Some elementary considerations in exact categories

Theo Bühler

Matthias Künzer

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Contents

We collect some elementary counterexamples and lemmata in exact categories; cf. [**5,** §2], [**1,** §2]. The counterexamples are for the most part easy, once known. Most of the lemmata are simple consequences of the Embedding Theorem of Gabriel-Quillen-Laumon, cf. [**1,** Th. A.1], but we want to give direct proofs.

Let $\mathcal{E}$ be an exact category equipped with a set of pure short exact sequences $\mathcal{S}$.

The notation follows [**3,** §A.2.1]. In particular, a morphism is called *pure* if it can be factored into a composite of a pure epimorphism ($\twoheadrightarrow$), followed by a pure monomorphism ($\rightarrowtail$).

1 Counterexamples

Given a ring R, we will always equip the category R-free of finitely generated free R-modules with the set of split short exact sequences as its set of pure short exact sequences.

The following assertions are false in general.

1.1 Pure morphisms

Assertion 1 *The composite of two pure morphisms in $\mathcal{E}$ is pure.*

Cf. also Lemma 19.

Counterexample. Let $\mathcal{E} = \mathbf{Z}$-free. The morphism $\mathbf{Z} \xrightarrow{(1\,2)} \mathbf{Z} \oplus \mathbf{Z}$ is purely monomorphic. The morphism $\mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\binom{0}{1}} \mathbf{Z}$ is purely epimorphic. But the composite $\mathbf{Z} \xrightarrow{2} \mathbf{Z}$ is not pure. ▫

Assertion 2 *Given pure morphisms $X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y$ in $\mathcal{E}$, the morphism $X \oplus X' \xrightarrow{\binom{f}{f'}} Y$ is pure.*

Counterexample. Let $\mathcal{E} = \mathbf{Z}$-free. The morphisms $\mathbf{Z} \xrightarrow{(1\,0)} \mathbf{Z} \oplus \mathbf{Z}$ and $\mathbf{Z} \xrightarrow{(1\,2)} \mathbf{Z} \oplus \mathbf{Z}$ are pure, but not $\mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\left(\begin{smallmatrix}1&0\\1&2\end{smallmatrix}\right)} \mathbf{Z} \oplus \mathbf{Z}$. ▫

Assertion 3 *Given pure morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ such that $fg = 0$, the morphism $X \oplus Y \xrightarrow{\left(\begin{smallmatrix}f&0\\1&-g\end{smallmatrix}\right)} Y \oplus Z$ is pure.*

Counterexample. Let $\mathcal{E} = \mathbf{Z}$-free. The morphisms $\mathbf{Z} \xrightarrow{(1\,2)} \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\binom{-2}{1}} \mathbf{Z}$ are both pure, but not

$$\mathbf{Z}^{\oplus 3} \xrightarrow{\left(\begin{smallmatrix}1&2&0\\1&0&2\\0&1&-1\end{smallmatrix}\right)} \mathbf{Z}^{\oplus 3}\,.$$

▫

Definition 4

(1) A complex $X \in \operatorname{Ob} \mathrm{C}(\mathcal{E})$ is called *pure* if all its differentials are pure.

(2) A complex $X \in \operatorname{Ob} \mathrm{C}(\mathcal{E})$ is called *purely acyclic* or a *pure acyclic complex* if we may factor $(X^{i-1} \xrightarrow{d} X^i) = (X^{i-1} \overset{\bar{d}}{\twoheadrightarrow} I^i \overset{\dot{d}}{\rightarrowtail} X^i)$ for $i \in \mathbf{Z}$ and $I^i \overset{\dot{d}}{\rightarrowtail} X^i \overset{\bar{d}}{\twoheadrightarrow} I^{i+1}$ is purely short exact for all $i \in \mathbf{Z}$; cf. [**1,** Def. 10.1]. In other words, a complex is purely acyclic if it is pieced together from pure short exact sequences.

(3) A complex $X \in \operatorname{Ob} \mathrm{C}(\mathcal{E})$ is called *split acyclic* if it is purely acyclic with respect to the split exact structure on $\mathcal{E}$, containing only the split short exact sequences; cf. [**3,** Ex. A.3].

So for a complex $X \in \operatorname{Ob} \mathrm{C}(\mathcal{E})$, we have the following implications.

$$\text{split acyclic} \quad \Longrightarrow \quad \text{purely acyclic} \quad \Longrightarrow \quad \text{pure}$$

The failure of Assertion 3 prevents the cone of (the identity of) a pure complex from being pure.

The counterexample to Assertion 3 can be used as another, but more complicated counterexample to Assertion 2. It can also be used as another, but more complicated counterexample to the following Assertion 5, taking into account the pointwise split monomorphism of a complex into its cone, completed to a short exact sequence of complexes.

Assertion 5 *Let $X' \longrightarrow X \longrightarrow X''$ be a pointwise split short exact sequence in* $\mathrm{C}(\mathcal{E})$*.*

If X' and X'' are pure, so is X.

Cf. also Lemma 28.

Counterexample. Let $\mathcal{E} = \mathbf{Z}$-free. We have the pointwise split short exact sequence of complexes

$$\begin{array}{ccccc} \mathbf{Z} & \xrightarrow{1} & \mathbf{Z} & \longrightarrow & 0 \\ \uparrow & & \uparrow{\scriptstyle 2} & & \uparrow \\ 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{1} & \mathbf{Z} \end{array}$$

concentrated in positions 0 and 1. $\square$

Assertion 6 *Let $X' \longrightarrow X \longrightarrow X''$ be a pointwise split short exact sequence in* $\mathrm{C}(\mathcal{E})$*.*

If X' and X are split acyclic, then X'' is pure.

Cf. Lemma 27.

Counterexample. The argument for the implication (i) $\Leftarrow$ (ii) in Lemma 27 below can be read as a counterexample to Assertion 6, when taking into account that there exists an exact category in which there exists a coretraction that is not purely monomorphic, as found in the counterexample to Assertion 9 below. $\square$

Assertion 7 *Suppose $\mathcal{E}$ is a full extension closed additive subcategory of an abelian category $\mathcal{A}$ such that $\mathcal{S}$ is the set of short exact sequences in $\mathcal{A}$ with all three objects in $\mathcal{E}$.*

That is, $\mathcal{E}$ is a fully exact *subcategory of $\mathcal{A}$; cf.* [**1,** Def. 10.21].

Suppose given a short exact sequence $X' \longrightarrow X \longrightarrow X''$ in $\mathcal{A}$ such that $X', X \in \mathrm{Ob}\,\mathcal{E}$, but such that X'' is not isomorphic in $\mathcal{A}$ to an object of $\mathcal{E}$. Then $X' \longrightarrow X$ does not have a cokernel in $\mathcal{E}$.

