}\tilde{a}^{i+1}\tilde{b}^{\frac{i(i+1)}{2}}\tilde{c}^{\frac{(i+1)i(i-1)}{6}}.
\end{aligned}$$

Hence the claim is correct for $i+1$ and therefore for all $i \in \mathbb{N}, i \geq 2$.

Ad(3) We immediately see from the definition of ψ that the claim is true for $i = 1$.
Let us assume that the claim is correct for some $i \in \mathbb{N}$ (†). Now do the step $\underline{i \mapsto i+1}$:
We have $\psi(\tilde{y})^{i+1}\left(\tilde{a}\right) \overset{(\dagger)}{=} \psi(\tilde{y})\left(\tilde{a}\tilde{d}^{i}\right) = \tilde{a}\tilde{d}\tilde{d}^{i} = \tilde{a}\tilde{d}^{i+1}$.
Hence the claim is correct for $i+1$ and therefore for all $i \in \mathbb{N}$.

Ad(4) We immediately see from the definition of ψ that the claim is true for $i = 1$.
Let us assume that the claim is correct for some $i \in \mathbb{N}$ (††). Now do the step $\underline{i \mapsto i+1}$:
We have $\psi(\tilde{y})^{i+1}\left(\tilde{x}\right) \overset{(\dagger\dagger)}{=} \psi(\tilde{y})\left(\tilde{x}\tilde{a}^{i}\tilde{d}^{\frac{i(i-1)}{2}}\right) = \left(\tilde{x}\tilde{a}\right)\left(\tilde{a}^{i}\tilde{d}^{i}\right)\left(\tilde{d}^{\frac{i(i-1)}{2}}\right) = \tilde{x}\tilde{a}^{i+1}\tilde{d}^{\frac{i(i+1)}{2}}$.
Hence the claim is correct for $i+1$ and therefore for all $i \in \mathbb{N}$.

The tensor decomposable character we found is a not trivially tensor decomposable in the meaning of 3.2.1. In order to see this we first claim that any normal subgroup $N \trianglelefteq G$ with $|N| = p^a$, $a \leq 4$ is abelian. Let us prove this claim shortly:
Because G is a p-group there is a normal subgroup $M \trianglelefteq G$ with $N \leq M$ and $|M| = p^4$. Thus $|G/M| = p^2$ and we conclude that G/M is abelian. But this also means that $G' \leq M$ and hence $G' = M$ since we showed earlier that G' is elementary abelian of order p^4. As $N \leq M = G'$ and G' is abelian it

follows that N is abelian and the claim is proven.
Now we come back to prove the original claim. Let us assume that there is a trivially tensor decomposable character $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$, i.e. there is a normal subgroup $N \trianglelefteq G$ such that $\chi_N = e\vartheta$ for some $\vartheta \in \mathrm{Irr}_{\mathbb{C}}(N)$, $e \in \mathbb{N}$ with $\vartheta(1) > 1$ and $e > 1$. Further, since N has an irreducible character of degree > 1, we deduce that N must be non-abelian. With the foregoing remark we now conclude that $|N| = p^a$ with $a \geq 5$. Hence G/N is cyclic. As $\chi_N = e\vartheta$ the inertia group of ϑ in G obviously is $I_G(\vartheta) = G$, i.e. ϑ is invariant in G. Now we apply 2.4.3 and obtain that ϑ is extendable to a character $\psi \in \mathrm{Irr}_{\mathbb{C}}(G)$. Consider the inner product $(\psi, \chi) = (\psi_N, \chi_N) = (\vartheta, e\vartheta) = e$. Yet ψ and χ are both irreducible characters of G which includes that $(\psi, \chi) \in \{0, 1\}$. Now we derive a constradiction since we assumed $e > 1$. Thus the tensor decomposable character we found cannot be trivially tensor decomposable.

Remark 3.3.2 In 3.3.1 we showed that for all prime numbers $p \geq 5$ there is a non-trivial example of a group of order p^6 possessing a tensor decomposable character.
However this is also true for $p = 2, 3$ as examples with GAP showed. For more details confer Chapter 5.

Chapter 4

The Character Table of G

As a last result we work out the character table of G. Afterwards we will easily see that G not only possesses one tensor decomposable character, but that all irreducible characters of degree p^2 are tensor decomposable.

Let us now give an overview about the steps we will pursue to reach our goal.

4.1 Outline of the determination of the character table of G

(1) Determine the character table of U and V via the following steps:

  (a) Compute the U- and V-conjugacy classes contained in G'.

  (b) Use Brauers permutation lemma in order to obtain the number of orbits and fixed points of the action of U on G'. Then determine the number of irreducible characters of U of degree 1 and of degree p and determine the conjugacy classes of U.

  (c) Obtain analogously the number of irreducible characters of V of degree 1 and of degree p and determine the conjugacy classes of V.

(d) Deduce all irreducible characters of U and V by extending or inducing characters from $\mathrm{Irr}_{\mathbb{C}}(G')$ to irreducible characters of U and V, respectively.

(2) Determine the G-conjugacy classes contained in V and obtain the number of orbits and fixed points under the action of G on $Cl(V)$.

(3) Use Brauers permutation lemma in order to obtain the number of orbits and fixed points on $\mathrm{Irr}_{\mathbb{C}}(V)$ under the action of G.

(4) Determine the number of irreducible characters of G of degree 1, p and p^2.

(5) Construct all irreducible characters of G by extending and inducing irreducible characters of V and U.

(6) Determine the conjugacy classes of G.

Notation 4.1.1 Let now for the whole chapter $I := \{0, 1, ..., p-1\}$ and $I^* := I \setminus \{0\}$.

4.2 The character table of U and V

Ad(1) (a)

Since G' is a normal subgroup of both U and V there is an action of U and V respectively on G' via conjugation, i.e.

$G' \times U \longrightarrow G',\ (g, u) \mapsto u^{-1}gu$ and analogously

$G' \times V \longrightarrow G',\ (g, v) \mapsto v^{-1}gv.$

Let us present the U- and V-conjugacy classes contained in G' in Tables 4.1 and 4.1 by giving a representative and the length for each class. Subsequently we will prove that the tables are correct.

Table 4.1: U-classes in G'

name	representative	parameter	length	number
$C_{\gamma\delta}$	$c^\gamma d^\delta$	$\gamma, \delta \in I$	1	p^2
$B_{\beta\delta}$	$b^\beta d^\delta$	$\beta \in I^*, \delta \in I$	p	$p(p-1)$
$A_{\alpha\gamma\delta}$	$a^\alpha c^\gamma d^\delta$	$\alpha \in I^*, \gamma, \delta \in I$	p	$p^2(p-1)$

Table 4.2: V-classes in G'

name	representative	parameter	length	number
$B_{\beta\gamma\delta}$	$b^\beta c^\gamma d^\delta$	$\beta, \gamma, \delta \in I$	1	p^3
$A_{\alpha\beta\gamma}$	$a^\alpha b^\beta c^\gamma$	$\alpha \in I^*, \beta, \gamma \in I$	p	$p^2(p-1)$

Proof of Tables 4.1 and 4.2:
Let us first collect some general information which is useful for both the U- and the V-conjugacy classes contained in G'.
Since G' is abelian G' stabilizes itself. We know that the length of the orbit of an element is equal to the index of its stabilizer in the group. Further the length of an orbit divides the group order. We have that U and V are both of order p^5, G' is of order p^4, hence each orbit is either of length 1 or of length p. Now $U = \langle x \rangle \ltimes G'$ and $V = \langle y \rangle \ltimes G'$, hence it is sufficient to investigate how x and y act on G'. Let us recall that the matrix of x on G' is given by

$$M_x = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For y we obtain the matrix

$$M_y = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Obviously we then obtain $\{c^\gamma d^\delta\}$, $\gamma, \delta \in I$ as orbits of length 1 under the action of U on G'. The remaining elements must all lie in orbits of length p. As we already saw in the proof of 3.3.1,

$$M_x^n = \begin{pmatrix} 1 & n & \frac{n(n-1)}{2} & 0 \\ 0 & 1 & n & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for all $n \in \mathbb{N}$. Therefore we can explicitly calculate the conjugacy class of an element of G' and we obtain $Cl_U(b^\beta d^\delta) = \{b^\beta c^{n\beta} d^\delta \mid n \in I\}$ for $\beta \in I^*$, $\delta \in I$ and $Cl_U(a^\alpha c^\gamma d^\delta) = \{a^\alpha b^{n\alpha} c^{\gamma+n(n-1)\alpha/2} d^\delta \mid n \in I\}$ for $\alpha \in I^*$, $\gamma, \delta \in I$. Hence we see that all elements in Table 4.1 listed in the column of the representatives lie in different conjugacy classes. Summing up we obtain p^4 elements and we conclude that we found all conjugacy classes.

We will procede analogously with the conjugacy classes of V. Obviously, for $\beta, \gamma, \delta \in I$, we obtain $\{b^\beta c^\gamma d^\delta\}$ as orbits of length 1. It is easy to see that

$$M_y^n = \begin{pmatrix} 1 & 0 & 0 & n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for all $n \in \mathbb{N}$.
Again we are able to explicitly calculate the conjugacy classes and we obtain $Cl_V(a^\alpha b^\beta c^\gamma) = \{a^\alpha b^\beta c^\gamma d^{\beta n} \mid n \in I\}$ for $\alpha \in I^*, \beta, \gamma \in I$. Hence we see that all elements in Table 4.2 in the column of the representatives lie in different conjugacy classes. Summing up we obtain p^4 elements and conclude that we found all conjugacy classes.

Ad(1) (b)
We also have an action of U and V respectively on $\mathrm{Irr}_{\mathbb{C}}(G')$ via $\varphi^w(g) = \varphi(wgw^{-1}), g \in G', w \in U$ and $w \in V$, respectively. Hence $\varphi^w(g) = \varphi({}^w g)$ for

all $g \in G', w \in U$ and $w \in V$, respectively and we can apply Lemma 2.3.11, Brauers permutation lemma.

Let us first stick to U. From Table 4.1 we deduce that G' has exactly p^2 fixed points under U. Hence we conclude that there are p^2 fixed irreducible characters of G' under the action of U, i.e. there are exactly p^2 orbits of length 1. Furthermore we deduce that $\mathrm{Irr}_{\mathbb{C}}(G')$ has exactly $p^2 + p(p^2 - 1)$ orbits under U. Since G' also stabilizes $\mathrm{Irr}_{\mathbb{C}}(G')$ the length of an orbit can again be either 1 or p.
We conclude that $\mathrm{Irr}_{\mathbb{C}}(G')$ has exactly $p(p^2 - 1)$ orbits of length p and p^2 orbits of length 1.
We now investigate the number of irreducible characters of U of degree 1 and p.

A character $\lambda \in \mathrm{Irr}_{\mathbb{C}}(G')$ is either extendable to a linear character of U or induces to an irreducible character of U of degree p. We claim that vice versa every irreducible character of U is either an extension of an irreducible character of G' or induced by one such. Any linear character of U clearly is an extension of an irreducible character of G'. In order to prove that any character of U of degree p is induced from G' consider $\chi \in \mathrm{Irr}_{\mathbb{C}}(U), \chi(1) = p$. Now let $\lambda \in \mathrm{Irr}_{\mathbb{C}}(G')$ such that $(\chi_{G'}, \lambda) \neq 0$, i.e. λ is a constituent of the restriction of χ to G'. Applying Frobenius reciprocity shows that $\left(\chi, \lambda^U\right) = (\chi_{G'}, \lambda) \neq 0$, hence χ is a constituent of λ^U. However $p = \chi(1) = [U : G']\, \lambda(1) = \lambda^U(1)$, i.e. λ^U and χ have the same degree. We conclude that $\chi = \lambda^U$ and the claim is proven.
Using 2.3.5 and 2.4.1 we deduce that there are exactly p^2 characters in $\mathrm{Irr}_{\mathbb{C}}(G')$ which can be extended to a linear character of U. These are just the ones which lie in orbits of length 1, which is equivalent to having inertia group U or being invariant under U. From 2.4.3 we conclude that each extendable linear character of G' has exactly p different extensions. Hence U has exactly p^3 irreducible characters of degree 1.
It is easy to see that two induced characters, say λ^U and μ^U, are the same if and only if λ and μ lie in the same orbit under the action of U on $\mathrm{Irr}_{\mathbb{C}}(G')$.

