Homework 9

To be handed in in the respective first tutorial in the week 01. May – 06. May.

Problem 42.

(1) Give the Taylor development of $f(x) = (\cos x)^{-1}$ in $x_0 = 0$ for $n = 4$. For $|x| \le \pi/4$, determine a constant $c > 0$ such that $|R_5(x)| \le c|x|^5$.

(2) Give the Taylor development of $f(x) = (2+x)^{4/5}$ in $x_0 = 0$ for $n = 3$. For $x \ge 0$, determine a constant $c > 0$ such that $|R_4(x)| \le cx^4$.

(3) Give the Taylor development of $f(x) = (\ln x)^2$ in $x_0 = 1$ for $n = 5$. For $x \ge 1$, determine a constant $c > 0$ such that $|R_6(x)| \le c(x-1)^6$.

Problem 43.

Determine the radius of convergence of the following power series.

(1) $\sum_{n=0}^{\infty} \dfrac{z^n}{n^2}$.

(2) $\sum_{n=0}^{\infty} \dfrac{(3n)^n}{n!} z^n$.

(3) $\sum_{n=1}^{\infty} n^n z^n$.

Problem 44.

Use the weak form of Stirling's formula to calculate the following limits.

(1) $\lim_{n\to\infty} (n!)^{1/n} n^{-1}$.

(2) $\lim_{n\to\infty} \binom{3n}{n}^{1/n}$.

Problem 45.

Use the power series definition of the complex exponential function together with a Cauchy product to show that $e^z e^w = e^{z+w}$ for all $z,\, w \in \mathbf{C}$.

Problem 46.

Calculate $\int e^x (\sin x)^3 \,\mathrm{d}x$ using $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$.

Problem 47. True or false?

Prove (by an argument) or disprove (by an example).

(1) If the Taylor series in x_0 of a function $f(x)$ converges in $x \ne x_0$, then the value of the Taylor series in x is equal to $f(x)$.

(2) Let $(a_n)_{n\ge 0}$ be a sequence of real numbers. If the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is > 1, then $\sum_{n=0}^{\infty} (-1)^n a_{2n}$ converges.