G. Hiß, M. Künzer

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Solution 1

Problem 1

(1) We have

$$\begin{array}{rcl} \mathrm{O}_2(\mathbf{R}) & = & \left\{\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} : \alpha \in \mathbf{R}\right\} \cup \left\{\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} : \alpha \in \mathbf{R}\right\} \\ \mathrm{SO}_2(\mathbf{R}) & = & \left\{\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} : \alpha \in \mathbf{R}\right\} \end{array}$$

Written as row vectors, the set of vertices of our n-gon is given by

$$\mathrm{V}_n := \left\{\begin{pmatrix} \cos(2\pi k/n) & \sin(2\pi k/n) \end{pmatrix} : k \in \mathbf{Z}\right\} \subseteq \mathbf{R}^{1\times 2} .$$

By definition, we have $\mathcal{D}_{2n} = \{S \in \mathrm{O}_2(\mathbf{R}) \ : \ \mathrm{V}_n S \subseteq \mathrm{V}_n\}$.
Note that for $\alpha, \beta \in \mathbf{R}$, we have

$$\begin{array}{rcl} \begin{pmatrix} \cos\beta & \sin\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} & = & \begin{pmatrix} \cos(\beta+\alpha) & \sin(\beta+\alpha) \end{pmatrix} \\ \begin{pmatrix} \cos\beta & \sin\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} & = & \begin{pmatrix} \cos(\beta-\alpha) & \sin(\beta-\alpha) \end{pmatrix} \end{array}$$

Hence for $S = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ our condition $\mathrm{V}_n S \subseteq \mathrm{V}_n$ reads

$$2\pi k/n + \alpha \in \frac{2\pi}{n}\mathbf{Z}$$

for all $k \in \mathbf{Z}$, i.e. $\alpha \in \frac{2\pi}{n}\mathbf{Z}$.
Moreover, for $S = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$ our condition $\mathrm{V}_n S \subseteq \mathrm{V}_n$ reads

$$2\pi k/n - \alpha \in \frac{2\pi}{n}\mathbf{Z}$$

for all $k \in \mathbf{Z}$, i.e. $\alpha \in \frac{2\pi}{n}\mathbf{Z}$. We obtain

$$\begin{array}{rcl} \mathcal{D}_{2n} & = & \left\{\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} : \alpha \in \frac{2\pi}{n}\mathbf{Z}\right\} \cup \left\{\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} : \alpha \in \frac{2\pi}{n}\mathbf{Z}\right\} \\ \mathcal{D}_{2n} \cap \mathrm{SO}_2(\mathbf{R}) & = & \left\{\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} : \alpha \in \frac{2\pi}{n}\mathbf{Z}\right\} \end{array}$$

(2) Write $a := \begin{pmatrix} \cos(2\pi/n) & \sin(2\pi/n) \\ -\sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$ and $b := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that

$$\begin{array}{rcl} a^k & = & \begin{pmatrix} \cos(2\pi k/n) & \sin(2\pi k/n) \\ -\sin(2\pi k/n) & \cos(2\pi k/n) \end{pmatrix} \\ ba^k & = & \begin{pmatrix} \cos(2\pi k/n) & \sin(2\pi k/n) \\ \sin(2\pi k/n) & -\cos(2\pi k/n) \end{pmatrix} \end{array}$$

for $k \in \mathbf{Z}$. Hence $\mathcal{D}_{2n} = \langle a, b\rangle = \{b^j a^i \ : \ 0 \le i \le n,\ 0 \le j \le 1\}$.
Note that

$$b^{-1}ab \ = \ \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix} \ = \ a^{-1} .$$

Hence

$$a^{-1}ba \ = \ ba^2.$$

We conclude that

$$\begin{array}{rcl} b^{-1}(b^j a^k)b & = & b^j a^{-k} \\ a^{-1}(b^j a^k)a & = & \begin{cases} a^k & \text{if } j = 0 \\ ba^{k+2} & \text{if } j = 1 \end{cases} \end{array}$$

for $k \in \mathbf{Z}$ and $j \in \{0,1\}$. This yields the following disjoint decompositions of $\mathcal{D}_{2n}$ into conjugacy classes.

Case $n \equiv_2 0$.

$$\mathcal{D}_{2n} = \{1\} \sqcup \{a, a^{-1}\} \sqcup \{a^2, a^{-2}\} \sqcup \cdots \sqcup \{a^{n/2-1}, a^{n/2+1}\} \sqcup \{a^{n/2}\} \sqcup \{b, ba^2, \ldots, ba^{n-2}\} \sqcup \{ba, ba^3, \ldots, ba^{n-1}\}$$

In particular, $\mathrm{Z}(\mathcal{D}_{2n}) = \{1, a^{n/2}\} = \langle a^{n/2} \rangle$.

Case $n \equiv_2 1$.

$$\mathcal{D}_{2n} = \{1\} \sqcup \{a, a^{-1}\} \sqcup \{a^2, a^{-2}\} \sqcup \cdots \sqcup \{a^{(n-1)/2}, a^{(n+1)/2}\} \sqcup \{b, ba, \ldots, ba^{n-1}\}$$

For the last conjugacy class we observe that the multiples of 2 run through all elements of $\mathbf{Z}/(n)$ in the exponent of a.

In particular, $\mathrm{Z}(\mathcal{D}_{2n}) = \{1\} = 1$.

Problem 2

(1) Assume that $G = G_m \cdot H$. We have to show that H acts transitively on M. Since G acts transitively on M by assumption, we have to show that given $g \in G$, there exists an $h \in H$ such that $m^h = m^g$.

In fact, write $g = ch$ with $c \in G_m$ and $h \in H$. Then $m^h = m^{ch} = m^g$.

