Finite Groups – killed by induction

Bulky volumes have been written on finite group theory. Here is the way to make a clean sweep of that bulk of sophistication.

Lemma. *Let G be a group, and let $H_1, \ldots, H_n$ be subgroups such that $G = H_1 \cup \cdots \cup H_n$. Then $G = H_i$ for some $i \in \{1, \ldots, n\}$.*

Proof. For $n = 1$, the lemma is trivial. Thus let us start with the case $n = 2$. Assume that $G = H_1 \cup H_2$, but $G \neq H_i$ for $i \in \{1, 2\}$. Then there are elements $g \in H_1 \setminus H_2$ and $h \in H_2 \setminus H_1$. By assumption, $gh \in H_1 \cup H_2$, say, $gh \in H_1$. Then $h = g^{-1}(gh) \in H_1$, a contradiction. Now let us assume that the lemma holds for a fixed n, and assume that $G = H_1 \cup \cdots \cup H_{n+1}$. By the inductive hypothesis, we can assume that $H_1 \cup \cdots \cup H_n \neq G$ and $H_{n+1} \neq G$. Arguing in the same way as before, we find elements $g \in (H_1 \cup \cdots \cup H_n) \setminus H_{n+1}$ and $h \in H_{n+1} \setminus (H_1 \cup \cdots \cup H_n)$. If $gh \in H_1 \cup \cdots \cup H_n$, we get $h = g^{-1}(gh) \in H_1 \cup \cdots \cup H_n$ since $g \in H_i$ for some $i \in \{1, \ldots, n\}$, and in case $gh \in H_{n+1}$, we get $g \in H_{n+1}$, which is both impossible. The lemma is proved. □

Recall that a group G is said to be *cyclic* if there is an element $g \in G$ such that no proper subgroup of G contains g.

Theorem. *Every finite group G is cyclic.*

Proof. For any $g \in G$, let H_g denote the subgroup generated by g. Thus $G = \bigcup_{g \in G} H_g$. Since G is finite, we can apply the lemma, which yields $G = H_g$ for some $g \in G$. Hence G is cyclic. □