

Solution to Final

Question 1.

(1) To determine a basis of U_1 , we consider the row echelon form

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence a basis of U_1 is given e.g. by $\left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}\right)$, so in particular $U_1 = \text{span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}\right)$.

To determine a basis of U_2 , we consider the row echelon form

$$\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence a basis of U_2 is given e.g. by $\left(\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)$, so in particular $U_2 = \text{span}\left(\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)$.

(2) To determine a basis for $U_1 \cap U_2$, we solve

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

to get the general solution

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = s \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad s, t \in \mathbf{R},$$

yielding e.g. the basis

$$\left(\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}\right)$$

for $U_1 \cap U_2$.

Finally, we calculate

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) = 3 + 3 - 2 = 4.$$

Question 2.

(1) For $m \neq 0$, we get

$$\begin{aligned} c_m &= \frac{1}{2\pi} \int_0^{2\pi} (x - \pi)^2 e^{-imx} dx \\ &= \frac{1}{2\pi} \left([(x - \pi)^2 \frac{i}{m} e^{-imx}]_0^{2\pi} - 2 \frac{i}{m} \int_0^{2\pi} (x - \pi) e^{-imx} dx \right) \\ &= -\frac{i}{\pi m} \left([(x - \pi) \frac{i}{m} e^{-imx}]_0^{2\pi} - \frac{i}{m} \int_0^{2\pi} e^{-imx} dx \right) \\ &= \frac{2}{m^2}. \end{aligned}$$

For $m = 0$, we get

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} (x - \pi)^2 dx = \frac{\pi^2}{3}.$$

Thus

$$f(x) \stackrel{\text{ess.}}{=} \frac{\pi^2}{3} + 2 \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{e^{imx}}{m^2} = \frac{\pi^2}{3} + 4 \sum_{m=1}^{\infty} \frac{\cos(mx)}{m^2}.$$

(2) Parseval's lemma gives

$$\sum_{m=-\infty}^{+\infty} |c_m|^2 = \frac{\pi^4}{9} + 2 \sum_{m=1}^{\infty} \frac{4}{m^4} = \frac{1}{2\pi} \int_0^{2\pi} (x - \pi)^4 dx = \frac{\pi^4}{5},$$

hence

$$\sum_{m=1}^{\infty} \frac{1}{m^4} = \frac{\pi^4}{90}.$$

Question 3.

(1) We get e.g.

$$\begin{aligned} \det A &= \det \begin{pmatrix} t-1 & -1 & 1 & 0 \\ -1 & 1 & t-1 & 1 \\ 0 & -1 & 1 & t \end{pmatrix} = \det \begin{pmatrix} t-1 & -1 & 1 & 0 \\ -1 & 1 & t-1 & 1 \\ -t & 0 & 0 & t \end{pmatrix} = \det \begin{pmatrix} t-1 & -1 & 1 & 0 \\ -2 & t-1 & 1 & 1 \\ 0 & 0 & 0 & t \end{pmatrix} \\ &= t \det \begin{pmatrix} t-1 & -1 & 1 \\ -2 & 1 & t \end{pmatrix} = t \det \begin{pmatrix} t-1 & 1 \\ t & t \end{pmatrix} = t \det \begin{pmatrix} t-1 & -2 & 1 \\ 0 & t+1 & -1 \\ 0 & 0 & t \end{pmatrix} \\ &= t^2((t^2 - 1) - (-6)) = t^2(t^2 + 5). \end{aligned}$$

(2) The matrix A is invertible for $\det A = t^2(t^2 + 5) \neq 0$, i.e. for $t \in \mathbf{R} \setminus \{0\}$.

Question 4.