Using this together with 2.3.5 we conclude that U has exactly $p(p^2-1)$ characters of degree p. Each of them is induced by any representative of an orbit of length p.

Since the number of conjugacy classes is equal to the number of irreducible characters of U we deduce that U has exactly $p^3+p^2=p^2(p-1)$ conjugacy classes.

Let us give an overview of the classes in Table 4.3 containing representatives together with the length of each class. Subsequently we will prove that the table is correct.

Table 4.3: $Cl(U)$

name	representative	parameter	length	number
$C_{\gamma\delta}$	$c^\gamma d^\delta$,	$\gamma, \delta \in I$	1	p^2
$B_{\beta\delta}$	$b^\beta d^\delta$,	$\beta \in I^*, \delta \in I$	p	$p(p-1)$
$A_{\alpha\gamma\delta}$	$a^\alpha c^\gamma d^\delta$,	$\alpha \in I^*, \gamma, \delta \in I$	p	$p^2(p-1)$
$X_{\xi\alpha\delta}$	$x^\xi a^\alpha d^\delta$,	$\xi \in I^*, \alpha, \delta \in I$	p^2	$p^2(p-1)$

Proof of Table 4.3:
Since we already deduced the U-classes contained in G' we only need to investigate the elements contained in $U \backslash G'$. Let us have a look at the conjugacy classes of elements $x^\xi a^\alpha d^\delta, \xi \in I^*, \alpha, \delta \in I$. We know that any conjugacy class is a subset of a coset uU' for some $u \in U'$. However we easily see that $U' = \langle b, c \rangle$ and therefore has order p^2. Conjugation with x, a and b shows that $Cl(x^\xi a^\alpha d^\delta)$ has more than p elements, hence it must have p^2 elements and thus be equal to uU' for some $u \in U$. Since $U' = \langle b, c \rangle$ we conclude that $Cl(x^\xi a^\alpha d^\delta) = \left\{ x^\xi a^\alpha d^\delta b^\beta c^\gamma \mid \beta, \gamma \in I \right\}$. Summing up we obtain $p^2(p-1)$ conjugacy classes, each of length p^2, hence $p^5 - p^4$ elements in total. Since $|U| = p^5$ and $|G'| = p^4$ we see that we found all conjugacy classes.

Ad(1) (c)
In order to obtain the irreducible characters of V we will use precisely the same arguments and obtain that $\mathrm{Irr}_{\mathbb{C}}(G')$ has $p^3 + p^2(p-1)$ orbits in total under V, p^3 of them have length 1 and $p^2(p-1)$ have length p. Hence V has p^4 linear characters and $p^2(p-1)$ characters of degree p. We conclude that V has $p^4 + p^2(p-1) = p^2(p^2 + p - 1)$ conjugacy classes. Let us again give an overview of the classes in Table 4.4 containing representatives together with the length of each class.

Table 4.4: $Cl(V)$

name	representative	parameter	length	number
$B_{\beta\gamma\delta}$	$b^\beta c^\gamma d^\delta$	$\beta, \gamma, \delta \in I$	1	p^3
$A_{\alpha\beta\gamma}$	$a^\alpha b^\beta c^\gamma$	$\alpha \in I^*, \beta, \gamma \in I$	p	$p^2(p-1)$
$Y_{\upsilon\alpha\beta\gamma}$	$y^\upsilon a^\alpha b^\beta c^\gamma,$	$\upsilon \in I^*, \alpha, \beta, \gamma \in I$	p	$p^3(p-1)$

Proof of Table 4.4:
Since we already deduced the V-classes contained in G' we only need to investigate the elements contained in $V \backslash G'$. Let us have a look at the conjugacy classes of elements $y^\upsilon a^\alpha b^\beta c^\gamma, \upsilon \in I^*, \alpha, \beta, \gamma \in I$. We know that any conjugacy class is a subset of a coset vV' for some $v \in V'$. We easily see that $V' = \langle d \rangle$ and therefore has order p. Conjugation with y and a shows that $Cl(y^\upsilon a^\alpha b^\beta c^\gamma)$ has more than one element, hence it must have p elements and be equal to vV' for some $v \in V$. Since
$V' = \langle d \rangle$ we conclude that $Cl(y^\upsilon a^\alpha b^\beta c^\gamma) = \{y^\upsilon a^\alpha b^\beta c^\gamma d^\delta \mid \delta \in I\}$. Summing up we obtain $p^3(p-1)$ conjugacy classes, each of length p, hence $p^5 - p^4$ elements in total. Since $|V| = p^5$ and $|G'| = p^4$ we see that we found all conjugacy classes.

Ad(1) (d)

Let us now present the character tables of U and V and subsequently prove that these tables are correct. Therefore we will introduce some notation.

Notation 4.2.1

(i) Let ζ be a complex root of unity of order p.

(ii) If $\lambda \in \mathrm{Irr}_{\mathbb{C}}(G')$ is invariant in $U = \langle x \rangle \ltimes G'$ or in $V = \langle y \rangle \ltimes G'$, we will use the notation $\hat{\lambda}$ for the extended linear character of $\mathrm{Irr}_{\mathbb{C}}(U)$ or $\mathrm{Irr}_{\mathbb{C}}(V)$, with respect to the construction given in the proof of 2.4.1.

(iii) By ε we will denote the linear character of $\mathrm{Irr}_{\mathbb{C}}(U/G')$ or $\mathrm{Irr}_{\mathbb{C}}(V/G')$, respectively, with $\varepsilon(x) = \zeta$ or $\varepsilon(y) = \zeta$, respectively.
Furthermore we also consider ε as a character of U or V, respectively.

(iv) For $s \in I^*$ we define

$$\rho_s := (-\zeta^s)^{(p^2-1)/8} \cdot \sqrt{(-1)^{(p-1)/2}} \cdot \sqrt{p} \in \mathbb{C},$$

where we use the positive branch of the root function.

Table 4.5: **The generic character table of U**

centralizer order			p^5	p^4	p^4	p^3
conjugacy class			$C_{\gamma\delta}$	$B_{\beta\delta}$	$A_{\alpha\gamma\delta}$	$X_{\xi\alpha\delta}$
parameter			$\gamma, \delta \in I$	$\beta \in I^*, \delta \in I$	$\alpha \in I^*, \gamma, \delta \in I$	$\xi \in I^*, \alpha, \delta \in I$
#	parameter	#	p^2	$p(p-1)$	$p^2(p-1)$	$p^2(p-1)$
p^3	$i, l, m \in I$	$\hat{\lambda}_{i,0,0,l} \cdot \varepsilon^m$	$\zeta^{l\delta}$	$\zeta^{l\delta}$	$\zeta^{i\alpha+l\delta}$	$\zeta^{m\xi+i\alpha+l\delta}$
$p^2(p-1)$	$i, l \in I, k \in I^*$	$\lambda^U_{i,0,k,l}$	$p\zeta^{k\gamma+l\delta}$	0	$\zeta^{i\alpha+k\gamma+l\delta} \cdot \rho_{k\alpha}$	0
$p(p-1)$	$l \in I, j \in I^*$	$\lambda^U_{0,j,0,l}$	$p\zeta^{l\delta}$	$p\zeta^{j\beta+l\delta}$	0	0

Table 4.6: **The generic character table of V**

centralizer order			p^5	p^4	p^4
conjugacy class			$B_{\beta\gamma\delta}$	$A_{\alpha\beta\gamma}$	$Y_{\upsilon\alpha\beta\gamma}$
parameter			$\beta, \gamma, \delta \in I$	$\alpha \in I^*, \beta, \gamma \in I$	$\upsilon \in I^*, \alpha, \beta, \gamma \in I$
#	parameter	#	p^3	$p^2(p-1)$	$p^3(p-1)$
p^4	$i, j, k, m \in I$	$\Theta_{ijkm} := \hat{\lambda}_{i,j,k,0} \cdot \varepsilon^m$	$\zeta^{j\beta+k\gamma}$	$\zeta^{i\alpha+j\beta+k\gamma}$	$\zeta^{m\upsilon+i\alpha+j\beta+k\gamma}$
$p^2(p-1)$	$j, k \in I, l \in I^*$	$\lambda^V_{0,j,k,l}$	$p\zeta^{j\beta+k\gamma+l\delta}$	0	0

In order to investigate the irreducible characters of U and V we will need the character table of G'.
By $\lambda_{i,j,k,l},\ i,j,k,l \in I$ we denote the linear character of G' with
$\lambda_{i,j,k,l}(a) = \zeta^i, \quad \lambda_{i,j,k,l}(b) = \zeta^j, \quad \lambda_{i,j,k,l}(c) = \zeta^k, \quad \lambda_{i,j,k,l}(d) = \zeta^k.$
Since $\{a,b,c,d\}$ is a generating set for G', a linear character is uniquely determined by the image of this set.

Theorem 4.2.2

$$\mathrm{Irr}_{\mathbb{C}}(G') = \{\lambda_{i,j,k,l} \,|\, i,j,k,l \in I\}\,.$$

Proof Since G' is elementary abelian of order p^4, the claim immediately follows from 2.1.5.

Let us now start to prove that the character table of U given in Table 4.5 is correct.

We already know that U has p^2 invariant linear characters in $\mathrm{Irr}_{\mathbb{C}}(G')$. Let us determine these characters.

Lemma 4.2.3 $\lambda_{i,j,k,l} \in \mathrm{Irr}_{\mathbb{C}}(G')$ *is invariant in* U *if and only if* $j = k = 0$.

Proof Since $x^{-1} \in U$ is of order p and therefore generates U/G', we know that $\lambda \in \mathrm{Irr}_{\mathbb{C}}(G')$ is invariant in U if and only if $\lambda^{x^{-1}} = \lambda$, i.e. if and only if $\lambda(x^{-1}gx) = \lambda(g)$ for all $g \in G'$. The set $\{a,b,c,d\}$ generates G', hence we only need to check that $\lambda(x^{-1}gx) = \lambda(g)$ for all $g \in \{a,b,c,d\}$. We obtain

$$\begin{aligned}
\lambda(a) &= \lambda(x^{-1}ax) = \lambda(ab), \text{ hence } \lambda(b) = 1,\\
\lambda(b) &= \lambda(x^{-1}bx) = \lambda(bc), \text{ hence } \lambda(c) = 1,\\
\lambda(c) &= \lambda(x^{-1}cx) = \lambda(c),\\
\lambda(d) &= \lambda(x^{-1}dx) = \lambda(d).
\end{aligned}$$

Thus $\lambda \in \mathrm{Irr}_{\mathbb{C}}(G')$ is invariant in U if and only if $\lambda(b) = \lambda(c) = 1$. This means that the character $\lambda_{i,j,k,l}$ of G' is invariant in U if and only if $j = k = 0$.