Counterexample. Let $\mathcal{E} = \mathbf{Z}$-free. Let $\mathcal{A} = \mathbf{Z}$-mod. Consider the short exact sequence $\mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{1} \mathbf{Z}/2$ in $\mathcal{A}$. The morphism $\mathbf{Z} \xrightarrow{2} \mathbf{Z}$ is epimorphic in $\mathcal{E}$, thus has cokernel $\mathbf{Z} \longrightarrow 0$ there. $\square$

Assertion 8 *Suppose given a morphism $X \longrightarrow Y$ in $\mathcal{E}$ that has a pure monomorphism as kernel in $\mathcal{E}$ and a pure epimorphism as cokernel in $\mathcal{E}$. Then $X \longrightarrow Y$ is pure.*

Counterexample. Let $\mathcal{E} = \mathbf{Z}$-free. The morphism $\mathbf{Z} \xrightarrow{2} \mathbf{Z}$ has kernel $0 \longrightarrow \mathbf{Z}$ and cokernel $\mathbf{Z} \longrightarrow 0$ in $\mathcal{E}$. $\square$

1.2 Idempotent completeness

Recall that an additive category is called *weakly idempotent complete* if each retraction has a kernel; cf. [**1,** Def. 7.2]. For an exact category this is equivalent to each retraction being purely epimorphic; cf. [**1,** Cor. 7.5].

Recall that an additive category is called *idempotent complete* (or *Karoubian*) if each idempotent has a kernel, cf. [**1,** Rem. 6.2]. This is equivalent to each idempotent being split in the sense described in [**1,** Def. 6.1].

Assertion 9 *An exact category is weakly idempotent complete.*

Counterexample. Let R be a ring such that there exists a finitely generated projective R-module P that is not free, but stably free, i.e. such that there exists a finitely generated free R-module F such that $F \oplus P$ is free.

We may take $R = \mathbf{R}[X,Y,Z]/(X^2+Y^2+Z^2-1)$ and

$$P \;=\; \{(f,g,h) \in R^3 \;:\; \bar{X}f + \bar{Y}g + \bar{Z}h = 0\} \;\subseteq\; R^3\;,$$

where $\bar{X}$ denotes the residue class of X in R, etc.; cf. [**2,** Th. 1].

Let $\mathcal{E} = R$-free. In $\mathcal{E}$, the morphism $F \xrightarrow{(1\,0)} F \oplus P$ is a coretraction. However, it is not a pure monomorphism. For if it were, there would be a split short exact sequence $F \xrightarrow{(1\,0)} F \oplus P \longrightarrow \tilde{F}$ in $\mathcal{E}$ with $\tilde{F} \in \operatorname{Ob}\mathcal{E}$, which remains short exact in R-mod. But then $P \simeq \tilde{F}$ would follow by comparison of cokernels, which can not be true.

Hence $\mathcal{E}$ is not weakly idempotent complete; cf. [**1,** Cor. 7.5]. □

From a geometric point of view, R-free is the category of trivial algebraic vector bundles over the sphere S^2, and P is the tangent bundle. By the Hairy Ball Theorem, P is non-trivial, hence P is not free as an R-module; however, $P \oplus N$ is trivial, where N is the normal bundle.

Assertion 10 *A weakly idempotent complete exact category is idempotent complete.*

Counterexample. Let $R = \mathbf{Q} \times \mathbf{Q}$. Let $\mathcal{E} = R$-free. A coretraction in $\mathcal{E}$ is split in R-mod. Since $(\mathbf{Q} \times \mathbf{Q})^{\oplus k} \oplus X \simeq (\mathbf{Q} \times \mathbf{Q})^{\oplus \ell}$ in R-mod implies $k \leq \ell$ and $X \simeq (\mathbf{Q} \times \mathbf{Q})^{\oplus(\ell-k)}$, we infer that it is purely monomorphic. Thus $\mathcal{E}$ is weakly idempotent complete; cf. [**1,** Cor. 7.5]. However, R-free is not idempotent complete, since the idempotent

$$\mathbf{Q} \times \mathbf{Q} \xrightarrow{1\times 0} \mathbf{Q} \times \mathbf{Q}$$

is not split. In fact, if it were split, it would decompose into a pure epimorphism, followed by a pure monomorphism. However, the only nonzero pure epimorphisms with source $\mathbf{Q} \times \mathbf{Q}$ are isomorphisms; and the only nonzero pure monomorphisms with target $\mathbf{Q} \times \mathbf{Q}$ are isomorphisms. So if a nonzero idempotent on $\mathbf{Q} \times \mathbf{Q}$ is split, it is an isomorphism. But our nonzero idempotent is not an isomorphism, hence not split. □

1.3 Diagonal matrices

Assertion 11 *Suppose given morphisms a, b in $\mathcal{E}$ such that $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ is purely monomorphic. Then a and b are purely monomorphic.*

Cf. also Lemma 20. Moreover, cf. [**4,** Rem. 1.9].

Counterexample. Let R be a ring such that there exists a finitely generated projective R-module P that is not free, but stably free, i.e. such that there exists a finitely generated free R-module F such that $F \oplus P$ is free; cf. [**2,** Th. 1]. Let $\mathcal{E} = R$-free. Then $F \xrightarrow{(1\,0)} F \oplus P$ is not purely monomorphic; cf. counterexample to Assertion 9. However,

$$F \xrightarrow{(1\,0\,0)} F \oplus P \oplus F \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} P \oplus F$$

is a pure short exact sequence. In particular, $F \oplus 0 \xrightarrow{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)} F \oplus P \oplus F$ is purely monomorphic. ▫

Assertion 12 *Suppose $\mathcal{E}$ to be weakly idempotent complete. Suppose given morphisms a, b in $\mathcal{E}$ such that $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ is pure. Then a and b are pure.*

Cf. also Lemma 21.

Counterexample. Let $\mathcal{E} = (\mathbf{Q} \times \mathbf{Q})$-free, which is a weakly idempotent complete exact category, as seen in the counterexample to Assertion 10. The commutative triangle

$$\begin{array}{ccc} (\mathbf{Q} \times \mathbf{Q})^{\oplus 2} & \xrightarrow{\left(\begin{smallmatrix} 1\times 0 & 0\times 0 \\ 0\times 0 & 0\times 1 \end{smallmatrix}\right)} & (\mathbf{Q} \times \mathbf{Q})^{\oplus 2} \\ {\scriptstyle \left(\begin{smallmatrix} 1\times 0 \\ 0\times 1 \end{smallmatrix}\right)} \searrow & & \nearrow {\scriptstyle (1\times 0 \;\; 0\times 1)} \\ & \mathbf{Q} \times \mathbf{Q} & \end{array}$$

together with $(1\times 0 \;\; 0\times 1)\left(\begin{smallmatrix} 1\times 0 \\ 0\times 1 \end{smallmatrix}\right) = 1 \times 1$ shows that $\left(\begin{smallmatrix} 1\times 0 & 0\times 0 \\ 0\times 0 & 0\times 1 \end{smallmatrix}\right)$ is pure; cf. [**1,** Cor. 7.5]. However, neither 1×0 nor 0×1 are pure; cf. the counterexample to Assertion 10. ▫

1.4 Pullbacks and pushouts

Assertion 13 *Suppose given a morphism of split short exact sequences*

$$\begin{array}{ccccc} X' & \rightarrowtail & X & \twoheadrightarrow & X'' \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \rightarrowtail & Y & \twoheadrightarrow & Y'' \end{array}$$

such that (X', X, Y', Y) is a pullback in $\mathcal{E}$ and such that $X' \rightarrowtail Y'$ and $X \rightarrowtail Y$ are split monomorphic. Then $X'' \longrightarrow Y''$ is pure.

