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Group theory, WS 07/08

Solution 13

Recall that a maximal subgroup of a group is, by definition, maximal amongst its proper subgroups.

Problem 47

(1) The assertion is false. Let $p = 2$ and $G = \mathcal{D}_8 \times \mathcal{C}_2$, where $\mathcal{D}_8 = \langle a, b \;:\; a^4,\, b^2,\, (ab)^2 \rangle$.

We have $\mathrm{K}_1(\mathcal{D}_8) = \mathcal{D}_8'$, which is generated by $[a,b] = a^2$ as a normal subgroup of $\mathcal{D}_8$; cf. proof of Theorem (3.21). However, $\langle a^2 \rangle$ is already normal in $\mathcal{D}_8$. Hence $\mathrm{K}_1(\mathcal{D}_8) = \langle a^2 \rangle$. Since a^2 is central in $\mathcal{D}_8$, we have $\mathrm{K}_2(\mathcal{D}_8) = [\mathcal{D}_8, \langle a^2 \rangle] = 1$.

Conversely, we have $\mathrm{Z}_1(\mathcal{D}_8) = \mathrm{Z}(\mathcal{D}_8) = \langle a^2 \rangle$; cf. Problem 1 (2). Thus $\mathcal{D}_8/\mathrm{Z}(\mathcal{D}_8)$ is of order 4, hence abelian. Since $\mathrm{Z}_2(\mathcal{D}_8)/\mathrm{Z}_1(\mathcal{D}_8) = \mathrm{Z}(\mathcal{D}_8/\mathrm{Z}(\mathcal{D}_8)) = \mathcal{D}_8/\mathrm{Z}(\mathcal{D}_8)$, we have $\mathrm{Z}_2(\mathcal{D}_8) = \mathcal{D}_8$.

So $\mathcal{D}_8$ does not give a counterexample.

But this also shows $G = \mathcal{D}_8 \times \mathcal{C}_2$ to be of nilpotency class $\ell = 2$.

Now $\mathrm{K}_1(\mathcal{D}_8 \times \mathcal{C}_2) = \mathrm{K}_1(\mathcal{D}_8) \times \mathrm{K}_1(\mathcal{C}_2) = \langle a^2 \rangle \times 1 \simeq \mathcal{C}_2$.

But $\mathrm{Z}_1(\mathcal{D}_8 \times \mathcal{C}_2) = \mathrm{Z}_1(\mathcal{D}_8) \times \mathrm{Z}_1(\mathcal{C}_2) = \langle a^2 \rangle \times \mathcal{C}_2 \simeq \mathcal{C}_2 \times \mathcal{C}_2$.

In particular, $\mathrm{K}_1(\mathcal{D}_8 \times \mathcal{C}_2) < \mathrm{Z}_1(\mathcal{D}_8 \times \mathcal{C}_2)$.

(2) The assertion is true. Since $\Phi(N)$ is characteristic in N, hence normal in G, it suffices to show that $\Phi(N) \le \Phi(G)$.

Assume that $\Phi(N) \not\le \Phi(G)$. Then there exists a maximal subgroup V of G such that $\Phi(N) \not\le V$, whence $\Phi(N)V = G$. This, however, implies $\Phi(N)(V \cap N) = \Phi(N)V \cap N = G \cap N = N$. If $V \cap N < N$, then $V \cap N$ is contained in a maximal subgroup of N, which also contains $\Phi(N)$, so that $\Phi(N)(V \cap N) < N$, which is not the case; cf. also Remark (4.12.2). Therefore, $V \cap N = N$, i.e. $N \le V$. But then $\Phi(N) \le N \le V$, and we have a *contradiction.*

(3) The assertion is false. Let p be a prime such that $p-1$ is not squarefree. For instance, p could be chosen to be equal to 5, 13, 17, 19 etc. Let k be an integer coprime to p such that its residue class in $\mathbf{F}_p$ generates the multiplicative group $\mathbf{F}_p^*$. So $k^s \not\equiv_p 1$ for all $s \in [1, p-2]$.

Write $\mathcal{C}_p = \langle b : b^p \rangle$ and $\mathcal{C}_{p-1} = \langle a : a^{p-1} \rangle$.

Let $\alpha : \mathcal{C}_{p-1} \longrightarrow \mathrm{Aut}\,\mathcal{C}_p$, $a \longmapsto (b \mapsto b^k)$. Let $G := \mathcal{C}_{p-1} \ltimes_\alpha \mathcal{C}_p$. So in G, we have $b^a = b^k$.

Let $U := \langle a \rangle \le G$. Then $\Phi(U) \ne 1$, for $p-1$ is not squarefree; cf. Problem 48 (1) and Theorem (4.16). (If one does not want to use Problem 48 (1), choose p in such a way that $p-1$ is a prime power with exponent ≥ 2, e.g. $p = 5$ or $p = 17$.)

We *claim* that $\Phi(G) = 1$. Since $[G : \langle a \rangle] = p$ is prime, $\langle a \rangle$ is a maximal subgroup in G. Since any conjugate of a maximal subgroup is a maximal subgroup, it suffices to show that $\langle a \rangle \cap \langle a^b \rangle = 1$.

Now for $s \in [1, p-2]$, we obtain

$$(a^b)^s \;=\; b^{-1}a^s b \;=\; a^s b^{-k^s} b \;=\; a^s b^{1-k^s} \;.$$

But since $k^s \not\equiv_p 1$, we conclude that $(a^b)^s \notin \langle a \rangle$. Since $\langle a^b \rangle = \{(a^b)^s \;:\; s \in [0, p-2]\}$, we conclude that $\langle a \rangle \cap \langle a^b \rangle = 1$. This proves the *claim.*

(4) The assertion is true. The proof is similar to the proof of (2). In fact, (2) is a particular case of (4).