Theorem 4.2.4 *The linear characters of* U *are given by*

$$\mathrm{Lin}_{\mathbb{C}}(U) = \left\{ \hat{\lambda}_{i,0,0,l} \cdot \varepsilon^m \,|\, i, l, m \in I \right\}.$$

Proof From 2.4.2, 2.4.3 and 4.2.3 it follows that any linear character of U is of the form $\hat{\lambda_{i,0,0,l}} \cdot \varepsilon$, where ε is a linear character of U/G'. We have $|U/G'| = p$, i.e. U/G' is abelian and it follows that $|\mathrm{Irr}_{\mathbb{C}}(U/G')| = p$. Hence $\mathrm{Irr}_{\mathbb{C}}(U/G') = \{\varepsilon^m \,|\, m \in I\}$. We conclude that any linear character of U is of the form $\hat{\lambda}_{i00l} \cdot \varepsilon^m,\ i, l, m \in I$. Summing up we see that we found exactly $p^2 \cdot |\mathrm{Irr}_{\mathbb{C}}(U/G')| = p^3$ linear characters of U which means we have found them all.

It remains to determine the irreducible characters of U of degree p. These are induced by linear characters $\lambda \in \mathrm{Irr}_{\mathbb{C}}(G')$ with inertia group $I_U(\lambda) = G'$. From 4.2.3 we conclude that $I_U(\lambda) = G'$ if and only if $j \neq 0$ or $k \neq 0$. Hence $\lambda^U_{i,j,k,l}$ is an irreducible character of U if and only if $j \neq 0$ or $k \neq 0$.

Lemma 4.2.5

(1) *Define* $\bar{x} := x^{-1}$. *Then we have* $\lambda^{\bar{x}^n}_{i,j,k,l} = \lambda_{i+jn+kn(n-1)/2,j+kn,k,l}$ *for all* $n \in \mathbb{N}, i, j, k, l \in I$.
Note that $\lambda_{i,j,k,l} = \lambda_{i(mod\, p),j(mod\, p),k(mod\, p),l(mod\, p)}$ *for all* $i, j, k, l \in \mathbb{N}$.

(2) *Let* $k \in I^*, i, j, l \in I$. *Then the orbit of* $\lambda_{i,j,k,l}$ *under* U *contains an element* $\lambda_{i',0,k,l}$ *for some* $i' \in I$.

(3) *Let* $j \in I^*, i, l \in I$. *Then the orbit of* $\lambda_{i,j,0,l}$ *under* U *contains* $\lambda_{0,j,0,l}$.

(4) *Let* $S_U := \{\lambda_{i,0,k,l} \,|\, i, l \in I, k \in I^*\}$ *and* $T_U := \{\lambda_{0,j,0,l} \,|\, j \in I^*, l \in I\}$. *Then none of the elements from* $S_U \cup T_U$ *are conjugate under* U.

Proof

(1) We will prove the claim by induction. Therefore let $\alpha, \beta, \gamma, \delta \in I$ and $i, j, k, l \in I$.
For $n = 1$ we obtain

$\lambda^{\bar{x}}_{i,j,k,l}(a^\alpha b^\beta c^\gamma d^\delta) = \lambda_{i,j,k,l}(x^{-1}a^\alpha b^\beta c^\gamma d^\delta x) = \lambda_{i,j,k,l}(a^\alpha b^{\beta+\alpha} c^{\gamma+\beta} d^\delta) = \zeta^{\alpha i+(\beta+\alpha)j+(\gamma+\beta)k+\delta l} = \zeta^{\alpha(i+j)+\beta(j+k)+\gamma k+\delta l} = \lambda_{i+j,j+k,k,l}(a^\alpha b^\beta c^\gamma d^\delta)$.

Thus the claim is correct for $n = 1$ (†).

Assume now that the claim is correct for some $n \in \mathbb{N}$ ($*$) and perform the step $n \mapsto n+1$:

We obtain

$\lambda^{\bar{x}^{n+1}}_{i,j,k,l} = \left(\lambda^{\bar{x}^n}_{i,j,k,l}\right)^{\bar{x}} \overset{(*)}{=} \left(\lambda_{i+jn+kn(n-1)/2),j+kn,k,l}\right)^{\bar{x}} \overset{(\dagger)}{=} \lambda_{i+j(n+1)+kn(n+1)/2),j+k(n+1),k,l}$.

We see that the claim is correct for $n+1$ and hence it is correct for all $n \in \mathbb{N}$.

(2) We choose $n \in \mathbb{N}$ such that $j + kn \equiv 0 \pmod p$. Then the claim immediately follows from part (1).

(3) We choose $n \in \mathbb{N}$ such that $i + jn \equiv 0 \pmod p$. Then the claim immediately follows from part (1).

(4) We know that $\lambda_{i,j,k,l}$ is invariant under G'. Hence we just need to consider elements of $U \backslash G'$ when investigating the action on $\lambda_{i,j,k,l}$. However $\bar{x}$ generates U/G', hence we only have to look the action of $\bar{x}^n$ on $\lambda_{i,j,k,l}$. Applying part (1) we can easily calculate the orbit of $\langle \bar{x} \rangle$ on $\lambda_{i,j,k,l}$.

For $\lambda_{i,0,k,l} \in S_U$ we obtain the orbit $\left\{\lambda_{i+kn(n-1)2,kn,k,l} \,|\, n \in I\right\}$. Hence none of the elements of S_U can be conjugate under U.

For $\lambda_{0,j,0,l} \in S_U$ we obtain the orbit $\{\lambda_{jn,j,0,l} \,|\, n \in I\}$. Hence none of the elements of T_U can be conjugate under U.

If we further compare S_U with T_U we immediately see that it is not possible for two elements of $S_U \cup T_U$ to be conjugate under U.

Corollary 4.2.6

$$\{\chi \in \mathrm{Irr}_{\mathbb{C}}(U) \,|\, \chi(1) = p\} = \left\{\lambda^U_{i,0,k,l} \,|\, i,l \in I, k \in I^*\right\} \cup \left\{\lambda^U_{0,j,0,l} \,|\, j \in I^*, l \in I\right\}.$$

The value of λ^U_{ijkl} *at an element* $u = x^\upsilon a^\alpha b^\beta c^\gamma d^\delta \in U$ *for* $\upsilon, \alpha, \beta, \gamma, \delta \in I$ *is given by:*

$$\lambda^U_{i,j,k,l}(u) = \begin{cases} 0, \ \textit{if} \ , \ \textit{if} \ \upsilon \neq 0, \\ \zeta^{\alpha i+\beta j+\gamma k+\delta l} \sum_{n\in I} \zeta^{\alpha(jn+kn(n-1)/2)+\beta(kn)}, \ \textit{if} \ \upsilon = 0. \end{cases}$$

Proof We know that two characters $\lambda^U_{i,j,k,l}$ and $\lambda^U_{i',j',k',l'}$ are the same if and only if $\lambda_{i,j,k,l}$ and $\lambda_{i',j',k',l'}$ lie in the same U-orbit. From 4.2.5 we conclude that all characters contained in $\left\{\lambda^U_{i,0,k,l} \,|\, i,l \in I, k \in I^*\right\} \cup \left\{\lambda^U_{0,j,0,l} \,|\, j \in I^*, l \in I\right\}$ are different to each other. Summing up we see that the set above contains exactly $p^2(p-1)+p(p-1) = p(p^2-1)$ characters. As we deduced earlier this is precisely the number of irreducible character of U of degree p.

Let us determine the value of $\lambda^U_{i,j,k,l}$ at $u \in U$.
Since $\lambda^U_{i,j,k,l}$ is induced from G' we have that $\lambda^U_{i,j,k,l}(u) = 0$, if $u \notin G'$.
For $u \in G'$, i.e. for $\xi = 0$ we obtain the value at u as follows:
$\lambda^U_{i,j,k,l}(a^\alpha b^\beta c^\gamma d^\delta) = \lambda^U_{i,j,k,l}(a^\alpha b^\beta c^\gamma d^\delta) = \sum_{n\in I} \lambda^{\bar{x}^n}_{i,j,k,l}(a^\alpha b^\beta c^\gamma d^\delta) \overset{4.2.5}{=}$
$\sum_{n\in I} \lambda_{i+jn+kn(n-1)/2,j+kn,k,l}(a^\alpha b^\beta c^\gamma d^\delta) = \sum_{n\in I} \zeta^{\alpha(i+jn+kn(n-1)/2)+\beta(j+kn)+\gamma k+\delta l} =$
$\zeta^{\alpha i+\beta j+\gamma k+\delta l} \sum_{n\in I} \zeta^{\alpha(jn+kn(n-1)/2)+\beta(kn)}$.

Now we are ready to fill in the entries in the character table of U.

Let us make a last useful remark to deduce the values for the character table of U.

Remark 4.2.7

(i) $\sum_{i=0}^{p-1} \zeta^r i = 0$ for all $r \in I^*$.

(ii) $\sum_{n=0}^{p-1} \zeta^{k\alpha\frac{n(n-1)}{2}} = \left(-\zeta^{k\alpha}\right)^{(p^2-1)/8} \cdot \sqrt{(-1)^{(p-1)/2}} \cdot \sqrt{p} = \rho_{k\alpha}$ for all $k, \alpha \in I^*$, where we defined ρ_s in 4.2.1.

Proof

(i) Since $\phi_p = X^{p-1} + X^{p-2} + ... + X^2 + X + 1$ is the minimal polynomial of a complex root of unity of order p we obtain that $\sum_{i=0}^{p-1} \zeta^{ri} = 0$. for all $r \in I^*$.

(ii) see [EM, Lemma 3, p. 288].

For each conjugacy class we now just insert i, j, k, l and $\upsilon, \alpha, \beta, \gamma, \delta$ in the formula given in 4.2.6 for characters of degree p or consider 4.2.4 for linear characters.

We now come to prove that the character table of V, given in Table 4.6, is correct. In order to do so we will follow precisely the same arguments as for the character table of U. Therefore we will omit the proofs at this stage. We have:

Lemma 4.2.8 *$\lambda_{i,j,k,l} \in \mathrm{Irr}_{\mathbb{C}}(G')$ is invariant in V if and only if $l = 0$.*

Theorem 4.2.9 *The linear characters of V are given by*

$$\mathrm{Lin}_{\mathbb{C}}(V) = \left\{ \Theta_{ijkm} \,|\, i, j, k, m \in I, \Theta_{ijkm} = \hat{\lambda}_{ijk0} \cdot \varepsilon^m \right\}.$$

It remains to determine the irreducible characters of V of degree p. Any such character is induced by a character $\lambda \in \mathrm{Irr}_{\mathbb{C}}(G')$. From 4.2.8 we deduce that $I_V(\lambda_{i,j,k,l}) = G'$ if and only if $l \neq 0$. Hence $\lambda_{i,j,k,l}^V$ is an irreducible character of V if and only if $l \neq 0$.

Lemma 4.2.10 *(1) Define $\bar{y} := y^{-1}$. Then we have ${\lambda_{i,j,k,l}}^{\bar{y}^n} = \lambda_{i+ln,j,k,l}$.*

(2) Let $S_V := \{\lambda_{0,j,k,l} \,|\, j, k \in I, l \in I^*\}$. Then none of the elements in S_V are conjugate under V.

Corollary 4.2.11 $\{\chi \in \mathrm{Irr}_{\mathbb{C}}(V) \,|\, \chi(1) = p\} = \left\{\lambda_{0,j,k,l}^V \,|\, j, k \in I, l \in I^*\right\}.$

The explicit value of $\lambda_{0,j,k,l}^V$ at an element $v = y^\upsilon a^\alpha b^\beta c^\gamma d^\delta \in V$ for $\upsilon, \alpha, \beta, \gamma, \delta \in I$ is given by:

$$\lambda^V_{0,j,k,l}(v) = \begin{cases} 0, & \textit{if } v \notin \langle b,c,d\rangle \\ p\zeta^{j\beta+k\gamma+l\delta}, & \textit{otherwise} \end{cases}$$

Analogously to U we can now fill in the character table of V with the corresponding character values.