Cf. also Lemma 28.

Counterexample. Let $\mathcal{E} = \mathbf{Z}$-free. Consider

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{1} & \mathbf{Z} \\ \downarrow & & \downarrow {\scriptstyle (1\,2)} & & \downarrow {\scriptstyle 2} \\ \mathbf{Z} & \xrightarrow{(1\,0)} & \mathbf{Z} \oplus \mathbf{Z} & \xrightarrow{\binom{0}{1}} & \mathbf{Z}\,. \end{array}$$

□

Assertion 14 *Suppose given a corner*

$$\begin{array}{ccc} Y & & \\ {\scriptstyle f} \uparrow & & \\ X & \xrightarrow{x} & X' \end{array}$$

in $\mathcal{E}$ in which f and x are split epimorphic. Then there exists a completion to a pushout

$$\begin{array}{ccc} Y & \xrightarrow{y} & Y' \\ {\scriptstyle f} \uparrow & & \uparrow {\scriptstyle f'} \\ X & \xrightarrow{x} & X' \end{array}$$

in $\mathcal{E}$.

Counterexample. Let $\mathcal{E} = (\mathbf{Z}/4)$-free. Consider the corner

$$\begin{array}{ccc} \mathbf{Z}/4 & & \\ {\scriptstyle \binom{1}{2}} \uparrow & & \\ \mathbf{Z}/4 \oplus \mathbf{Z}/4 & \xrightarrow{\binom{1}{0}} & \mathbf{Z}/4\,. \end{array}$$

A pushout would yield a cokernel to $\mathbf{Z}/4 \oplus \mathbf{Z}/4 \xrightarrow{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)} \mathbf{Z}/4 \oplus \mathbf{Z}/4$. This morphism is isomorphic to $\mathbf{Z}/4 \oplus \mathbf{Z}/4 \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)} \mathbf{Z}/4 \oplus \mathbf{Z}/4$. Hence a pushout would yield a cokernel to $\mathbf{Z}/4 \xrightarrow{2} \mathbf{Z}/4$. *Assume* that $\mathbf{Z}/4 \xrightarrow{2} \mathbf{Z}/4$ has a cokernel in $\mathcal{E}$. Write it in the form

$$\mathbf{Z}/4 \xrightarrow{2} \mathbf{Z}/4 \xrightarrow{(a_1 \,\dots\, a_k)} (\mathbf{Z}/4)^{\oplus k}$$

for some $k \geq 0$ and some $a_i \in \mathbf{Z}/4$. By composition, we conclude that $a_i \in \{0, 2\}$ for $1 \leq i \leq k$. Since cokernels are epimorphic, composition with $(\mathbf{Z}/4)^{\oplus k} \xrightarrow{\left(\begin{smallmatrix} 2 & & \\ & \ddots & \\ & & 2 \end{smallmatrix}\right)} (\mathbf{Z}/4)^{\oplus k}$ implies $k = 0$. But $\mathbf{Z}/4 \longrightarrow 0$ is not a cokernel of $\mathbf{Z}/4 \xrightarrow{2} \mathbf{Z}/4$, since the latter is not epimorphic. We have arrived at a *contradiction.* □

Assertion 15 *Suppose given a bicartesian quadrangle*

$$\begin{array}{ccc} Y & \xrightarrow{\ y\ } & Y' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ X & \xrightarrow{\ x\ } & X' \end{array}$$

in $\mathcal{E}$ in which f and x are split epimorphic. Then f' is purely epimorphic.

Counterexample. Let $\mathcal{E} = \mathbf{Z}$-mod, equipped with the set $\mathcal{S}$ of split exact sequences. The morphism of short exact sequences

$$\begin{array}{ccccc} \mathbf{Z} & \xrightarrow{\ 2\ } & \mathbf{Z} & \xrightarrow{\ 1\ } & \mathbf{Z}/2 \\ {\scriptstyle 1}\uparrow & & {\scriptstyle \binom{1}{2}}\uparrow & & \uparrow{\scriptstyle 1} \\ \mathbf{Z} & \xrightarrow{(0\ 1)} & \mathbf{Z}\oplus\mathbf{Z} & \xrightarrow{\binom{1}{0}} & \mathbf{Z} \end{array}$$

shows $(\mathbf{Z}\oplus\mathbf{Z},\ \mathbf{Z},\ \mathbf{Z},\ \mathbf{Z}/2)$ to be a bicartesian square in $\mathcal{E}$. Its lower and left morphisms are split epimorphic, but its upper and right morphisms are not. ▫

Recall that a commutative quadrangle (A, B, A', B') in $\mathcal{E}$ is called a *pure square* if its diagonal sequence $A \longrightarrow A' \oplus B \longrightarrow B'$ is purely short exact; cf. [**3,** §A.4].

Assertion 16 *Suppose given a pure square*

$$\begin{array}{ccc} B & \xrightarrow{\ b\ } & B' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ A & \xrightarrow[\ a\ ]{} & A' \end{array}$$

in $\mathcal{E}$. If a is pure, then b is pure.

Cf. also Lemmata 25 and 26.

Counterexample. Let R be a ring such that there exists a finitely generated projective R-module P that is not free, but stably free, i.e. such that there exists a finitely generated free R-module F such that $F \oplus P$ is free; cf. [**2,** Th. 1]; counterexample to Assertion 9. Let $\mathcal{E} = R$-free. Then $F \oplus P \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)} F \oplus P$ is not pure. For if it were, its image, taken in R-mod, would be free. Its image P, however, is not free.

The commutative quadrangle

$$\begin{array}{ccc} F \oplus P & \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)} & F \oplus P \\ {\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)}\uparrow & & \uparrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} \\ F \oplus P \oplus F & \xrightarrow{\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} & F \oplus P \oplus F \end{array}$$

is a bicartesian square in R-mod. Since its objects are free, it is thus a pure square in R-free.

Consider the lower morphism. Its image, taken in R-mod, is the free module $P \oplus F$. Its kernel and its cokernel, taken in R-mod, is the free module F. Thus it is pure in R-free.

Note that in addition, the vertical morphisms are purely epimorphic. □

1.5 Frobenius categories

The following examples are essentially due to Sebastian Thomas.

An object $B \in \operatorname{Ob} \mathcal{E}$ is said to be *bijective* if $\operatorname{Hom}_{\mathcal{E}}(B, -)$ and $\operatorname{Hom}_{\mathcal{E}}(-, B)$ map pure short exact sequences to short exact sequences of abelian groups.

A morphism $X \xrightarrow{f} Y$ in $\mathcal{E}$ is said to be *stably zero* if f factors over a bijective object.

Suppose $\mathcal{E}$ to be Frobenius, i.e. suppose that for all $X \in \operatorname{Ob} \mathcal{E}$ there exists $B' \twoheadrightarrow X \rightarrowtail B''$ with bijective objects B', $B'' \in \operatorname{Ob} \mathcal{E}$.