Write $F := \bigcap_{g \in G} \Phi(U)^g$. Since F is normal in G, it suffices to show that $F \le \Phi(G)$.

Assume that $F \not\le \Phi(G)$. Then there exists a maximal subgroup V of G such that $F \not\le V$, whence $FV = G$.

Forming the intersection in $F = \bigcap_{g \in G} \Phi(U)^g$ is only needed in order to have a normal subgroup F of G, so that FV actually becomes a subgroup of G as well.

This, however, implies $F(V\cap U) = FV\cap U = G\cap U = U$. Since $F\leq \varphi(U)$, this implies $V\cap U = U$, i.e. $U\leq V$; cf. Remark (4.12.2). But then $F\leq U\leq V$, and we have a *contradiction.*

(5) The assertion is false. Let G be as in the solution to (3). Let $N := \langle b\rangle$. Then $\Phi(G/N)\simeq\Phi(\langle a\rangle)\neq 1$. However, $\Phi(G)N/N = 1$ since $\Phi(G) = 1$; cf. solution to (3).

(6) The assertion is true. Let U/N be a maximal subgroup of G/N. Then U is a maximal subgroup of G, which, moreover, contains N. In particular, $\Phi(G)N\leq U$, and so $\Phi(G)N/N\leq U/N$. Intersection over all these maximal subgroups yields $\Phi(G)N/N\leq\Phi(G/N)$.

If G is a p-group for some prime p, then the assertions (3) and (5) hold as well; cf. Theorem (4.16.3).

Problem 48

(1) Suppose given a maximal subgroup U of G. We *claim* that $U\times H$ is a maximal subgroup of $G\times H$.

Let $(x,y)\in(G\times H)\smallsetminus(U\times H)$. Note that $x\in G\smallsetminus U$, whence, by maximality of U, we have $G=\langle U,x\rangle$.

We have to show that $\langle U\times H,(x,y)\rangle = G\times H$. Since $G\times H = (G\times 1)(1\times H)$, and since $1\times H\leq U\times H$, it suffices to show that $G\times 1\leq\langle U\times H,(x,y)\rangle$. Now $(x,1) = (x,y)(1,y^{-1})\in\langle U\times H,(x,y)\rangle$. Hence $G\times 1 = \langle U\times 1,(x,1)\rangle\leq\langle U\times H,(x,y)\rangle$. This proves the *claim.*

The proof of this claim has been simplified by David Lorch.

We *claim* that $\Phi(G\times H)\leq\Phi(G)\times\Phi(H)$. Suppose given a maximal subgroup $U\leq G$. By intersection and symmetry in G and H, it suffices to show that $\Phi(G\times H)\leq U\times H$. By the preceding claim, however, we know that $U\times H$ is a maximal subgroup of $G\times H$, and the *claim* ensues.

We *claim* that $\Phi(G\times H)\geq\Phi(G)\times\Phi(H)$. Since $G\times 1\trianglelefteq G\times H$ we have, by Problem 47 (2), $\Phi(G\times 1)\leq\Phi(G\times H)$. Now the image of $\Phi(G)$ under the isomorphism $G\xrightarrow{\sim} G\times 1$, $g\longmapsto(g,1)$, is on the one hand equal to $\Phi(G)\times 1$, and on the other hand equal to $\Phi(G\times 1)$. Similarly for H. Hence

$$\Phi(G)\times\Phi(H) \;=\; (\Phi(G)\times 1)\cdot(1\times\Phi(H)) \;=\; \Phi(G\times 1)\cdot\Phi(1\times H) \;\leq\; \Phi(G\times H)\,.$$

This proves the *claim.*

The last two claims show that $\Phi(G\times H) = \Phi(G)\times\Phi(H)$.

(2) If G is nilpotent, then it is a direct product of its p-Sylow subgroups; cf. Theorem (4.9.1,5). Since given groups U and V, we have $(U\times V)' = U'\times V'$ and, by (1), $\Phi(U\times V) = \Phi(U)\times\Phi(V)$, we may assume that G is a p-group for some prime p. But then $G'\leq G'G^p = \Phi(G)$ by Theorem (4.16.1).

Conversely, suppose that $G'\leq\Phi(G)$. Then $G/\Phi(G)$ is abelian, hence nilpotent. By Theorem (4.14) we conclude that G is nilpotent.

Problem 49

If G is cyclic of prime power order for some prime p, generated by $g\in G$, then $\langle g^p\rangle$ is the unique maximal subgroup of G.

Conversely, suppose that G has a unique maximal subgroup U.

Assume that $|G|$ is not a prime power. Then U contains all proper subgroups of G, and so in particular, all Sylow subgroups of G. These are therefore Sylow subgroups of U as well, whence the product of their orders, taken over the different prime divisors of $|G|$, is equal to $|G|$ and equal to $|U|$. This *contradicts* $U<G$.

Therefore, G is a p-group for some prime p. Now $\Phi(G)$ equals the only maximal subgroup U of G. Hence $G/\Phi(G)\simeq\mathcal{C}_p$, for otherwise there would exist a nontrivial subgroup of $G/\Phi(G)$, yielding a subgroup of G strictly in between $\Phi(G)$ and G.

In particular, $G/\Phi(G)$ is generated by a single element, say $x\Phi(G)$, where $x\in G$. Hence $G=\langle x,\Phi(G)\rangle$. Now Remark (4.12.1) yields $G=\langle x\rangle$, thus solving the Problem.