Mathematics for Engineers II (Math203)

Makeup Exam

Net duration 180 minutes.
We allow 1 formula sheet (of maximal size A3), but no calculator.
The total sum of points is 50.

Question 1 (3+5 points).

Let $U_1 = \operatorname{span}\left(\begin{pmatrix}1\\1\\2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\2\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix}\right)$ and $U_2 = \operatorname{span}\left(\begin{pmatrix}-1\\2\\2\\4\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\2\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\2\\1\end{pmatrix}\right)$ be subspaces of the vector space $\mathbf{R}^5$ over $\mathbf{R}$.

(1) Determine bases for U_1 and U_2.

(2) Determine a basis for $U_1 \cap U_2$. Evaluate $\dim(U_1 + U_2)$.

Question 2 (8 points).

Let $f(x) = \frac{x^2}{(1+x^2)^2}$. Calculate the volume $\mathrm{V}(f, 0, 1)$ of the rotation solid between 0 and 1.

Question 3 (4+2+2 points).

Let $f(x) = e^x \sin x$ for $x \in [0, 2\pi)$, continued 2π-periodically to $\mathbf{R}$.

(1) Determine the complex Fourier series of $f(x)$.

(2) Use Parseval's Lemma to evaluate $\sum_{m=1}^{\infty} \frac{1}{m^4+4}$.

(3) Use the value of the Fourier series at $x = 0$ together with (2) to evaluate $\sum_{m=1}^{\infty} \frac{m^2}{m^4+4}$.

Question 4 (5 points).

Find the solution of the differential equation

$$x(e^y + e^{-y})y' = e^y - e^{-y}$$

that satisfies $y(1) = \ln 2$.

Question 5 (7 points).

Let

$$f(x, y) = \det \begin{pmatrix} 1 & x & 0 & 0 \\ y & 1 & x & 0 \\ 0 & y & 1 & x \\ 0 & 0 & y & 1 \end{pmatrix}$$

define a function f from $\mathbf{R}^2$ to $\mathbf{R}$.

Calculate the principal minors of the Hesse matrix $\mathrm{H}_f(x, y)$ for two different critical points of f. Is it possible to decide whether the respective critical point is an extremum?

Question 6 (3+1 points).

Let $f(x,y) = \begin{pmatrix} x^2-y^2 \\ x+y \\ x-y \end{pmatrix}$, let $g(u,v,w) = \frac{uw}{v}$.

(1) Calculate $(g \circ f)'(x,y)$ using the Chain Rule in several variables.

(2) Calculate $(g \circ f)'(x,y)$ again, without use of the Chain Rule.

Question 7 (4 points).

Determine the radius of convergence R of the power series

$$\sum_{n=1}^{\infty} \left(\frac{1+i}{n} \right)^n \cdot n! \cdot z^n \, .$$

Question 8 (3+2+1 points).

Let $f(x) = x^2 e^x$.

(1) Determine $f^{(n)}(x)$ for $n \geq 0$.

(2) Determine the Taylor series of $f(x)$ in $x_0 = 0$ using (1).

(3) Determine the Taylor series of $f(x)$ in $x_0 = 0$ again, using the Taylor series of e^x.