Assertion 17 *Suppose given a morphism*

$$\begin{array}{ccccc} X' & \rightarrowtail & X & \twoheadrightarrow & X'' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} \\ Y' & \rightarrowtail & Y & \twoheadrightarrow & Y'' \end{array}$$

of pure short exact sequences in $\mathcal{E}$. If f' and f are stably zero, then so is f''.

Counterexample. Let $\mathcal{E} = \mathbf{Z}/4$-mod, with all short exact sequences declared to be pure. The object $\mathbf{Z}/4$ is bijective, but $\mathbf{Z}/2$ is not. Consider

$$\begin{array}{ccccc} \mathbf{Z}/2 & \overset{2}{\rightarrowtail} & \mathbf{Z}/4 & \overset{1}{\twoheadrightarrow} & \mathbf{Z}/2 \\ \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ 0 & \rightarrowtail & \mathbf{Z}/2 & \overset{1}{\twoheadrightarrow} & \mathbf{Z}/2\,. \end{array}$$

□

Assertion 18 *Suppose given a commutative quadrangle*

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{y} & Y \end{array}$$

in $\mathcal{E}$ such that f' and f are stable zero. Then there exists a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ B' & \xrightarrow{b} & B \\ {\scriptstyle v'}\downarrow & & \downarrow{\scriptstyle v} \\ Y' & \xrightarrow{y} & Y \end{array}$$

in $\mathcal{E}$ such that $u'v' = f'$ and $uv = f$ and such that B' and B are bijective.

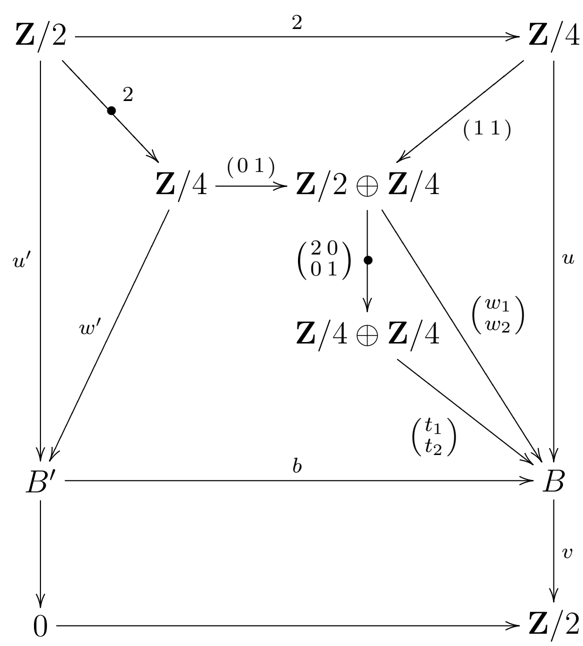
Counterexample. Let $\mathcal{E} = \mathbf{Z}/4$-mod, with all short exact sequences declared to be pure. Consider the commutative quadrangle

$$\begin{array}{ccc} \mathbf{Z}/2 & \xrightarrow{2} & \mathbf{Z}/4 \\ \downarrow & & \downarrow{\scriptstyle 1} \\ 0 & \longrightarrow & \mathbf{Z}/2\,. \end{array}$$

Assume that there exists a commutative diagram

$$\begin{array}{ccc} \mathbf{Z}/2 & \xrightarrow{2} & \mathbf{Z}/4 \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ B' & \xrightarrow{b} & B \\ \downarrow & & \downarrow{\scriptstyle v} \\ 0 & \longrightarrow & \mathbf{Z}/2\,. \end{array}$$

in $\mathcal{E}$ such that B' and B are bijective and such that $(\mathbf{Z}/4 \xrightarrow{uv} \mathbf{Z}/2) = (\mathbf{Z}/4 \xrightarrow{1} \mathbf{Z}/2)$. We construct the following commutative diagram.



The factorisation $u' = 2w'$ exists by bijectivity of B'. We have inserted the pushout $\mathbf{Z}/2 \oplus \mathbf{Z}/4$. The morphism $\binom{w_1}{w_2}$ such that $(1\,1)\binom{w_1}{w_2} = u$ and $(0\,1)\binom{w_1}{w_2} = w'b$ is induced by this pushout. The factorisation $\binom{w_1}{w_2} = \left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)\binom{t_1}{t_2}$ exists by bijectivity of B.

Now $1 = uv = (1\,1)\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)\binom{t_1}{t_2} v = 2t_1 v + t_2 v = t_2 v = (0\,1)\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)\binom{t_1}{t_2} v = (0\,1)\binom{w_1}{w_2} v = w'bv = 0$, as morphisms from $\mathbf{Z}/4$ to $\mathbf{Z}/2$, which is a *contradiction.* ▫

2 Lemmata

2.1 Counterexamples extended to characterisations

Lemma 19 *The composite of each composable pair of pure morphisms in $\mathcal{E}$ is pure if and only if $\mathcal{E}$ is abelian, with $\mathcal{S}$ being the set of all short exact sequences.*

Cf. also Assertion 1.

Proof. If $\mathcal{E}$ is abelian, then every morphism in $\mathcal{E}$ is pure with respect to the set $\mathcal{S}$ of all short exact sequences.

Conversely, suppose that the composite of each composable pair of pure morphisms in $\mathcal{E}$ is pure. Note that if all morphisms in $\mathcal{E}$ are pure, then $\mathcal{E}$ is abelian and $\mathcal{S}$ is the set of all short exact sequences.

Suppose given a morphism f in $\mathcal{E}$. Then $\left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} f \\ 1 \end{smallmatrix}\right)$ are pure; cf. [**1,** Prop. 2.9], and note that $\left(\begin{smallmatrix} f \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & f \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. We conclude that $f = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} f \\ 1 \end{smallmatrix}\right)$ is pure. □

Lemma 20 *The following are equivalent.*

(i) *The exact category $\mathcal{E}$ is weakly idempotent complete.*

(ii) *Given morphisms a, b in $\mathcal{E}$ such that $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ is purely monomorphic, then a and b are purely monomorphic.*

Cf. also Assertion 11.

Proof.

Ad (i) $\Rightarrow$ (ii). Consider the following commutative quadrangle.

$$\begin{array}{ccc} A \oplus B & \xrightarrow[\bullet]{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)} & A' \oplus B' \\ {\scriptstyle (1\,0)}\Big\uparrow\bullet & & \Big\uparrow{\scriptstyle (1\,0)} \\ A & \xrightarrow{\;\;a\;\;} & A' \end{array}$$

Thus a is purely monomorphic by [**1,** Cor. 7.7].