4.3 The determination of the irreducible characters of G

Ad(2)
The task is now to determine the conjugacy classes of V under G in order to obtain the number of orbits and fixed points under the action of G on $Cl(V)$. Since V is normal in G we have an action of G on V via conjugation. In order to obtain the G-classes in V we need to consider how $Cl(V)$, the conjugacy classes of V, behave under conjugation with x, because x generates G/V. Some of the classes may fall together whereas some may remain as they are under conjugation with V. We will first present the conjugacy classes of V under G in Table 4.7 with representatives and length. Subsequently we will give a proof that the table is correct.

Table 4.7: G-classes in V

name	representative	parameter	length	number
$C_{\gamma\delta}$	$c^\gamma d^\delta$	$\gamma, \delta \in I$	1	p^2
$V_{\beta\delta}$	$b^\beta d^\delta$	$\beta \in I^*, \delta \in I$	p	$p(p-1)$
$A_{\alpha\gamma}$	$a^\alpha c^\gamma$	$\alpha \in I^*, \gamma \in I$	p^2	$p(p-1)$
$Y_{\upsilon\beta\gamma}$	$y^\upsilon b^\beta c^\gamma$	$\upsilon \in I^*, \beta, \gamma \in I$	p^2	$p^2(p-1)$

Proof As already mentioned we will just have to investigate how the conjugacy classes of V behave under conjugation with x. We recall that the matrix corresponding to the conjugation with x^n is given by

$$M_x^n = \begin{pmatrix} 1 & n & \frac{n(n-1)}{2} & 0 \\ 0 & 1 & n & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for all $n \in \mathbb{N}$.
Furthermore, since $|G/V| = p$, we note that the length of a conjugacy class under G either is the same as it was under conjugation with V or it is multiplied by p.

Now we can start to investigate the different classes. We know that $Z(G) = \langle c, d \rangle$, hence we still have $\left\{c^\gamma d^\delta\right\}$, $\gamma, \delta \in I$ as classes of length 1 under conjugation with G.
However, applying the matrix M_x^n, we see that under conjugation with x^n, $n \in I$ we have $b^\beta d^\delta \sim b^\beta c^{n\beta} d^\delta$. Thus we obtain that $Cl(b^\beta d^\delta) = \left\{b^\beta c^\gamma d^\delta \mid \gamma \in I\right\}$, $\beta \in I^*, \delta \in I$ is a conjugacy class of length p under G.
Analogously we see that $a^\alpha c^\gamma \sim a^\alpha b^\alpha c^\gamma$ under conjugation with x. Looking at Table 4.4 we see that $Cl_V(a\alpha c\gamma) \neq Cl_V(a\alpha b\alpha c\gamma)$. This implies that $Cl(a^\alpha c^\gamma)$ has more than p elements under the action of G. As it had length p under conjugation with V it follows that it has length p^2 under conjugation with G. Looking at Table 4.4, the table of $Cl(V)$, we see that for every $\alpha \in I^*, \beta \in I$, the p conjugacy classes $A_{\alpha\beta\gamma}$, $\gamma \in I$ of V (see Table 4.4) fuse to one conjugacy class of G.

Now we have $y \sim ya^{-1}$ under conjugation with x. Hence the same as above happens to each of the conjugacy classes of the third row in Table 4.4. We find that for $\upsilon \in I^*, \beta, \gamma \in I$ the p classes $Y_{\upsilon\alpha\beta\gamma}, \alpha \in I$ fuse together to one G-conjugacy class with representative $y^\upsilon b^\beta c^\gamma$.
Summing up we see that we obtain $p^5 = |V|$ elements, hence the investigation of the G-classes in V is complete.

Ad(3)
We also have an action of G on $Cl(V)$. From Table 4.7 and from Table 4.4 we can deduce the number of fixed points and orbits from this action. We obtain:

(i) p^2 orbits of length 1, where a set of representatives are the classes $B_{0\gamma\delta}, \gamma, \delta \in I$,

(ii) $p(p-1)$ orbits of length p, where a set of representatives are the classes $B_{\beta 0\delta}, \beta \in I^*, \delta \in I$ (the class of $b^\beta d^\delta$ is conjugated to the classes $b^\beta c^\gamma d^\delta$, $\gamma \in I$),

(iii) $p(p-1)$ orbits of length p, where a set of representatives are the classes $A_{\alpha 0\gamma}, \alpha \in I^*, \gamma \in I$ (the class of $a^\alpha c^\gamma$ is conjugated to the classes $a^\alpha b^\beta c^\gamma$, $\beta \in I$),

(iv) $p^2(p-1)$ orbits of length p, where a set of representatives are the classes $Y_{\upsilon 0\beta\gamma}, \upsilon \in I^*, \beta, \gamma \in I$ (the class of $y^\upsilon b^\beta c^\gamma$ is conjugated to the classes $y^\upsilon a^\alpha b^\beta c^\gamma$, $\alpha \in I$).

All in all we obtain p^2 orbits of length 1 and $p^3 + p^2 - 2p$ orbits of length p. We again apply Lemma 2.3.11, Brauers permutation lemma, and conclude that the action of G on $\mathrm{Irr}_{\mathbb{C}}(V)$ yields p^2 orbits of length 1 and $p^3 + p^2 - 2p$ orbits of length p.

Ad(4)
Now we are concerned with the determination of the number of linear characters of G, the number of characters of degree p and of degree p^2.

Lemma 4.3.1 *Any $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ is either an extension of a character of V, i.e. $\chi_V \in \mathrm{Irr}_{\mathbb{C}}(V)$, or we have that $\chi = \varphi^G$ for some $\varphi \in \mathrm{Irr}_{\mathbb{C}}(V)$.*

Proof [I, (6.19), p.86].

Theorem 4.3.2 *G has exactly:*

- *p^2 characters of degree 1,*
- *$2p^3 - p^2 - 1$ irreducible characters of degree p, where $p^2(p-1)$ of them are extensions of characters from $\mathrm{Irr}_{\mathbb{C}}(V)$ and the rest, i.e. $p^3 - 1$ characters, are induced from a linear character of V,*
- *$(p-1)^2$ irreducible characters of degree p^2.*

Proof Obviously each linear character of G is an extension of a linear character of V. Furthermore we deduce from 2.4.3 that each linear character $\hat{\lambda} \in \mathrm{Irr}_{\mathbb{C}}(V)$ with inertia group $I_G(\hat{\lambda}) = G$ has exactly p different extensions of λ to a linear character of G.
From the proof of 3.3.1 we know that $|G/G'| = p^2$. Hence it follows that G has exactly p^2 linear characters. Therefore p of the p^2 orbits of length 1 from the action of G on $\mathrm{Irr}_{\mathbb{C}}(V)$ contain characters of degree 1, and the rest, i.e. $p(p-1)$ orbits, contain characters of degree p. Thus G has p^2 linear characters and $p^2(p-1)$ characters of degree p which are extensions of irreducible characters of V. All other characters of $\mathrm{Irr}_{\mathbb{C}}(G)$ are induced from a character of V which also means that their restriction to V is not irreducible.
How many of the remaining $p^3 + p^2 - 2p$ characters of $\mathrm{Irr}_{\mathbb{C}}(V)$, which induce to pairwise different irreducible characters of G, are of degree 1 and how many are of degree p? In order to answer this question we will use two facts:
Let $m = |\chi \in \mathrm{Irr}_{\mathbb{C}}(G) \,|\, \chi(1) = p, \chi_V \notin \mathrm{Irr}_{\mathbb{C}}(V)|$
and $n = |\chi \in \mathrm{Irr}_{\mathbb{C}}(G) \,|\, \chi(1) = p^2, \chi_V \notin \mathrm{Irr}_{\mathbb{C}}(V)|$.
Then

- $n + m = p^3 + p^2 - 2p$,

- $np^2 + mp^4 = p^6 - p^5 + p^4 - p^2$.

  This follows from the fundamental formula 3.1, which in this case yields $\sum_{\chi \in \mathrm{Irr}_{\mathbb{C}}(G)} \chi(1)^2 = p^6$. Hence

$$np^2 + mp^4 = \sum_{\substack{\chi \in \mathrm{Irr}_{\mathbb{C}}(G), \\ \chi(1) \neq 1, \chi_V \notin \mathrm{Irr}_{\mathbb{C}}(V)}} \chi(1)^2 = p^6 - \underbrace{p^2}_{(\dagger)} - p^2 \underbrace{(p^2(p-1))}_{(\dagger\dagger)}$$
$$= p^6 - p^5 + p^4 - p^2.$$

  Ad ($\dagger$): G has p^2 linear characters.

  Ad ($\dagger\dagger$): G has $p^2(p-1)$ characters of degree p which are induced from V.

From the second equation we deduce that $n + mp^2 = p^4 - p^3 + p^2 - 1$, hence $n = p^4 - p^3 + p^2 - 1 - mp^2$. Inserting this into the first equation we obtain $p^4 - p^3 + p^2 - 1 - mp^2 + m = p^3 + p^2 - 2p$.
Thus $m = (p-1)^2$ and $n = p^3 - 1$.
This means that G has $(p-1)^2$ irreducible characters of degree p^2 and $p^3 - 1$ irreducible characters of degree p which are induced from a linear character of V.

Ad(5)

In Table 4.8 we list the conjugacy classes of G and in Table 4.3we present the generic character table of G. We will afterwards first prove that the table of $Cl(G)$ is correct and subsequently we will prove that the given character table of G is correct.

Table 4.8: $Cl(G)$

name	representative	parameter	length	number
$C_{\gamma\delta}$	$c^\gamma d^\delta$	$\gamma, \delta \in I$	1	p^2
$B_{\beta\delta}$	$b^\beta d^\delta$,	$\beta \in I^*, \delta \in I$	p	$p(p-1)$
$A_{\alpha\gamma}$	$a^\alpha c^\gamma$,	$\alpha \in I^*, \gamma \in I$	p^2	$p(p-1)$
$Y_{\upsilon\beta\gamma}$	$y^\upsilon b^\beta c^\gamma$,	$\upsilon \in I^*, \beta, \gamma \in I$	p^2	$p^2(p-1)$
$X_{\upsilon\xi\delta}$	$y^\upsilon x^\xi d^\delta$,	$\xi \in I^*, \upsilon, \delta \in I$	p^3	$p^2(p-1)$

Notation 4.3.3

(i) Let ζ be a the complex root of unity of order p as in 4.2.1.

(ii) If $\Theta_{ijkm} \in \mathrm{Lin}_{\mathbb{C}}(V)$ is invariant in G, we will use the notation $\hat{\Theta}_{ijkm}$ for the extended linear character of $\mathrm{Irr}_{\mathbb{C}}(G)$, with respect to the construction given in the proof of 2.4.1.

(iii) By η we will denote the linear character of $\mathrm{Irr}_{\mathbb{C}}(G/V)$ with $\eta(x) = \zeta$. Furthermore we also consider η as character of G.

(iv) Let ρ_s be as in 4.2.1.

(v) For $l \in \{2, 3\}$ we define $\tau_{s,t}^{(l)} = \sum_{r \in I} \zeta^{sr + t\binom{r}{l}}$.

