Kimmerle, Künzer; Mathematics for Engineers II (Math203)

Final Exam

Net duration 180 minutes.
We allow 1 formula sheet (of maximal size A3), but no calculator.
The total sum of points is 40.

Question 1 (8 points).

Let $U_1 = \operatorname{span}(\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\\0\end{pmatrix})$ and $U_2 = \operatorname{span}(\begin{pmatrix}0\\0\\0\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\\0\\0\end{pmatrix})$
be subspaces of the vector space $\mathbf{R}^5$ over $\mathbf{R}$.

(1) Determine bases for U_1 and U_2.

(2) Determine a basis for $U_1 \cap U_2$. Evaluate $\dim(U_1 + U_2)$.

Question 2 (7 points).

Let $f(x) = (x - \pi)^2$ for $x \in [0, 2\pi)$, continued 2π-periodically to $\mathbf{R}$.

(1) Determine the **complex** and the **real** Fourier series of $f(x)$.
In the solution for the real Fourier series, the complex number i must not show up.

(2) Use Parseval's Lemma to evaluate $\sum_{m=1}^{\infty} \frac{1}{m^4}$.

Question 3 (5 points).

Let $t \in \mathbf{R}$ be a parameter. Let

$$A = \begin{pmatrix} t & -1 & 1 & 0 \\ 1 & t & -1 & 1 \\ -1 & 1 & t & -1 \\ 0 & -1 & 1 & t \end{pmatrix}.$$

(1) Evaluate $\det A$.

(2) For which values of $t \in \mathbf{R}$ is A an invertible matrix?

Question 4 (4 points).

Use the chain rule to evaluate

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{x^2}^{0}(\tan t)e^{x\cos t}\,\mathrm{d}t\right)$$

for $x \in \mathbf{R}$ with $|x| < \sqrt{\pi/2}$.

Question 5 (7 points).

Let $f(x, y, z) = (x^2 + y^2 + z^2)e^{x+y+z}$ define a function from $\mathbf{R}^3$ to $\mathbf{R}$.

(1) Determine the critical points of $f(x, y, z)$.

(2) Calculate the principal minors of the Hesse matrix at the critical points of f. Find a local maximum or a local minimum of $f(x, y, z)$.

Question 6 (5 points).

(1) Determine the radius of convergence R of the power series $\sum_{n=0}^{\infty} 2^n i^n z^n$. Evaluate the series in $z = \frac{1}{4}$.

(2) Determine the radius of convergence R of the power series $\sum_{n=2}^{\infty} (n!)^{1/\ln n} z^n$.

Question 7 (4 points).

Determine a constant $c > 1$ such that

$$\ln(\ln x) \geq \frac{1}{e}(x-e) - \frac{1}{e^2}(x-e)^2 + \frac{c}{(x \ln x)^3}\,(x-e)^3$$

for $x \geq e$.