Ad (i) $\Leftarrow$ (ii). Suppose that (i) does not hold. Then there exists a coretraction $X \xrightarrow{s} Y$ which is not purely monomorphic; cf. [**1,** Cor. 7.5]. There exists $X \xleftarrow{t} Y$ such that $st = 1$. We have an involutive automorphism $\left(\begin{smallmatrix} 0 & s \\ t & ts-1 \end{smallmatrix}\right)$ of $X \oplus Y$; inspired by [**4,** Rem. 1.9]. Hence $\left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & s \\ t & ts-1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & s \end{smallmatrix}\right)$ is purely monomorphic; cf. [**1,** Prop. 2.9]. Therefore, $0 \oplus X \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 0 & s \end{smallmatrix}\right)} X \oplus Y$ is purely monomorphic. Hence (ii) does not hold. □

Lemma 21 *The following are equivalent.*

(i) *The exact category $\mathcal{E}$ is idempotent complete.*

(ii) *Given morphisms a, b in $\mathcal{E}$ such that $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ is pure, then a and b are pure.*

Cf. also Assertion 12.

Proof.

Ad (i) $\Rightarrow$ (ii). Consider the following commutative diagram.

$$\begin{array}{ccccc} A \oplus B & \rightarrowtail & I & \twoheadrightarrow & A' \oplus B' \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\big\downarrow & & \xi \big\downarrow & & \big\downarrow \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \\ A \oplus B & \rightarrowtail & I & \twoheadrightarrow & A' \oplus B' \end{array},$$

where the horizontal composites equal $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$. We conclude that $\xi^2 = \xi$. Since $\mathcal{E}$ is idempotent complete, we may assume, by isomorphic substitution, that $\xi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. This yields a diagram as follows.

$$\begin{array}{ccccc} A \oplus B & \rightarrowtail & M \oplus N & \twoheadrightarrow & A' \oplus B' \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\big\downarrow & & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\big\downarrow & & \big\downarrow \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \\ A \oplus B & \rightarrowtail & M \oplus N & \twoheadrightarrow & A' \oplus B' \end{array},$$

Commutativity of the quadrangles forces the horizontal morphisms to be of the form

$$A \oplus B \xrightarrow{\left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)} M \oplus N \xrightarrow{\left(\begin{smallmatrix} i & 0 \\ 0 & j \end{smallmatrix}\right)} A' \oplus B' .$$

By Lemma 20 and its dual, the morphisms p and q are purely epimorphic, and the morphisms i and j are purely monomorphic. So $a = pi$ and $b = qj$ are pure.

Ad (i) $\Leftarrow$ (ii). Suppose given an idempotent morphism $X \xrightarrow{e} X$ in $\mathcal{E}$. Write $f := 1 - e$. We have an involutive automorphism $\left(\begin{smallmatrix} e & f \\ f & e \end{smallmatrix}\right)$ of $X \oplus X$. Since $\left(\begin{smallmatrix} e & f \\ f & e \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right)\left(\begin{smallmatrix} e & f \\ f & e \end{smallmatrix}\right)$ and since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)$ is pure, the morphism $\left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right)$ is pure. By (ii), we conclude that e is pure, hence split; cf. [**1,** Rem. 6.2]. ▫

A further characterisation of weakly idempotent complete exact categories can be found in Lemma 27.

2.2 Pure squares

Recall that a commutative quadrangle (A, B, A', B') in $\mathcal{E}$ is called a *pure square* if its diagonal sequence $A \longrightarrow A' \oplus B \longrightarrow B'$ is purely short exact; cf. [**3,** §A.4].

Lemma 22 *Suppose given a composition*

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \\ {\scriptstyle x}\uparrow & & {\scriptstyle y}\uparrow & & {\scriptstyle z}\uparrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

of commutative quadrangles in $\mathcal{E}$. If two out of the three quadrangles (X, Y, X', Y'), (Y, Z, Y', Z'), (X, Z, X', Z') are pure squares, so is the third.

Proof.

Suppose (X, Y, X', Y') and (Y, Z, Y', Z') to be pure squares. We *claim* that (X, Z, X', Z') is a pure square.

Since (X, Y, X', Y') and (Y, Z, Y', Z') are bicartesian, so is (X, Z, X', Z'), i.e. the sequence

$$X \xrightarrow{(x\; fg)} X' \oplus Z \xrightarrow{\left(\begin{smallmatrix} f'g' \\ -z \end{smallmatrix}\right)} Z'$$

is short exact. So in order to prove that it is purely short exact, it remains to show that $(x\; fg)$ is purely monomorphic.

Consider the following commutative diagram.

$$\begin{array}{ccccc} X' \oplus Y & & \xrightarrow{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & y & g \end{smallmatrix}\right)} & & X' \oplus Y' \oplus Z \\ {\scriptstyle (x\; f)}\uparrow & & & & \wr\downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & -f' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} \\ X & & \xrightarrow{(x\; 0\; fg)} & & X' \oplus Y' \oplus Z \\ & {\scriptstyle (x\; fg)}\searrow & & \nearrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} & \\ & & X' \oplus Z & & \end{array}$$

By the Obscure Axiom [**1,** Prop. 2.16], we conclude that $(x\; fg)$ is purely monomorphic. This proves the *claim.*

Suppose (X, Y, X', Y') and (X, Z, X', Z') to be pure squares. We *claim* that (Y, Z, Y', Z') is a pure square.

Since (X, Y, X', Y') and (X, Z, X', Z') are pushouts, so is (Y, Z, Y', Z'). Hence, in the sequence

$$Y \xrightarrow{(y\; g)} Y' \oplus Z \xrightarrow{\left(\begin{smallmatrix} g' \\ -z \end{smallmatrix}\right)} Z' ,$$

$\left(\begin{smallmatrix} g' \\ -z \end{smallmatrix}\right)$ is a cokernel of $(y\; g)$. It remains to show that $(y\; g)$ is purely monomorphic.

Consider the following morphism of pure short exact sequences.

$$\begin{array}{ccccc} X & \xrightarrow{(x\; f)} & X' \oplus Y & \xrightarrow{\left(\begin{smallmatrix} f' \\ -y \end{smallmatrix}\right)} & Y' \\ \| & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right)} & & \downarrow{\scriptstyle g'} \\ X & \xrightarrow{(x\; fg)} & X' \oplus Z & \xrightarrow{\left(\begin{smallmatrix} f'g' \\ -z \end{smallmatrix}\right)} & Z' \end{array}$$

By [**1,** Prop. 2.12], the diagonal sequence of its right hand side quadrangle is purely short exact. Consider the following commutative quadrangle.

$$\begin{array}{ccc} X' \oplus Y & \xrightarrow[\bullet]{\left(\begin{smallmatrix} 1 & 0 & f' \\ 0 & g & -y \end{smallmatrix}\right)} & X' \oplus Z \oplus Y' \\ {\scriptstyle (0\,1)}\uparrow\bullet & & \uparrow{\scriptstyle \left(\begin{smallmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \end{smallmatrix}\right)} \\ Y & \xrightarrow{(y\,g)} & Y' \oplus Z \end{array}$$

By the Obscure Axiom [**1,** Prop. 2.16], we conclude that $(y\,g)$ is purely monomorphic. This proves the *claim.* ▫

Lemma 23 *Suppose given a commutative quadrangle*

$$\begin{array}{ccc} B & \xrightarrow{b} & B' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ A & \xrightarrow[a]{} & A' \end{array}$$

and a morphism $B \xrightarrow{g} C$ *in* $\mathcal{E}$*. Then* (A, B, A', B') *is a pure square if and only if*

$$\begin{array}{ccc} B & \xrightarrow{(b\,g)} & B' \oplus C \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle \left(\begin{smallmatrix} f' & 0 \\ 0 & 1 \end{smallmatrix}\right)} \\ A & \xrightarrow[(a\,fg)]{} & A' \oplus C \end{array}$$

is a pure square.