(vi) For $j, n \in I, l \in I^*$ we define $\chi_{j,l,n}$ as follows:
For $v \in V$ we define $\chi_{j,l,n}(v) = \lambda_{0,j,0,l}^{V}(v)$.
Now choose $r_{jl} \in I$ such that $j + r_{jl} \equiv 0 \pmod p$.
Then for $g = y^\upsilon x^\xi a^\alpha b^\beta c^\gamma d^\delta \in X_{\upsilon\xi\delta},\ \upsilon, \delta \in I,\ \xi \in I^*$ and α, β, γ such that $g \in X_{\upsilon\xi\delta}$ we define

$$\chi_{j,l,n}(g) = \begin{cases} 0, \text{ if } \upsilon \neq \xi \cdot r_{jl}, \\ p\zeta^{l\delta + n\upsilon} \cdot \tau^{(2)}_{j\binom{\xi}{2}, l\xi}, \text{ if } \upsilon = \xi \cdot r_{jl} \end{cases}.$$

Table 4.9: **The generic character table of G**

centralizer order			p^6	p^5	p^4	p^4	p^4
conjugacy class			$C_{\gamma\delta}$	$B_{\beta\delta}$	$A_{\alpha\gamma}$	$Y_{\upsilon\beta\gamma}$	$X_{\upsilon\xi\delta}$
parameter			$\gamma, \delta \in I$	$\beta \in I^*, \delta \in I$	$\alpha \in I^*, \gamma \in I$	$\upsilon \in I^*, \beta, \gamma \in I$	$\xi \in I^*, \upsilon, \delta \in I$
#	parameter		p^2	$p(p-1)$	$p(p-1)$	$p^2(p-1)$	$p^2(p-1)$
p^2	$m, n \in I$	$\hat{\Theta}_{0,0,0,m} \cdot \eta^n$	1	1	1	$\zeta^{n\upsilon}$	$\zeta^{m\xi+n\upsilon}$
$p^2(p-1)$	$j, n \in I, l \in I^*$	$\chi_{0,j,0,l,n}$	$p\zeta^{l\delta}$	$p\zeta^{j\beta+l\delta}$	0	0	0, if $\upsilon \neq \xi \cdot r_{jl}$ $\zeta^{l\delta+n\upsilon} \cdot \tau^{(2)}_{j\binom{\xi}{2},l\xi}$, if $\upsilon = \xi \cdot r_{jl}$
$p-1$	$i \in I^*$	$\Theta^G_{i,0,0,0}$	p	p	$p\zeta^{i\alpha}$	0	0
$p^2(p-1)$	$i, m \in I, k \in I^*$	$\Theta^G_{i,0,k,m}$	$p\zeta^{k\gamma}$	0	$\zeta^{i\alpha+k\gamma} \cdot \rho_{k\alpha}$	$\tau^{(3)}_{k\beta-i\upsilon,-k\upsilon}$	0
$p(p-1)$	$j \in I^*, m \in I$	$\Theta^G_{0,j,0,m}$	p	$p\zeta^{j\beta}$	0	$\zeta^{j\beta+m\upsilon} \cdot \rho_{-j\upsilon}$	0
$(p-1)^2$	$k, l \in I^*$	$\lambda^G_{0,0,k,l}$	$p^2\zeta^{k\gamma+l\delta}$	0	0	0	0

Proof of Table 4.8:

We first point out that we already deduced the G-conjugacy classes contained in V in Table 4.7. This are the first four rows of the table. Hence we just have to investigate $G\backslash V$ under the action of G. Now let us determine the conjugacy class of an element $g_{rst} := y^r x^s d^t \in G\backslash V$ for some $r, t \in I, s \in I^*$. We shall use the following Lemma.

Fix $l \in \mathbb{N}$. Then for all $r, s, t \in I$ we have

Lemma 4.3.4 *(1)* $y^{-l} g_{rst} y^l = y^r x^s a^{ls} b^{l\frac{s(s-1)}{2}} c^{l\frac{s(s-1)(s-2)}{6}} d^{s\frac{l(l-1)}{2}+t}$

(2) $b^l g_{rst} b^{-l} = y^r x^s c^{sl} d^t$

(3) $a^l g_{rst} a^{-l} = y^r x^s b^{sl} c^{l\frac{s(s-1)}{2}} d^{rl+t}$

Proof We will prove these properties briefly:

Let $r, s, t \in \mathbb{N}$.

Ad(1) First we show by induction that for all $s \in \mathbb{N}$ we have $y^{-l} x^s y^l = x^s a^{sl} b^{l\frac{s(s-1)}{2}} c^{l\frac{s(s-1)(s-2)}{6}} d^{s\frac{l(l-1)}{2}}$:

Using induction (4) at the end of the proof of 3.3.1 we see that the claim is true for $s = 1$.

Let us assume that the claim is correct for some $s \in \mathbb{N}$ (†). Now do the step $s \mapsto s + 1$:

We have

$$
\begin{aligned}
&y^{-l} x^{s+1} y^l = (y^{-l} x^s y^l)(y^{-l} x y^l) \\
&\overset{(\dagger)}{=} \left(x^s a^{sl} b^{l\frac{s(s-1)}{2}} c^{l\frac{s(s-1)(s-2)}{6}} d^{s\frac{l(l-1)}{2}}\right) \left(x a^l d^{\frac{l(l-1)}{2}}\right) \\
&\overset{(c,d\in Z(G))}{=} x^s (x x^{-1}) a^{sl} (x x^{-1}) b^{l\frac{s(s-1)}{2}} x c^{l\frac{s(s-1)(s-2)}{6}} d^{s\frac{l(l-1)}{2}} a^l d^{\frac{l(l-1)}{2}} \\
&= x^{s+1} (\underbrace{x^{-1} a x}_{=ab})^{sl} (\underbrace{x^{-1} b x}_{=bc})^{l\frac{s(s-1)}{2}} a^l c^{l\frac{s(s-1)(s-2)}{6}} d^{(s+1)\frac{l(l-1)}{2}} \\
&= x^{s+1} a^{(s+1)l} b^{l\frac{s(s+1)}{2}} c^{l\frac{s(s-1)(s+1)}{6}} d^{(s+1)\frac{l(l-1)}{2}}.
\end{aligned}
$$

Hence the claim is correct for $s + 1$ and therefore for all $s \in \mathbb{N}$.

Now using the above induction we obtain $y^{-l} g_{rst} y^l = y^r y^{-l} x^s y^l d^t = y^r x^s a^{sl} b^{l\frac{s(s-1)}{2}} c^{l\frac{s(s-1)(s-2)}{6}} d^{s\frac{l(l-1)}{2}+t}$ and the claim is proven.

Ad(2) We know that x^s acts on G' via the matrix

$$M_x^s = \begin{pmatrix} 1 & s & \frac{s(s-1)}{2} & 0 \\ 0 & 1 & s & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We obtain $x^{-s}b^l x^s = b^l c^{sl}$, hence we have $b^l x^s b^{-l} = x^s c^{ls}$. It follows that $b^l g_{rst} b^{-l} = b^l y^r x^s d^t b^{-l} = y^r(b^l x^s b^{-l})d^t = y^r x^s c^{ls} d^t$ and the claim is proven.

Ad(3) We further know that y^r acts on G' via the matrix

$$M_y^r = \begin{pmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Analogously to case (2) we obtain $x^{-s}a^l x^s = a^l b^{sl} c^{l\frac{s(s-1)}{2}}$ as well as $y^{-r}a^l y^r = a^l d^{rl}$, hence we have $a^l x^s a^{-l} = x^s b^{sl} c^{l\frac{s(s-1)}{2}}$ as well as $a^l y^r a^{-l} = y^r d^{lr}$. It follows that $a^l g_{rst} a^{-l} = (a^l y^r a^{-l})(a^l x^s a^{-l}) a^l d^t a^{-l} = y^r x^s b^{ls} c^{l\frac{s(s-1)}{2}} d^{rl+t}$ and the claim is proven.

With the information gained we now prove that the length of the conjugacy class $Cl(g_{rst})$ is at least equal to p^3. We show that $g_{rst} \sim y^r x^s a^\alpha b^\beta c^\gamma d^{\tilde{t}}$ for all $\alpha, \beta, \gamma \in I$ and for some $\tilde{t} \in I$. The value of $\tilde{t}$ does not matter here. Now fix $\alpha, \beta, \gamma \in I$.
Using (1) choose $l \in I$ such that $y^{-l} g_{rst} y^l = y^r x^s a^\alpha z$ for some $z \in \langle b, c, d \rangle$.
Using (3) choose $l' \in I$ such that $b^{-l'} y^{-l} g_{rst} y^l b^{l'} = y^r x^s a^\alpha b^\beta z'$ for some $z' \in \langle c, d \rangle$.
Using (2) choose $l'' \in I$ such that $a^{-l''} b^{-l'} y^{-l} g_{rst} y^l b^{l'} a^{l''} = y^r x^s a^\alpha b^\beta c^\gamma d^{\tilde{t}}$ for some $\tilde{t} \in I$.
Thus $|Cl(g_{rst})| \geq p^3$. However we also have $|Cl(g_{rst})| \leq p^3$. This is because

$Z(G) = \langle c, d \rangle \subseteq C_G(g_{rst})$ of g_{rst} in G as well as $\langle g_{rst} \rangle \subseteq C_G(g_{rst})$. Hence $\langle g_{rst}, c, d \rangle \subseteq C_G(g_{rst})$ and it follows that $|C_G(g_{rst})| \geq p^3$, i.e. $|Cl(g_{rst})| = [G\colon C_G(g_{rst})] \leq p^3$.

All in all we conclude that $\{Cl(g_{rst}) \,|\, r, t \in I, s \in I^*\}$ yield $p^2(p-1)$ different conjugacy classes of length p^3. Summing up the lengths and numbers of conjugacy classes of the above table we see that we obtain $p^6 = |G|$ elements. Hence the investigation of the conjugacy classes of G is complete.

In order to prove that Table 4.3 is correct we will begin with the determination of the linear characters of G.

Theorem 4.3.5

$$\mathrm{Lin}_{\mathbb{C}}(G) := \left\{ \hat{\Theta}_{0,0,0,m} \cdot \eta^n \,|\, m, n \in I \right\}.$$

Proof Every linear character of G is an extension of a linear character $\Theta_{i,j,k,m} \in \mathrm{Lin}_{\mathbb{C}}(V)$. Hence we need to check which linear characters of V we can extend to G, i.e. which linear characters of V have inertia group G. These are just the ones which are trivial when restricted to G', i.e. this is the set $\{\Theta_{0,0,0,m} \,|\, m \in I\}$.
From 2.4.2 and 2.4.3 we now deduce that every linear character of G is of the form $\hat{\Theta}_{0,0,0,m} \cdot \eta^n$, $m, n \in I$.
We have $|G/V| = p$, hence $|\mathrm{Irr}_{\mathbb{C}}(G/V)| = p$. We already saw that there are exactly p characters $\Theta \in \mathrm{Irr}_{\mathbb{C}}(V)$ satisfying $\Theta_{G'} = 1_{G'}$. Thus we found exactly $p \cdot |\mathrm{Irr}_{\mathbb{C}}(U/G')| = p^2$ linear characters of G which means we have found them all.

Now we will determine the irreducible characters of G of degree p. First we investigate the irreducible characters of G of degree p which are induced from a linear character of V. From 4.3.2 we know that these are precisely $p^3 - 1$ characters.

Let us therefore recall the linear characters of V. These are given by:

$$\mathrm{Lin}_{\mathbb{C}}(V) = \{\Theta_{i,j,k,m} \,|\, i,j,k,m \in I\}$$

, where $\Theta_{i,j,k,m} = \hat{\lambda}_{i,j,k,0} \cdot \varepsilon^m$.

Lemma 4.3.6 *Let $\alpha, \beta, \gamma, \delta, \upsilon \in I$. Then*
$x^{-r} y^{\upsilon} a^{\alpha} b^{\beta} c^{\gamma} d^{\delta} x^r = y^{\upsilon} a^{\alpha - \upsilon r} b^{\alpha r + \beta - \upsilon \binom{r}{2}} c^{\alpha \binom{r}{2} + \beta r + \gamma - \upsilon \binom{r}{3}} d^{\alpha r + \delta}$ *for all $r \in \mathbb{N}$.*

Proof Induction.

Notation 4.3.7 Let in the following $\bar{x} := x^{-1}$.