Conversely, assume that H acts transitively on M. We have to show that $G = G_m \cdot H$. Suppose given $g \in G$. Since H acts transitively on M, there exists an $h \in H$ such that $m^g = m^h$. Hence $c := gh^{-1} \in G_m$, and in fact $g = ch$.

(2) We restrict the action of G on $U\backslash G$ to U. This U-set contains the fixed point U. The U-orbits in $U\backslash G \smallsetminus \{U\}$ have a length that divides $|U|$, which is thus either equal to 1 or $\geq p$. Since by assumption $p > [G : U] - 1 = |U\backslash G \smallsetminus \{U\}|$, we can conclude that these U-orbits have length 1. In other words, for all $g \in G$ and all $u \in U$, we have $Ugu = Ug$; that is, $gug^{-1} \in U$. Hence $U \trianglelefteq G$.

Problem 3

We have an embedding

$$\begin{array}{ccc} G & \xrightarrow{\mu} & \mathcal{S}_G \\ g & \longmapsto & \big(\mu(g) : x \longmapsto xg\big) \ ; \end{array}$$

cf. Cayley's Theorem, which states that any group is isomorphic to a subgroup of a symmetric group.

Since the order of G is even, there exists an element $y \in G$ of order 2; cf. Cauchy's Theorem or Sylow's Theorem.

Since $\mu(y)$ is of order 2, its cycle type consists of 2-cycles. But since $\mu(y)$ does not have a fixed point, the number of 2-cycles in its cycle type is n.

Consider the sign (or signum) morphism

$$\begin{array}{ccc} \mathcal{S}_G & \xrightarrow{\mathrm{sgn}} & \{\pm 1\} \\ \sigma & \longmapsto & \mathrm{sgn}\,\sigma \ . \end{array}$$

Since $\mathrm{sgn}(\mu(y)) = -1$, we see that $\mathrm{sgn} \circ \mu : G \longmapsto \{\pm 1\}$ is surjective. Let $N \trianglelefteq G$ be the kernel of $\mathrm{sgn} \circ \mu$. We have $[G : N] = |\{\pm 1\}| = 2$.

Problem 4

The group $\mathrm{GL}_2(\mathbf{F}_3)$ acts on the set

$$L := \big\{ \langle (\begin{smallmatrix} 1 & 0 \end{smallmatrix}) \rangle,\ \langle (\begin{smallmatrix} 1 & 1 \end{smallmatrix}) \rangle,\ \langle (\begin{smallmatrix} 1 & -1 \end{smallmatrix}) \rangle,\ \langle (\begin{smallmatrix} 0 & 1 \end{smallmatrix}) \rangle \big\}$$

of one-dimensional subspaces of $\mathbf{F}_3^2$ by multiplication from the right. This yields a morphism

$$\begin{array}{rcl} \mathrm{GL}_2(\mathbf{F}_3) & \xrightarrow{\varphi} & \mathcal{S}_L \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \Big(\langle (\,x\;y\,)\rangle \longmapsto \langle (\,x\;y\,)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\rangle\Big)\ . \end{array}$$

Its kernel consists of those invertible matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that leave all one-dimensional subspaces invariant. Matrices of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, where $\lambda \in \mathbf{F}_3^\times := \mathbf{F}_3 \smallsetminus \{0\}$, fulfill this condition. Conversely, for a matrix that fulfills this condition, we have

$$\begin{array}{rcl} (\,1\;0\,)\begin{pmatrix} a & b \\ c & d \end{pmatrix} & = & (\,\mu\;0\,) \\ (\,0\;1\,)\begin{pmatrix} a & b \\ c & d \end{pmatrix} & = & (\,0\;\nu\,) \end{array}$$

for some $\mu,\ \nu \in \mathbf{F}_3^\times$, and thus $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix}$. Finally,

$$(\,1\;1\,)\begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix} \;=\; (\,\mu\;\nu\,) \;\in\; \langle (\,1\;1\,)\rangle$$

shows that $\mu = \nu$. Hence

$$\operatorname{Kern}\varphi \;=\; \Big\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \;:\; \lambda \in \mathbf{F}_3^\times\Big\} \;=\; \mathrm{Z}(\mathrm{GL}_2(\mathbf{F}_3))\ .$$

Now $|\,\mathrm{GL}_2(\mathbf{F}_3)| = (3^2-1)(3^2-3) = 48$ and $|\operatorname{Kern}\varphi| = 2$ shows that the image of φ has order $48/2 = 24$. Therefore, φ is surjective.

Since there is an isomorphism $\mathcal{S}_L \simeq \mathcal{S}_4$ induced by a bijection $L \xrightarrow{\sim} \{1,2,3,4\}$, by composition we also obtain an epimorphism $\mathrm{GL}_2(\mathbf{F}_3) \longrightarrow \mathcal{S}_4$ with kernel $\mathrm{Z}(\mathrm{GL}_2(\mathbf{F}_3))$.

As to the image of $\mathrm{SL}_2(\mathbf{F}_3)$ under φ, we claim that it equals the alternating group $\mathcal{A}_L$. In fact, since $\mathrm{Z}(\mathrm{GL}_2(\mathbf{F}_3)) \leq \mathrm{SL}_2(\mathbf{F}_3)$, we have $|\varphi\,(\mathrm{SL}_2(\mathbf{F}_3))| = 24/2 = 12$. Since $\mathcal{A}_L$ is the only subgroup of $\mathcal{S}_L$ of order 12, we conclude that $\varphi\,(\mathrm{SL}_2(\mathbf{F}_3)) = \mathcal{A}_L$.

Finally, under the chosen isomorphism $\mathcal{S}_L \simeq \mathcal{S}_4$, the subgroup $\mathcal{A}_L$ is mapped to the subgroup $\mathcal{A}_4$.