Proof. Consider the following isomorphism of sequences.

$$\begin{array}{ccccc} A & \xrightarrow{(a\,fg\,f)} & A' \oplus C \oplus B & \xrightarrow{\left(\begin{smallmatrix} f' & 0 \\ 0 & 1 \\ -b & -g \end{smallmatrix}\right)} & B' \oplus C \\ \Vert & & {\scriptstyle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -g \end{smallmatrix}\right)}\downarrow\wr & & \Vert \\ A & \xrightarrow{(a\,f\,0)} & A' \oplus B \oplus C & \xrightarrow{\left(\begin{smallmatrix} f' & 0 \\ -b & 0 \\ 0 & 1 \end{smallmatrix}\right)} & B' \oplus C \end{array}$$

The upper sequence is purely short exact if and only if the lower one is. The lower one in turn is purely short exact if and only if

$$A \xrightarrow{(a\,f)} A' \oplus B \xrightarrow{\left(\begin{smallmatrix} f' \\ -b \end{smallmatrix}\right)} B'$$

is; cf. [**1,** Prop. 2.9., Cor. 2.18]. ▫

Lemma 24 *Suppose given a commutative quadrangle*

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ {\scriptstyle a}\uparrow & & \uparrow{\scriptstyle b} \\ A & \xrightarrow{f} & B \end{array}$$

in $\mathcal{E}$ with f and f' pure. The following are equivalent.

(i) *The quadrangle (A, B, A', B') is bicartesian.*

(ii) *The quadrangle (A, B, A', B') is a pure square.*

(iii) *There exists a commutative diagram*

$$(*) \qquad \begin{array}{ccccccc} K' & \xrightarrow{i'} & A' & \xrightarrow{f'} & B' & \xrightarrow{p'} & C' \\ {\scriptstyle k}\uparrow\wr & & {\scriptstyle a}\uparrow & & \uparrow{\scriptstyle b} & & \wr\uparrow{\scriptstyle c} \\ K & \xrightarrow{i} & A & \xrightarrow{f} & B & \xrightarrow{p} & C \end{array}$$

in which i is a kernel and p is a cokernel of f, and in which i' is a kernel and p' is a cokernel of f'.

Proof. First, we remark that the quadrangle (A, B, A', B') decomposes into

$$(**) \qquad \begin{array}{ccccc} A' & \xrightarrow{\bar{f}'} & I' & \xrightarrow{\dot{f}'} & B' \\ {\scriptstyle a}\uparrow & & {\scriptstyle \xi}\uparrow & & \uparrow{\scriptstyle b} \\ A & \xrightarrow{\bar{f}} & I & \xrightarrow{\dot{f}} & B \end{array}$$

Ad (ii) $\Rightarrow$ (i). This follows a fortiori.

Ad (i) $\Rightarrow$ (iii). Suppose (A, B, A', B') to be bicartesian. Since (A, I, A', I') is a pullback and (I, B, I', B') is a pushout, a diagram of the form $(*)$ follows from [**1,** Prop. 2.12].

Ad (iii) $\Rightarrow$ (ii). Suppose a diagram of the form $(*)$ to exist. Decomposing (A, B, A', B') into quadrangles as in $(**)$, it follows by [**1,** Prop. 2.12] that (A, I, A', I') is a pullback and that (I, B, I', B') is a pushout. From [**1,** Cor. 2.14] we conclude that (A, B, A', B') is a pure square. ▫

Lemma 25 *Suppose given a pure square*

$$\begin{array}{ccc} B & \xrightarrow{b} & B' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ A & \xrightarrow[a]{} & A' \end{array}$$

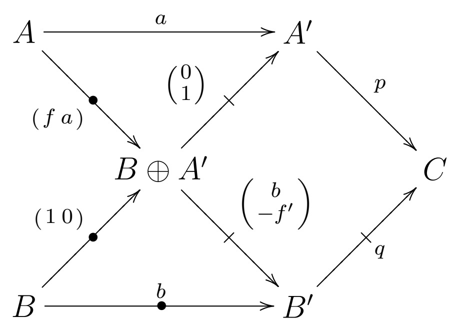
in $\mathcal{E}$.

Then a is purely monomorphic if and only if b is purely monomorphic.

Dually, a is purely epimorphic if and only if b is purely epimorphic.

Proof. If a is purely monomorphic, so is b by [**1,** Def. 2.1] and the uniqueness of the pushout completion up to isomorphism.

Suppose b to be purely monomorphic. Consider the following diagram, in which $(B,\ B \oplus A',\ A')$, (B, B', C) and $(A,\ B \oplus A',\ B')$ are purely short exact, and in which $p := -f'q$.

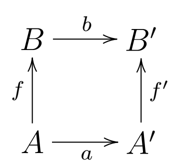


We *claim* that a is a kernel of p. Suppose given a morphism t such that $tp = 0$. Then $tf'q = 0$, whence there exists a morphism s such that $sb = tf'$, i.e. $\begin{pmatrix} s & t \end{pmatrix}\begin{pmatrix} b \\ -f' \end{pmatrix} = 0$. Hence there exists a morphism u such that $\begin{pmatrix} s & t \end{pmatrix} = u \begin{pmatrix} f & a \end{pmatrix}$. In particular, $t = ua$. To show uniqueness of the factorisation of t over a, suppose given a morphism v such that $va = 0$. Then $0 = vaf' = vfb$, whence $vf = 0$. So $v \begin{pmatrix} a & f \end{pmatrix} = 0$, and therefore $v = 0$. This proves the *claim.*

By [**1,** Prop. 2.16], p is purely epimorphic. Consequently, a is purely monomorphic. □

Concerning the proof of Lemma 25, cf. also [**1,** Ex. 3.11.(i)].

Lemma 26 *Suppose $\mathcal{E}$ to be weakly idempotent complete. Suppose given a pure square*

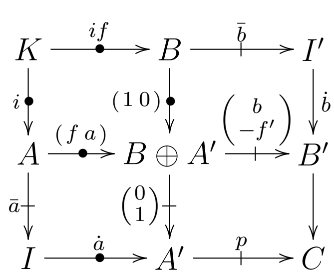


in $\mathcal{E}$. Then a is pure if and only if b is pure.

Cf. also Assertion 16.

Proof. By duality, it suffices to show that if a is pure, then b is pure.

Consider the following diagram, in which $a = \bar{a}\dot{a}$, in which i is the kernel of $\bar{a}$, in which if is purely monomorphic by [**1,** Prop. 7.6] and by weak idempotent completeness of $\mathcal{E}$, and in which $\bar{b}$ is the cokernel of if, and p the cokernel of $\dot{a}$.