Corollary 4.3.8 $(\Theta_{i,j,k,m})^{\bar{x}^r} = \Theta_{i+jr+k\binom{r}{2}, j+kr, k, m-(ir+j\binom{r}{2}+k\binom{r}{3})}$ *for all $i,j,k,m \in I$.*

Proof Let $v = y^{\upsilon} a^{\alpha} b^{\beta} c^{\gamma} d^{\delta} \in V,\ \upsilon, \alpha, \beta, \gamma, \delta \in I$. We obtain:
$(\Theta_{i,j,k,m})^{\bar{x}^r}(v) = \hat{\lambda}_{i,j,k,0}(x^{-r} v x^r) \cdot \varepsilon^m (x^{-r} v x^r) \overset{(4.3.6)}{=}$
$\hat{\lambda}_{i,j,k,0}(y^{\upsilon} a^{\alpha - \upsilon r} b^{\alpha r + \beta - \upsilon \binom{r}{2}} c^{\alpha \binom{r}{2} + \beta r + \gamma - \upsilon \binom{r}{3}} d^{\alpha r + \delta}) \cdot \varepsilon^m(y^{\upsilon}) =$
$\Theta_{i+jr+k\binom{r}{2}, j+kr, k, m-(ir+j\binom{r}{2}+k\binom{k}{3})}(v)$.

Corollary 4.3.9 *$\Theta_{i,j,k,m} \in \mathrm{Irr}_{\mathbb{C}}(V)$, $i,j,k,m \in I$, induces to an irreducible character of G if and only if $i \neq 0$ or $j \neq 0$ or $k \neq 0$.*

Theorem 4.3.10 *For $i,j,k,m \in I$ we define $S_{i,j,k,m} := \left\{\Theta^G_{i,j,k,m}\right\}$.*
Then $\{\chi \in \mathrm{Irr}_{\mathbb{C}}(G) \,|\; , \chi$ is induced from a linear character of $V\} = \{S_{i,0,k,m} \,|\, i,m \in I, k \in I^*\} \cup \{S_{0,j,0,m} \,|\, j \in I^*, m \in I\} \cup \{S_{i,0,0,0} \,|\, i \in I^*\}$.

The value at an element $g = y^{\upsilon} x^{\xi} a^{\alpha} b^{\beta} c^{\gamma} d^{\delta} \in G$ of $\chi \in S_{i,j,k,m}$ in each of the cases above is given as follows:

We have $\chi(g) = 0$, if $g \notin V$, i.e. if $\xi \neq 0$.

Let now $\xi = 0$, i.e. $g \in V$.

For $\chi \in \{S_{i,0,k,m} \,|\, i, m \in I, k \in I^*\}$ we have
$\chi(g) = \zeta^{i\alpha+k\gamma+mv} \sum_{r\in I} \zeta^{k\alpha\binom{r}{2}+k\beta r-ivr-kv\binom{r}{3}}$.

For $\chi \in \{S_{0,j,0,m} \,|\, j \in I^*, m \in I\}$ we have
$\chi(g) = \zeta^{j\beta+mv} \sum_{r\in I} \zeta^{j\alpha r-jv\binom{r}{2}}$.

For $\chi \in \{S_{i,0,0,0} \,|\, i \in I^*\}$ we have
$\chi(g) = \zeta^{i\alpha} \sum_{r\in I} \zeta^{ivr}$.

(In order to fill in the character values into the table for these cases also consider 4.2.7).

Proof Follows from 4.3.9 and 4.3.8.

Now we are concerned with characters $\chi \in \mathrm{Irr}_{\mathbb{C}}(G),\ \chi(1) = p$ which are extensions of irreducible characters of V. From 4.3.2 we know that we are looking for precisely $p^2(p-1)$ characters.

Lemma 4.3.11 *For all $j \in I, l \in I^*$ we have $I_G(\lambda^V_{0,j,k,l}) = G$ if and only if $k = 0$.*

Proof Let $j \in I, l \in I^*$. Since xV generates G/V we have that $I_G(\lambda^V_{0,j,k,l}) = G$ if and only if $\lambda^V_{0,j,k,l}$ is invariant under x.
From Table 4.6 we know that the value of $\lambda^V_{0,j,k,l}$ at an element $v = y^v a^\alpha b^\beta c^\gamma d^\delta \in V$ is given by:
$\lambda^V_{0,j,k,l}(v) = \begin{cases} 0, \text{ if } v \notin \langle b, c, d\rangle, \\ p\zeta^{j\beta+k\gamma+l\delta}, \text{ otherwise.} \end{cases}$

Thus we have to check that $\lambda^V_{0,j,k,l}\left(x^{-1}(b^\beta c^\gamma d^\delta)x\right) = \lambda^V_{0,j,k,l}\left(b^\beta c^\gamma d^\delta\right)$ for all $\beta, \gamma, \delta \in I$. However $x^{-1}(b^\beta c^\gamma d^\delta)x = b^\beta c^{\gamma+\beta} d^\delta$ and we obtain the equation

$\lambda^V_{0,j,k,l}\left(b^\beta c^{\gamma+\beta} d^\delta\right) = \lambda^V_{0,j,k,l}\left(b^\beta c^\gamma d^\delta\right)$. Since $\lambda_{0,j,k,l}(c) = \zeta^k$ we conclude that $\lambda_{0,j,k,l}$ is invariant under x if and only if $k = 0$.

Corollary 4.3.12 *For all $j \in I, l \in I^*$ we have that $\lambda^V_{0,j,0,l}$ has exactly p different extensions to an irreducible character of G.*

We will now determine the extensions of $\lambda^V_{0,j,0,l}$ for $j \in I, l \in I^*$.

Lemma 4.3.13

(1) $(y^r x)^i = y^{ri} x^i a^{ri(i-1)/2} b^{ri(i-1)(2i-1)/6} c^{ri(i-1)(i-2)(3i-1)/24} d^{i(i-1)(2r(i-1)-1)r/4}$.

(2) $y^r x \in G$ *has order* p *for all* $r \in I$.

Proof (1) Induction.

(2) Follows from part (1) since $p \geq 5$.

Corollary 4.3.14 *For $r \in I^*$ we define $W_r = \langle y^r x, a, b, c, d \rangle$.*
Then $W_r \cong \langle y^r x \rangle \ltimes G' \cong C_p \ltimes C_p^4$

Lemma 4.3.15 *Let $j \in I, l \in I^*$ and let $r_{jl} \in I$ be such that $j + r_{jl} l \equiv 0$ (mod p). Then for $\lambda_{0,j,0,l} \in \mathrm{Irr}_{\mathbb{C}}(G')$ we have $I_{W_{r_{jl}}}(\lambda_{0,j,0,l}) = W_{r_{jl}}$ and $\lambda_{0,j,0,l}$ is extendable to a linear character of $W_{r_{jl}}$.*
We will use the notation $\hat{\lambda}_{0,j,0,l}$ for the extended linear character of $\mathrm{Irr}_{\mathbb{C}}(W_{r_{jl}})$ with respect to the construction given in the proof of 2.4.1.

Proof Fix $j \in I, l \in I^*$ and let $z = (y^{r_{jl}} x)^{-1}$. Using the relations of G we obtain for $\alpha, \beta, \gamma, \delta \in I$ that $(\lambda_{0,j,0,l})^z(a^\alpha b^\beta c^\gamma d^\delta) = \lambda_{0,j,0,l}(a^\alpha b^{\alpha+\beta} c^{\beta+\gamma} d^{r_{jl}\alpha+\delta}) = \lambda_{0,j,0,l}(a^\alpha b^\beta c^\gamma d^\delta) \cdot \zeta^{j+r_{jl}l}$. Since $j + r_{jl} l \equiv 0$ (mod p) we conclude that $\lambda_{0,j,0,l}$ is invariant in $W_{r_{jl}}$. Using 4.3.14 the second assertion follows immediately from 2.4.1.

Lemma 4.3.16

(1) *For all* $r_{jl} \in I^*$ *there is a character* $\varepsilon_{r_{jl}} \in \mathrm{Lin}_{\mathbb{C}}(G/W_{r_{jl}})$ *with* $\varepsilon_{r_{jl}}(y) = \zeta$.

(2) $\mathrm{Lin}_{\mathbb{C}}(G/W_{r_{jl}}) = \left\{ \varepsilon_{r_{jl}}^n \,|\, n \in I \right\}$ *for all* $r_{jl} \in I^*$.

(3) *Let* $r_{jl} \in I$ *be such that* $j + r_{jl}l \equiv 0$. *Then* $\left\{ \hat{\lambda}_{0,j,0,l} \cdot \varepsilon_{r_{jl}}^n \,|\, j, n \in I, l \in I^* \right\}$ *are* $p^2(p-1)$ *different linear characters of* $W_{r_{jl}}$.

Proof

(1) Since $y \notin W_{r_{jl}}$ we have that $\{y^r \,|\, r \in I\}$ is a set of coset representatives for $G/W_{r_{jl}}$. Now the claim follows since $G/W_{r_{jl}} \cong C_p$ and thus $\mathrm{Irr}_{\mathbb{C}}(G/W_{r_{jl}}) \cong C_p$.

(2) This follows from part (1).

(3) This follows from part (2) and 2.4.3.

Theorem 4.3.17 *Let* $j, n \in I, l \in I^*$ *and let* $r_{jl} \in I^*$ *such that* $j + r_{jl}l \equiv 0$ *(mod p). Then the linear character* $\hat{\lambda}_{0,j,0,l} \cdot \varepsilon_{jl}^n$ *of* $W_{r_{jl}}$ *induces to an irreducible character of* G.
Furthermore $\hat{\lambda}_{0,j,0,l} \cdot \varepsilon_{jl}^n = \chi_{j,l,n}$ *and* $\chi_{j,l,n}$ *is an extension of* $\lambda_{0,j,0,l}^V$, *where* $\chi_{j,l,n}$ *is defined in 4.3.3. The characters* $\chi_{j,l,n}$, $j, n \in I, l \in I^*$ *are pairwise different.*

Proof Fix $j, n \in I, l \in I^*$ and let us look at the orbit of $\hat{\lambda}_{0,j,0,l} \cdot \varepsilon_{jl}^n$ under $G/W_{r_{jl}}$. We already saw that $\{y^t \,|\, t \in I\}$ is a set of representatives for the cosets of $G/W_{r_{jl}}$.
Hence the orbit of $\hat{\lambda}_{0,j,0,l} \cdot \varepsilon_{jl}^n$ is given by $\left\{ (\hat{\lambda}_{0,j,0,l})^{y^t} \cdot (\varepsilon_{jl}^n)^{y^t} \,|\, t \in I^t \,|\, t \in I \right\}$.
Now $(\hat{\lambda}_{0,j,0,l})^{y^{-t}} = \hat{\lambda}_{tl,j,0,l}$ and ε_{jl}^n is invariant under y^t. Hence $\hat{\lambda}_{0,j,0,l} \cdot \varepsilon_{jl}^n$ is not invariant in G and therefore it induces to an irreducible character of G.
Using the fact that $V \cdot W_{r_{jl}} = G$ and $V \cap W_{r_{jl}} = G'$ we conclude by applying 2.2.9, Mackeys Subgroup Theorem, that $\hat{\lambda}_{0,j,0,l} \cdot \varepsilon_{jl}^n$ is an extension of $\lambda_{0,j,0,l}^V$.