Let $h(y, z) \stackrel{\text{def}}{=} \int_y^0 (\tan t) e^{z \cos t} dt$, let $g(x) \stackrel{\text{def}}{=} \begin{pmatrix} x^2 \\ x \end{pmatrix}$. Since $(h \circ g)(x) = \int_{x^2}^0 (\tan t) e^{x \cos t} dt$, we have to calculate

$$(h \circ g)'(x) = h'(g(x)) \cdot g'(x).$$

Now $g'(x) = \begin{pmatrix} 2x \\ 1 \end{pmatrix}$ and

$$\begin{aligned} h'(y, z) &= \begin{pmatrix} -(\tan y) e^{z \cos y} & \int_y^0 \frac{\partial}{\partial z} (\tan t) e^{z \cos t} dt \\ -(\tan y) e^{z \cos y} & \int_y^0 (\sin t) e^{z \cos t} dt \end{pmatrix} \\ &= \begin{pmatrix} -(\tan y) e^{z \cos y} & \frac{1}{z} \int_0^y (-z \sin t) e^{z \cos t} dt \\ -(\tan y) e^{z \cos y} & \frac{1}{z} (e^{z \cos y} - e^z) \end{pmatrix}. \end{aligned}$$

So

$$\begin{aligned} (h \circ g)'(x) &= h'(g(x)) \cdot g'(x) = \begin{pmatrix} -\tan(x^2) e^{x \cos(x^2)} & \frac{1}{x} (e^{x \cos(x^2)} - e^x) \end{pmatrix} \cdot \begin{pmatrix} 2x \\ 1 \end{pmatrix} \\ &= -2x \tan(x^2) e^{x \cos(x^2)} + \frac{1}{x} (e^{x \cos(x^2)} - e^x). \end{aligned}$$

Question 5.

(1) To determine the critical points of $f(x, y, z)$, we solve

$$f'(x, y, z) = \begin{pmatrix} (2x+x^2+y^2+z^2)e^{x+y+z} & (2y+x^2+y^2+z^2)e^{x+y+z} & (2z+x^2+y^2+z^2)e^{x+y+z} \end{pmatrix} \stackrel{!}{=} (0 \ 0 \ 0).$$

We obtain first $2x = 2y = 2z = -x^2 + y^2 + z^2$, hence $x = y = z$.

From $2x + x^2 + y^2 + z^2 \stackrel{!}{=} 0$ we can now take that $2x = -3x^2$, i.e. that either $x = 0$ or $x = -\frac{2}{3}$.

Therefore, $f(x, y, z)$ has the critical points $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2/3 \\ -2/3 \\ -2/3 \end{pmatrix}$.

(2) The Hesse matrix of f is calculated to be

$$H_f(x, y, z) = e^{x+y+z} \begin{pmatrix} 2+4x+x^2+y^2+z^2 & 2y+2x+x^2+y^2+z^2 & 2z+2x+x^2+y^2+z^2 \\ 2x+2y+x^2+y^2+z^2 & 2+4y+x^2+y^2+z^2 & 2z+2y+x^2+y^2+z^2 \\ 2x+2z+x^2+y^2+z^2 & 2y+2z+x^2+y^2+z^2 & 2+4z+x^2+y^2+z^2 \end{pmatrix}.$$

Hence, for the critical point $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ we get

$$H_f(0, 0, 0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} .$$

Its principal minors are 2, 4 and 8, and therefore $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is a local minimum.

For the critical point $\begin{pmatrix} -2/3 \\ -2/3 \\ -2/3 \end{pmatrix}$ we get

$$H_f(-2/3, -2/3, -2/3) = e^{-2} \begin{pmatrix} 2/3 & -4/3 & -4/3 \\ -4/3 & 2/3 & -4/3 \\ -4/3 & -4/3 & 2/3 \end{pmatrix} .$$

Its principal minors are $2/3e^{-2}$, $-4/3e^{-4}$ and $-8e^{-6}$, and therefore we can not decide whether $\begin{pmatrix} -2/3 \\ -2/3 \\ -2/3 \end{pmatrix}$ is a local extremum.

Question 6.