By [**1,** Cor. 3.6], the cokernel-induced morphism $\dot{b}$ is purely monomorphic. Therefore, $b = \bar{b}\dot{b}$ is pure. □

2.3 Pure acyclic complexes

Lemma 27 *The following are equivalent.*

(i) *The exact category $\mathcal{E}$ is weakly idempotent complete.*

(ii) *Suppose given a sequence $X' \xrightarrow{f} X \xrightarrow{g} X''$ in $\mathrm{C}(\mathcal{E})$ that is pointwise purely short exact. If X' and X are purely acyclic, so is X''.*

If we supposed resp. asserted only purity instead of pure acyclicity in (ii), the implication (i) $\Rightarrow$ (ii) would not hold; cf. counterexample to Assertion 13.

Proof.

Ad (i) $\Rightarrow$ (ii). For $i \in \mathbf{Z}$, we factor $(X'^{i-1} \xrightarrow{d'} X'^{i}) = (X'^{i-1} \overset{\bar{d}'}{\twoheadrightarrow} I'^{i} \overset{\dot{d}'}{\rightarrowtail} X'^{i})$ and $(X^{i-1} \xrightarrow{d} X^{i}) = (X^{i-1} \overset{\bar{d}}{\twoheadrightarrow} I^{i} \overset{\dot{d}}{\rightarrowtail} X^{i})$. We obtain the following diagram.

$$\begin{array}{ccccc}
I'^{i} & \rightarrowtail & I^{i} & \twoheadrightarrow & J^{i} \\
{\scriptstyle \dot{d}'}\downarrow & & {\scriptstyle \dot{d}}\downarrow & & \downarrow \\
X'^{i} & \overset{f^i}{\rightarrowtail} & X^{i} & \overset{g^i}{\twoheadrightarrow} & X''^{i} \\
{\scriptstyle \bar{d}'}\downarrow & & {\scriptstyle \bar{d}}\downarrow & & \downarrow \\
I'^{i+1} & \rightarrowtail & I^{i+1} & \twoheadrightarrow & J^{i+1}
\end{array}$$

Note that the induced morphism $I'^{i} \longrightarrow I^{i}$ is purely monomorphic since it composes to $I'^{i} \rightarrowtail X^{i}$ and since $\mathcal{E}$ is weakly idempotent complete; cf. [**1,** Prop. 7.6.(ii)]. The middle row is purely short exact by assumption. The upper and the lower row are purely short exact by construction. The left and the middle column are purely short exact by assumption. Now the sequence $J^{i} \longrightarrow X''^{i} \longrightarrow J^{i+1}$ is purely short exact by [**1,** Cor. 3.6.(i)]. Pure acyclicity of X'' ensues.

Ad (i) $\Leftarrow$ (ii). Suppose $\mathcal{E}$ not to be weakly idempotent complete. By [**1,** Cor. 7.5], there exist morphisms $X \xrightarrow{s} Y$ and $X \xleftarrow{t} Y$ in $\mathcal{E}$ such that $st = 1_X$, but such that s is not purely monomorphic.

Consider the following pointwise pure short exact sequence of complexes.

$$\begin{array}{ccccc}
0 & \longrightarrow & X & \xrightarrow{1} & X \\
\downarrow & & \downarrow{\scriptstyle (0\ s)} & & \downarrow{\scriptstyle s} \\
X & \xrightarrow{(1\ 0)} & X \oplus Y & \xrightarrow{\binom{0}{1}} & Y \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1-ts \end{smallmatrix}\right)} & & \downarrow{\scriptstyle 1-ts} \\
X & \xrightarrow{(1\ 0)} & X \oplus Y & \xrightarrow{\binom{0}{1}} & Y \\
\downarrow & & \downarrow{\scriptstyle \binom{0}{t}} & & \downarrow{\scriptstyle t} \\
0 & \longrightarrow & X & \xrightarrow{1} & X
\end{array}$$

The left hand side complex is split acyclic, hence purely acyclic.

The middle complex is split acyclic, hence purely acyclic, as the following isomorphism of complexes shows.

$$
\begin{array}{ccccccc}
X & \xrightarrow{(0\;s)} & X \oplus Y & \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1-ts \end{smallmatrix}\right)} & X \oplus Y & \xrightarrow{\left(\begin{smallmatrix} 0 \\ t \end{smallmatrix}\right)} & X \\
{\scriptstyle 1}\uparrow & & {\scriptstyle \left(\begin{smallmatrix} 0 & s \\ t & 1 \end{smallmatrix}\right)}\uparrow\wr & & \wr\uparrow{\scriptstyle \left(\begin{smallmatrix} -1 & s \\ t & 1-ts \end{smallmatrix}\right)} & & \uparrow{\scriptstyle 1} \\
X & \xrightarrow{(1\;0)} & X \oplus Y & \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)} & X \oplus Y & \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} & X
\end{array}
$$

Note that $\left(\begin{smallmatrix} 0 & s \\ t & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & s \\ t & 1-ts \end{smallmatrix}\right)$ are mutually inverse automorphisms of $X \oplus Y$.

The right hand side complex is not pure, hence not purely acyclic, since s is not pure. ▫

Lemma 28 *Let $X' \xrightarrow{f} X \xrightarrow{g} X''$ be a sequence in $\mathrm{C}(\mathcal{E})$ that is pointwise purely short exact. If X' and X'' are purely acyclic, so is X.*

If we supposed resp. asserted only purity instead of pure acyclicity, the assertion would not hold; cf. Assertion 5.

Proof. The following construction will be summarised in the diagram below.

For $i \in \mathbf{Z}$, we factor $(X'^{i-1} \xrightarrow{d'} X'^{i}) = (X'^{i-1} \xrightarrow{\bar{d}'} \!\!\!\!\twoheadrightarrow I'^{i} \xrightarrow{\dot{d}'} \!\!\!\!\rightarrowtail X'^{i})$ and $(X''^{i-1} \xrightarrow{d} X''^{i}) = (X''^{i-1} \xrightarrow{\bar{d}''} \!\!\!\!\twoheadrightarrow I''^{i} \xrightarrow{\dot{d}''} \!\!\!\!\rightarrowtail X''^{i})$.

By [**1,** Props. 2.12, 2.15, 3.1], we insert pushouts $(X'^{i-1}, I'^{i}, X^{i-1}, P)$ and (I'^{i}, X'^{i}, P, Q) and pullbacks $(R, X^{i}, I''^{i}, X''^{i})$ and $(Q, R, X''^{i-1}, I''^{i})$ as depicted below. There are pure short exact sequences (I'^{i}, P, X''^{i-1}), (X'^{i}, Q, X''^{i-1}) and (X'^{i}, R, I''^{i}). The triangles (X^{i-1}, P, X''^{i-1}), (P, Q, X''^{i-1}), (X'^{i}, Q, R) and (X'^{i}, R, X^{i}) commute.

Let $I''^{i-1} \rightarrowtail Q \twoheadrightarrow R$ be a completion to a pure short exact sequence such that the triangle $(I''^{i-1}, Q, X''^{i-1})$ commutes; let $P \rightarrowtail Q \twoheadrightarrow I'^{i+1}$ be a completion to a pure short exact sequence such that the triangle (X'^{i}, Q, I'^{i+1}) commutes; cf. [**1,** Prop. 2.12].