In order to finally see that $\hat{\lambda}_{0,j,0,l} \cdot \varepsilon_{jl}^{n} = \chi_{j,l,n}$ we determine the value of $\hat{\lambda}_{0,j,0,l} \cdot \varepsilon_{jl}^{n}$ at $g = y^{\upsilon}x^{\xi}a^{\alpha}b^{\beta}c^{\gamma}d^{\delta} \in X_{\upsilon\xi\delta}$ for $\upsilon, \delta \in I, \xi \in I^*$ and α, β, γ such that $g \in X_{\upsilon\xi\delta}$:
Using 4.3.13 we see that $g \notin W_{r_{jl}}$ if $\upsilon \neq \xi \cdot r_{jl}$. Hence we obtain:
$(\hat{\lambda}_{0,j,0,l} \cdot \varepsilon_{jl}^{n})^{G}(g) = 0,$ if $\upsilon \neq \xi \cdot r_{jl}$.
Let now $\upsilon = \xi \cdot r_{jl}$. Using 4.3.4 we obtain:
$(\hat{\lambda}_{0,j,0,l} \cdot \varepsilon_{jl}^{n})^{G}(g) = (\hat{\lambda}_{0,j,0,l} \cdot \varepsilon_{jl}^{n})^{G}(y^{\upsilon}x^{\xi}d^{\delta}) =$
$\sum_{t\in I}(\hat{\lambda}_{0,j,0,l}(y^{-t}y^{\upsilon}x^{\xi}d^{\delta}y^{t}) \cdot \varepsilon_{jl}^{n}(y^{-t}y^{\upsilon}x^{\xi}d^{\delta}y^{t}) =$
$\zeta^{n\upsilon}\sum_{t\in I}\hat{\lambda}_{0,j,0,l}(y^{\upsilon}x^{\xi}a^{t\xi}b^{t\binom{\xi}{2}}c^{t\binom{\xi}{3}}d^{\delta+\xi\binom{t}{2}}) =$
$\zeta^{l\delta+n\upsilon}\sum_{t\in I}\zeta^{jt\binom{\xi}{2}+l\xi\binom{t}{2}} = \zeta^{l\delta+n\upsilon} \cdot \tau^{(2)}_{j\binom{\xi}{2},l\xi} = \chi_{j,l,n}(g).$

Considering the values of $\chi_{j,l,n}$ we see that $\chi_{j,l,n},\ j,n \in I, l \in I^*$ are pairwise different characters.

The investigation of the irreducible characters of degree p is now complete.
It only remains to determine the irreducible characters of degree p^2 of G. We recall that we are looking for $(p-1)^2$ characters.

Lemma 4.3.18 *Each irreducible character χ of degree p^2 is induced by a linear character of G'.*

Proof We consider an irreducible character $\lambda \in \mathrm{Irr}_{\mathbb{C}}(G')$ such that $(\chi_{G'}, \lambda) \neq 0$. Using Frobenius reciprocity we obtain $\left(\chi, \lambda^{G}\right) \neq 0$. Since χ is irreducible and both χ and λ^{G} are of degree p we conclude that $\chi = \lambda^{G}$.

Theorem 4.3.19

$$\left\{\chi \in \mathrm{Irr}_{\mathbb{C}}(G) \,|\, \chi(1) = p^2\right\} = \left\{\lambda^{G}_{0,0,k,l} \,|\, k, l \in I^*\right\}.$$

For $g = y^{\upsilon}x^{\xi}a^{\alpha}b^{\beta}c^{\gamma}d^{\delta} \in G$ we have $\lambda^{G}_{0,0,k,l}(g) = \begin{cases} 0, & \text{if } g \notin \langle c, d\rangle, \\ p\zeta^{k\gamma+l\delta}, & \text{otherwise.} \end{cases}$

Proof First let us prove that $\lambda_{0,0,k,l}^G \in \mathrm{Irr}_{\mathbb{C}}(G)$ for all $k, l \in I^*$. By 2.2.6 we obtain $\lambda_{0,0,k,l}^G = \left(\lambda_{0,0,k,l}^V\right)^G$. From 4.2.11 it we know that $\lambda_{0,0,k,l}^V \in \mathrm{Irr}_{\mathbb{C}}(V)$ for all $k, l \in I^*$. Furthermore we follow from 4.3.11 that a character $\lambda_{0,j,k,l}^V \in \mathrm{Irr}_{\mathbb{C}}(V)$ is invariant in G if and only if $k = 0$. However in our case we have $k \neq 0$, and we conclude that $I_G\left(\lambda_{0,0,k,l}^V\right) = V$. This implies that $\lambda_{0,0,k,l}^V$ induces to an irreducible character of G.

The center of G is given by $\langle c, d\rangle$. Hence $\lambda_{0,0,k,l}^G(c) = p^2\lambda_{0,0,k,l}(c) = p^2\zeta^k$ and $\lambda_{0,0,k,l}^G(d) = p^2\lambda_{0,0,k,l}(d) = p^2\zeta^l$. Therefore each of the characters above yields a different character $\lambda_{0,0,k,l}^G \in \mathrm{Irr}_{\mathbb{C}}(G)$. Furthermore we have $k, l \in I^*$, hence we obtain $(p-1)^2$ characters from the construction above. This is exactly the number of irreducible characters of G of degree p^2 and we conclude that we found all the characters we were looking for.

Let us determine the value of $\lambda_{0,0,k,l}^G$ at g:
Since $\lambda_{0,0,k,l} \in \mathrm{Irr}_{\mathbb{C}}(G')$ we clearly have $\lambda_{0,0,k,l}^G(g) = 0$, if $g \notin G'$, i.e. if $\xi \neq 0$ or if $\upsilon \neq 0$.
Now let $g \in G'$.
We have that $\{y^s x^r \mid r, s \in I\}$ is a set of coset representatives for G/G'. Hence we obtain

$$\lambda_{0,0,k,l}^G(g) = \sum_{r\in I}\sum_{s\in I}\lambda_{0,0,k,l}(x^{-r}y^{-s}gy^sx^r) =$$
$$\sum_{r\in I}\sum_{s\in I}\lambda_{0,0,k,l}(x^{-r}y^{-s}a^\alpha b^\beta c^\gamma d^\delta y^s x^r) \overset{(Induction)}{=}$$
$$\sum_{r\in I}\sum_{s\in I}\lambda_{0,0,k,l}(x^{-r}a^\alpha b^\beta c^\gamma d^{\delta+s\alpha}x^r) = \sum_{s\in I}\zeta^{s\alpha}\sum_{r\in I}\lambda_{0,0,k,l}(x^{-r}a^\alpha b^\beta c^\gamma d^\delta x^r)$$
$$\overset{(4.2.7)}{=} \begin{cases} 0, \text{ if } \alpha \neq 0 \\ p\sum_{r\in I}\lambda_{0,0,k,l}(b^{\beta+r\alpha}c^{\gamma+r\beta}d^\delta), \text{ if } \alpha = 0. \end{cases}$$

Together we obtain $\lambda_{0,0,k,l}^G(g) = \begin{cases} 0, \text{ if } g \notin \{c, d\}, \\ p\zeta^{k\gamma+l\delta}, \text{ otherwise.} \end{cases}$

The investigation of the irreducible characters of G is now complete.

Let us finally state a very nice conclusion, which we easily obtain by looking at the generic character table of G.

Theorem 4.3.20 *All irreducible characters of G of degree p^2 are tensor decomposable.*

Proof We look at Table 4.3, the generic character table of G. Every irreducible character of degree p^2 (confer the last row) is a product of a character from the second row with a character from the fourth row, hence it is tensor decomposable.

Chapter 5

GAP

During my work on this thesis I gained some valuable results using the computer algebra system GAP [GAP] whose development has been started at Lehrstuhl D für Mathematik, RWTH Aachen. GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory. It provides a programming language, a library of thousands of functions implementing algebraic algorithms written in the GAP language as well as large data libraries of algebraic objects. For this work the most useful one was the SmallGroups library, which is written by E. A. O'Brien, B. Eick, and H. U. Besche.

The SmallGroups library contains all groups of certain "small" orders. The word 'small' is used to mean orders less than a certain bound and orders whose prime factorisation is small in some sense. The groups are sorted by their orders and they are listed up to isomorphism; that is, for each of the available orders a complete and irredundant list of isomorphism type representatives of groups is given. Currently, the library contains the following groups:

- those of order at most 2000 except 1024 (423 164 062 groups)
- those of cubefree order at most 50 000 (395 703 groups)
- those of order p^n for $n \leq 6$ and an arbitray prime p

- those of order $q^n \cdot p$ where q^n divides $2^8, 3^6, 5^5$ or 7^4 and p is an arbitrary prime not equal to q
- those of squarefree order
- those whose order factorises into at most 3 primes

The library also has an identification function: it returns the library number of a given group. Currently, this function is available for all orders in the library except for the orders 512 and 1536 and for p^5 and p^6 above 2000.

I used GAP to investigate p-groups, looking for the existence of irreducible tensor decomposable characters. In the following we shall present observations and results as well as the way we gained those by using GAP.

5.1 Check if there are groups of certain order possessing a tensor decomposable character

In order to check if there are groups of order n possessing a tensor decomposable character I wrote a small GAP-program consisting of defined GAP routines. This program is particularly written for groups of order p^5 and p^6. In the following we will first see the GAP code, which already contains some comments, indicated by #. Subsequently we will give a more detailed explanation of the source code and provide an overview which results I gained using this program.

```
tensorsmall:=function(n, d1)
# returns a list S consisting of the ids of the groups of
# order n which possess a tensor decomposable character.
# The tensored character then has degree d1*d1 and is the
# product of two characters, each of degree d1.

```

```
local S, ids, i, g, ct, irr, ChOfSmallDeg, ChOfLargeDeg,
      TensCh, M, Nil;

S:=[];
ids:=IdsOfAllSmallGroups(n);

for i in [1..Length(ids)] do
    g:=SmallGroup(ids[i]);

    if IsAbelian(g) = false
      then
      ct:=CharacterTable(g);
      irr:=Irr(ct);
      ChOfSmallDeg:=Filtered(irr, c -> Degree(c) = d1);
      ChOfLargeDeg:=Filtered(irr, c -> Degree(c) = d1*d1);

      if (Length(ChOfLargeDeg)>0 and Length(ChOfSmallDeg)>0)
        then
        TensCh:=Tensored(ChOfSmallDeg,ChOfSmallDeg);
        M:=MatScalarProducts(TensCh, ChOfLargeDeg);
        Nil:=NullMat(Length(ChOfLargeDeg),Length(TensCh));

        if((M=Nil) = false)
          then
          Append(S,[ids[i]]);
        fi;
      fi;
    fi;
od;

return S;
end;;
```

After defining local variables in Line 7 we generate by `IdsOfAllSmallGroups` a list of the library numbers of all groups of order n (Line 11), where n is given by the input of the user.

Now for every library number we perform the following procedure:

Generate the group corresponding to the respective library number by the command `SmallGroup` (Line 14).

Next we want to reduce the number of groups we are working with by fast methods. Therefore we first check by the command `IsAbelian` whether the generated group is abelian or not (Line 16).

We continue to investigate this group just in case it is not abelian, since abelian groups only have irreducible characters of degree 1, hence they do not have tensor decomposable characters.

We go on and generate the character table of g in Line 18.

However this does not give us all information we need so that we generate all irreducible characters of g with the command `Irr(ct)` (Line 19).

Subsequentely, in Lines 20 and 21 respectively, we produce lists `ChOfSmallDeg` and `ChOfLargeDeg` which consist of all irreducible characters of g of degree $d1$ and $d1 * d1$ respectively.

Now we check by `Length(ChOfLargeDeg)>0 and Length(ChOfSmallDeg)>0)` (Line 23), if the group possesses at least one irreducible character of degree $d1$ and one of degree $d1 * d1$. For groups of order p^5 or p^6 the only possibility for a tensor decomposable character is a character of degree p^2, which then must be a product of two characters each of degree p. Since every group has at least one linear character, the trivial character, we are looking for a group which has irreducible characters of degree $1, p$ and p^2. Thus we can eliminate all groups whose irreducible characters have only two or less different degrees.