(1) The radius of convergence is

$$R = \left(\lim_{n \rightarrow \infty} |2^n i^n|^{1/n} \right)^{-1} = \left(\lim_{n \rightarrow \infty} |2i| \right)^{-1} = \frac{1}{2} .$$

Moreover, the geometric series yields for $z = \frac{1}{4}$ the value

$$\sum_{n=0}^{\infty} 2^n i^n \left(\frac{1}{4} \right)^n = \sum_{n=0}^{\infty} \left(\frac{i}{2} \right)^n = \frac{1}{1 - i/2} = \frac{4 + 2i}{5} .$$

(2) We will calculate the radius of convergence as

$$R = \left(\lim_{n \rightarrow \infty} |(n!)^{1/\ln n}|^{1/n} \right)^{-1} = \left(\lim_{n \rightarrow \infty} (n!)^{1/(n \ln n)} \right)^{-1} .$$

By the weak form of Stirling's formula, we get

$$(e^{n \ln n + 1 - n})^{1/(n \ln n)} \leq (n!)^{1/(n \ln n)} \leq (e^{(n+1) \ln(n+1) - n})^{1/(n \ln n)} ,$$

whence

$$e^{1 + \frac{1}{n \ln n} - \frac{1}{\ln n}} \leq (n!)^{1/(n \ln n)} \leq e^{\frac{n+1}{n} \frac{\ln(n+1)}{\ln n} - \frac{1}{\ln n}} .$$

Now the exponent $1 + \frac{1}{n \ln n} - \frac{1}{\ln n}$ tends to 1 as n tends to ∞ .

Moreover, the exponent $\frac{n+1}{n} \frac{\ln(n+1)}{\ln n} - \frac{1}{\ln n}$ tends to $1 \cdot 1 + 0 = 1$ as n tends to ∞ .

Hence both the lower and the upper bound tend to e , so that also the middle sequence tends to e .

Inverting finally yields $R = 1/e$.

Question 7.

We calculate the Taylor development of $f(x) \stackrel{\text{def}}{=} \ln(\ln(x))$ in $x_0 = e$ for $n = 2$.

$$\begin{aligned} f(x) &= \ln(\ln(x)) & f(e) &= 0 \\ f'(x) &= \frac{1}{x} \frac{1}{\ln x} & f'(e) &= \frac{1}{e} \\ f''(x) &= -\frac{1}{x^2} \left(\frac{1}{\ln x} + \frac{1}{(\ln x)^2} \right) & f''(e) &= -\frac{2}{e^2} \\ f'''(x) &= \frac{1}{x^3} \left(\frac{2}{\ln x} + \frac{3}{(\ln x)^2} + \frac{2}{(\ln x)^3} \right) \end{aligned}$$

So for $x \geq e$ we have

$$\ln(\ln(x)) = \frac{1}{e}(x - e) - \frac{1}{e^2}(x - e)^2 + \frac{1}{\xi^3} \left(\frac{2}{\ln \xi} + \frac{3}{(\ln \xi)^2} + \frac{2}{(\ln \xi)^3} \right) \frac{(x - e)^3}{6}$$

for some ξ with $e \leq \xi \leq x$. In particular, $\ln \xi \geq 1$ and $\ln \xi \leq \ln x$, so that we can conclude that

$$\begin{aligned} \ln(\ln(x)) &\geq \frac{1}{e}(x - e) - \frac{1}{e^2}(x - e)^2 + \frac{1}{\xi^3} \left(\frac{2}{(\ln \xi)^3} + \frac{3}{(\ln \xi)^3} + \frac{2}{(\ln \xi)^3} \right) \frac{(x - e)^3}{6} \\ &\geq \frac{1}{e}(x - e) - \frac{1}{e^2}(x - e)^2 + \frac{1}{x^3} \frac{7}{(\ln x)^3} \frac{(x - e)^3}{6} , \end{aligned}$$

allowing us to choose $c = 7/6$.