The pentagon $(X^{i-2}, X^{i-1}, X''^{i-2}, I''^{i-1}, Q)$ commutes, as one verifies by postcomposing with $Q \longrightarrow X^{i}$ and with $Q \twoheadrightarrow X''^{i-1}$. Dually, the pentagon $(Q, X^{i}, X^{i+1}, I'^{i+1}, X'^{i+1})$ commutes.

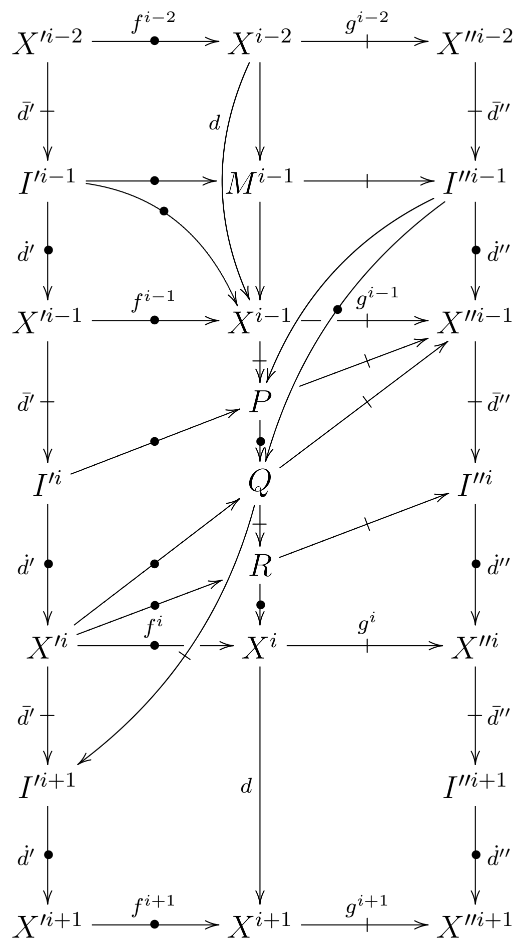
Now $(I''^{i-1} \rightarrowtail Q \twoheadrightarrow I'^{i+1}) = (I''^{i-1} \xrightarrow{0} I'^{i+1})$, as one sees by precomposing with $(X^{i-2} \twoheadrightarrow I''^{i-1})$ and postcomposing with $(I'^{i+1} \rightarrowtail X^{i+1})$. Thus there exists $(I''^{i-1} \longrightarrow P)$ such that the triangle (I''^{i-1}, P, Q) commutes. Note that the pentagon $(X^{i-2}, X^{i-1}, X''^{i-2}, I''^{i-1}, P)$ commutes, as verified by postcomposing with $P \rightarrowtail Q$. Let $(M^{i-1}, I''^{i-1}, X^{i-1}, P)$ be a pullback.

Let $I'^{i-1} \rightarrowtail X^{i-1} \twoheadrightarrow P$ be a completion to a pure short exact sequence such that the triangle $(I'^{i-1}, X'^{i-1}, X^{i-1})$ commutes; let $I'^{i-1} \rightarrowtail M^{i-1} \twoheadrightarrow I''^{i-1}$ be a completion to a pure short exact sequence such that the triangle $(I'^{i-1}, M^{i-1}, X^{i-1})$ commutes; cf. [**1,** Prop. 2.12]. In particular, the quadrangle $(I'^{i-1}, M^{i-1}, X'^{i-1}, X^{i-1})$ commutes.

Since the pentagon $(X^{i-2}, X^{i-1}, X''^{i-2}, I''^{i-1}, P)$ commutes, we have an induced morphism $X^{i-2} \longrightarrow M^{i-1}$ such that the triangle $(X^{i-2}, M^{i-1}, X^{i-1})$ and the quadrangle $(X^{i-2}, X''^{i-2}, M^{i-1}, I''^{i-1})$ commute.

The quadrangle $(M^{i-1}, I''^{i-1}, X^{i-1}, X''^{i-1})$ commutes since it is a pullback, postcomposed with $P \longrightarrow X''^{i-1}$.

The quadrangle $(X'^{i-2}, X^{i-2}, I'^{i-1}, M^{i-1})$ commutes, as one verifies by postcomposing with $M^{i-1} \longrightarrow X^{i-1}$ and with $M^{i-1} \xrightarrow{\;\vert\;} I''^{i-1}$.



We have obtained the following commutative diagrams for $i \in \mathbf{Z}$, which, pieced together, yield the sequence of complexes $X' \xrightarrow{f} X \xrightarrow{g} X''$. Its outer columns and its rows are purely short exact.

$$\begin{array}{ccccc}
I'^{i-1} & \xrightarrow{\;\bullet\;} & M^{i-1} & \xrightarrow{\;\vert\;} & I''^{i-1} \\
{\scriptstyle \dot{d}'}\downarrow{\scriptstyle\bullet} & & \downarrow & & {\scriptstyle\bullet}\downarrow{\scriptstyle \dot{d}''} \\
X'^{i-1} & \xrightarrow[\bullet]{f^{i-1}} & X^{i-1} & \xrightarrow[\vert]{g^{i-1}} & X''^{i-1} \\
{\scriptstyle \bar{d}'}\downarrow{\scriptstyle\vert} & & \downarrow & & {\scriptstyle\vert}\downarrow{\scriptstyle \bar{d}''} \\
I'^{i} & \xrightarrow{\;\bullet\;} & M^{i} & \xrightarrow{\;\vert\;} & I''^{i}
\end{array}$$

By [**1,** Cor. 3.2], $M^{i-1} \longrightarrow X^{i-1}$ is purely monomorphic and $X^{i-1} \longrightarrow M^i$ is purely epimorphic. Now $(M^{i-1} \xrightarrow{\;\bullet\;} X^{i-1} \xrightarrow{\;\vert\;} M^i) = (M^{i-1} \xrightarrow{0} M^i)$, as precomposition with $X^{i-2} \xrightarrow{\;\vert\;} M^{i-1}$ and postcomposition with $M^i \xrightarrow{\;\bullet\;} X^i$ shows. Thus $M^{i-1} \xrightarrow{\;\bullet\;} X^{i-1} \xrightarrow{\;\vert\;} M^i$ is purely short exact by [**1,** Cor. 3.6.(ii)]. Altogether, X is purely acyclic. ▫

Corollary 29 *Suppose $\mathcal{E}$ to be weakly idempotent complete. Let $X' \longrightarrow X \longrightarrow X''$ be a sequence in $\mathrm{C}(\mathcal{E})$ that is pointwise purely short exact.*

If two out of the three objects X', X and X'' are purely acyclic, so is the third.

Proof. This follows from Lemma 27 and Lemma 28. □

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Theo Bühler
Forschungsinstitut für Mathematik
Rämistrasse 101
CH-8092 ETH Zürich
Switzerland
theo@math.ethz.ch

Matthias Künzer
Lehrstuhl D für Mathematik
RWTH Aachen
Templergraben 64
D-52062 Aachen
kuenzer@math.rwth-aachen.de
www.math.rwth-aachen.de/∼kuenzer