If both lists contain at least one entry, which means that there is at least one irreducible character of degree $d1$ and one of degree $d1 * d1$, then we investigate if one of the characters in the list `ChOfLargeDeg` is a product of two characters contained in `ChOfSmallDeg` (Lines 24-29). This will be done using the following commands:

The function `Tensored` (Line 25) creates a new list of characters out of the parameters we pass on, i.e. out of two lists of characters. The result is a list containing all products of characters of the two lists we passed on. This list, named here with the variable `TensCh`, is now passed on together with the list `ChOfLargeDeg` to the function `MatScalarProducts` (Line 26). We obtain a matrix whose entries are the scalar products of each pair of characters, where one of the characters comes from `TensCh` and the other one from `ChOfLargeDeg`. Hence we compute scalar products of two characters both of degree $d1^2$, where the one from the list `ChOfLargeDeg` is an irreducible character. Hence the only possible result for each entry of this matrix is 1 or 0. We obtain 1 if and only if the two characters are the same, and 0, if and only if they are different from each other. Comparing this matrix with the zero matrix (Line 29) we can test, if one of the characters in `ChOfLargeDeg` is a product of two characters contained in `ChOfSmallDeg`. In case the matrix `M` of all scalar products is not the zero matrix we append the library number of the respective group we were working with in this step to the list `S` (Line 31). Finally `S` contains all library numbers of groups of order n which have a tensor decomposable character which is of degree $d1 \cdot d1$ and we return `S` (Line 37).

<u>Results</u>:

First I used the program for all groups of order p^5 for various primes p and checked, if any of them possessed a tensor decomposable character. However for all p-groups I considered, which were all groups of order p^5 with $p \leq 17$, I saw that neither of them has such a character. Hence I wondered whether this is the case for all groups of order p^5, where p is an arbitrary prime number and tried to prove this claim which finally worked out. The proof can be seen in 3.1.1.
After having proven that there are no groups of order p^5 which have a tensor decomposable character I started to investigate groups of order p^6. Obviously, using the above program for p^6, the amount of time to obtain results became longer and longer. However I saw that there are groups of order p^6 for $p \in \{2, 3, 5, 7, 11\}$ with the desired property. This led me to conjecture

that for any prime number p there is a group of order p^6 possessing a tensor decomposable character. The result can be seen in 3.3.1. Furthermore I observed that for $p \in \{5, 7, 11\}$ all groups G of order p^6 with $Z(G) \cong C_p \times C_p$ and derived subgroup $G' \cong C_p \times C_p \times C_p \times C_p$ have a tensor decomposable character. Therefore I started to wonder if possibly all groups with these properties have a tensor decomposable character. Yet this problem seemed to be anything but obvious.

Let us finally present the output of the above program for p^6 with $p = 2$ and $p = 3$ in order to get a better understanding of the whole context and moreover to prove that there are tensor decomposable characters of groups of order 2^6 and 3^6. For $p = 2$ we obtain the following output:

```
[ [ 64, 8 ], [ 64, 9 ], [ 64, 10 ], [ 64, 11 ], [ 64, 12 ],
[ 64, 13 ], [ 64, 14 ], [ 64, 128 ], [ 64, 129 ], [ 64, 130 ],
[ 64, 131 ], [ 64, 132 ], [ 64, 133 ], [ 64, 140 ], [ 64, 141 ],
[ 64, 142 ], [ 64, 143 ], [ 64, 144 ], [ 64, 145 ], [ 64, 155 ],
[ 64, 156 ], [ 64, 157 ], [ 64, 158 ], [ 64, 159 ], [ 64, 160 ],
[ 64, 161 ], [ 64, 162 ], [ 64, 163 ], [ 64, 164 ], [ 64, 165 ],
[ 64, 166 ], [ 64, 226 ], [ 64, 227 ], [ 64, 228 ], [ 64, 229 ],
[ 64, 230 ], [ 64, 231 ], [ 64, 232 ], [ 64, 233 ], [ 64, 234 ],
[ 64, 235 ], [ 64, 236 ], [ 64, 237 ], [ 64, 238 ], [ 64, 239 ],
[ 64, 240 ] ]
```

Analogously we obtain a similar output for $p = 3$:

```
[ [ 729, 40 ], [ 729, 41 ], [ 729, 42 ], [ 729, 43 ], ... ]
```

This shows that there indeed are groups of order 2^6 and 3^6 which have a tensor decomposable character. However we are looking for a group which is not the direct product of two non-abelian groups of order 2^3 respectively 3^3. Otherwise we could easily construct a tensor decomposable character (refer to 3.2.1). Therefore we will use the command
`ClassPositionsOfDirectProductDecompositions(CharacterTable(G))`.
This returns a list of all those pairs $[L_1, L_2]$ where L_1 and L_2 are lists of

class positions of normal subgroups N_1, N_2 of the group G such that G is their direct product and such that $|N_1| \leq |N_2|$ holds. For the group `G=SmallGroup([64,8])` we obtain the output: `[ ]`. Hence this group is not a direct product at all.

For the group `G=SmallGroup([729,40])` we also obtain the empty set here. Thus, in the sense of 3.2.1, we found a non-trivial example of a group of order 2^6 and a group of order 3^6 possessing a tensor decomposable character, namely the groups of the GAP 'SmallGroups'-library with the library number $[64, 8]$ and $[729, 40]$.

5.2 Obtain a representation of a group contained in the GAP library

I proved that for any prime number $p \geq 5$ there always is a group of order p^6 possessing a tensor decomposable character by giving an explicit construction of such a group using power commutator presentations. The resulting group, including the proof that it actually is a group with the desired properties, is given in 3.3.1.
In order to find an appropriate presentation I used some GAP code which will be described subsequently. With the program of the previous section I obtained library numbers of groups contained in the GAP library for $p \in \{2, 3, 5, 7, 11\}$. The task now was to find a suitable presentation of one of these groups which I could use to find a general group of order p^6 possessing a tensor decomposable character. I gathered some further general information about such a group in 3.2.4. Having obtained these results I tried an approach via finding generators for the derived group and the commutator factor group, taking representatives of each generator of the commutator factor group and determining the order and all commutators of these elements. I aimed at being able to generalize the order, e.g., if for $p = 3$, the order of an element is 9, I defined in general the order of this generator element to be p^2, and so on.
As mentioned earlier I observed that for $p \in \{5, 7, 11\}$ all groups G of order p^6 with $Z(G) \cong C_p \times C_p$ and derived subgroup $G' \cong C_p \times C_p \times C_p \times C_p$ have

a tensor decomposable character. Therefore the GAP code was particularly written for groups with these properties. We will see it in the following:

```
g:=SmallGroup(<libary number>);
cg:=DerivedSubgroup(g);
cfgg:=FactorGroup(g,cg);
genscg:=GeneratorsOfGroup(cg);
a:=genscg[1];
b:=genscg[2];
c:=genscg[3];
d:=genscg[4];
genscfgg:=GeneratorsOfGroup(cfgg);
x:=genscfgg[1];
y:=genscfgg[2];
preimagex:=PreImages(NaturalHomomorphismByNormalSubgroup(g,cg),x);
preimagey:=PreImages(NaturalHomomorphismByNormalSubgroup(g,cg),y);
x_:=Representative(preimagex);
y_:=Representative(preimagey);
gnew:=Subgroup(g,[a,b,c,d,x_,y_]);
hom:=EpimorphismFromFreeGroup(gnew:names:=["r","s","t","u","v","w"]);
Print(Order(cg),"\n",
      Order(a),",",Order(b),",",Order(c),",",Order(d),",",
      Order(x_),",",Order(y_),"\n",
      PreImagesRepresentative(hom,Comm(a,x_)),"\n",
      PreImagesRepresentative(hom,Comm(b,x_)),"\n",
      PreImagesRepresentative(hom,Comm(c,x_)),"\n",
      PreImagesRepresentative(hom,Comm(d,x_)),"\n",
```

```
31          PreImagesRepresentative(hom,Comm(a,y_)),"\n",
32          PreImagesRepresentative(hom,Comm(b,y_)),"\n",
33          PreImagesRepresentative(hom,Comm(c,y_)),"\n",
34          PreImagesRepresentative(hom,Comm(d,y_)),"\n",
35          PreImagesRepresentative(hom,Comm(x_,y_)),"\n"
36  );
```

By the `routine SmallGroup` (Line 1) GAP generates a group, here named `g`, with the respective library number which the user has to pass on as a parameter. As explained above I tried to find generators for the derived subgroup and the commutator factor group, here named by `cg` and `cfgg` respectively (Lines 2,3). This can be easily done by using the command `GeneratorsOfGroup` (Line 5). I named the generators of the commutator group `a,b,c,d` and the generators of the commutator factor group `x,y`. However I did not exactly need generators of the commutator factor group, but elements of the group so that their image in the factor group are generators. Hence I had to do some more work.
With the command `NaturalHomomorphismByNormalSubgroup(g,cg)` in Line 15 I first created the natural group homomorphism of the group `g` into the factor group `g/cg` and then passed it on as a paramenter to the routine `PreImages(f,el)`. The routine `PreImages(f,el)` returns a preimage of the element `el` under the homomorphism `f` (in case `f` is a homomorphism and `el` contained in its image). Using these two commands I was able to obtain cosets which generate the commutator factor group. Finally, with the command `Representative`, I obtained the desired elements (Lines 18,19).
We now want to know the order of our elements and their commutators. Deducing the order of elements is rather easy. It just requires to enter the command `Order` (Lines 24,25). Yet it was not that simple to obtain useful outputs for the commutators. A small trick was necessary. First define a group `gnew` as a group generated by `a,b,c,d,x_,y_`. We do this by using the command `Subgroup` (Line 21) and generate `gnew` as a subgroup of the group `g`. Then we define a homomorphism by the command `EpimorphismFromFreeGroup` (Line 22). With a known generating set (which here consists of the generators of

the derived subgroup `a,b,c,d` and the elements obtained from the generators of the commutator factor group `x_,y_`), this routine returns a homomorphism from a free group that maps the free generators (here `r,s,t,u,v,w`) to the groups generators. Now we can represent `a` by `r`, `b` by `s`, `c` by `t` etcetera. The function `Comm` (Lines 27-35) returns the commutator of two elements. Using the routine `PreImagesRepresentative` (Lines 27-35) I then obtained the results for the commutators of all pairs of elements in the variables `r,s,t,u,v,w`. Thereby I could easily guess what the commutator in the group of order p^6 with any arbitrary prime p might look like.

We now give an example and present the output of the above program for the library number [15625,555] (which is the library number of a group of order 5^6 possessing a tensor decomposable character, having an elementary abelian derived subgroup of order p^4 and having an elementary abelian center of order p^2):

```
625
5,5,5,5,5,5
s
<identity ...>
<identity ...>
<identity ...>
t
<identity ...>
u
<identity ...>
r^4
```

This means that the order of the commutator group is `Order(cg)=625`, the order of all `a,b,c,d,x_,y_=5`, the commutator `Comm(a,x_)=s` which means `[a,x]=b`, `Comm(b,x_)=<identity>` which means `[b,x]=1`,..., and the commutator `Comm(x_,y_)=r^4` meaning `[x,y]=a^4`.

Generalising this we obtain the following presentation:

$$\langle a, b, c, d, x, y \mid a^p, b^p, c^p, d^p, x^p, y^p,\ [a,b]\ ,\ [a,c]\ ,\ [a,d]\ ,\ [b,c]\ ,\ [b,d]\ ,\ [c,d]\ ,$$
$$[a,x]\, b^{-1},\ [b,x]\ ,\ [c,x]\ ,\ [d,x]\ ,\ [a,y]\, c^{-1},\ [b,y]\ ,\ [c,y]\, d^{-1},\ [d,y]\ , [x,y]\, a\rangle.$$

At this point it is worth pointing out that lots of attempts did not work out in general. However I gained more and more experience which relations might be suitable and which not. Finally I found more or less coincidently a presentation which actually fulfilled all our needs. This presentation with a subsequent proof that this indeed is a group of order p^6 possessing a tensor decomposable character can be found in 3.3.1.

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Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig und nur unter Benutzung der angegebenen Hilfsmittel angefertigt habe.

Aachen, den 11. Juli 2008

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