G. Hiß, M. Künzer Due Friday, 11.01.08

Group theory, WS 07/08

Sheet 9

Problem 31 (8 points)

Let G be a group. Let $H \leq G$ be such that $[G : H] < \infty$. Let A be a G-module. Let T be a system of left coset representatives for H in G containing 1. For $g \in G$, we write $g = \xi(g) \cdot \eta(g)$, where $\xi(g) \in T$ and $\eta(g) \in H$. Show that

$$\begin{array}{rcl} \mathrm{Z}^2(H, A|_H) & \longrightarrow & \mathrm{Z}^2(G, A) \\ \alpha & \longmapsto & \left(\alpha\uparrow_H^G \;:\; (g, g') \;\mapsto\; \sum_{t \in T} \alpha\big(\eta(gg't)\eta(g't)^{-1},\, \eta(g't)\big) t^{-1} \right) \end{array}$$

maps in fact to $\mathrm{Z}^2(G, A)$, and that it induces a morphism $\mathrm{Cores}\uparrow_H^G : \mathrm{H}^2(H, A|_H) \longrightarrow \mathrm{H}^2(G, A)$ of abelian groups. Show that

$$\mathrm{Cores}\uparrow_H^G \circ \mathrm{Res}\downarrow_H^G \;=\; [G : H] \cdot \mathrm{id}_{\mathrm{H}^2(G,A)} \;.$$

Problem 32 (6 points)

Let Q be a divisible abelian group. Let $B \xrightarrow{f} A$ be an injective morphism of abelian groups. Show that the induced morphism

$$\begin{array}{rcl} \mathrm{Hom}(A, Q) & \longrightarrow & \mathrm{Hom}(B, Q) \\ g & \longmapsto & g \circ f \end{array}$$

is surjective.

Problem 33 (4+2+2 points)

Let A be a finite abelian group.

(1) Show that $A \simeq \hat{A} = \mathrm{Hom}(A, \mathbf{C}^*)$.

(2) Let $B \leq A$ be a subgroup. Show that B is isomorphic to a quotient of A.

(3) Show that the evaluation map $\varepsilon : A \longrightarrow \hat{\hat{A}}$, $a \longmapsto (\lambda \mapsto \lambda(a))$ is an isomorphism.

Problem 34 (6+3 points)

Let p be a prime. Let $n \geq 0$.

(1) Let G be a finite group such that $|G/\mathrm{Z}(G)| = p^n$. Show that $|G'|$ divides $p^{n(n-1)/2}$.
(Hint: Induction on n. If $n \geq 1$, then choose $x \in G \smallsetminus \mathrm{Z}(G)$ such that $x\mathrm{Z}(G) \in \mathrm{Z}(G/\mathrm{Z}(G))$ and consider $N = \{[x, g] \;:\; g \in G\} \trianglelefteq G$.)

(2) Let P be a group of order p^n. Show that $|\mathrm{M}(P)|$ divides $p^{n(n-1)/2}$.
(Hint: Apply (1) to a representation group of P.)

Christmas Problem (10 bonus points)

Let G be a finite nonabelian simple group. Show that G has a nonabelian proper subgroup.