Solution to Makeup

Question 1.

(1) To determine a basis of U_1, we consider the row echelon form

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Hence a basis of U_1 is given e.g. by $\left(\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)$, so in particular $U_1 = \operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)$.

To determine a basis of U_2, we consider the row echelon form

$$\begin{pmatrix} -1 & -1 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 4 & 1 & 2 & 2 \\ -1 & -1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Hence a basis of U_2 is given e.g. by $\left(\begin{pmatrix} -1 \\ 2 \\ 2 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}\right)$, so in particular $U_2 = \operatorname{span}\left(\begin{pmatrix} -1 \\ 2 \\ 2 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}\right)$.

(2) To determine a basis for $U_1 \cap U_2$, we solve

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 1 & 2 & 1 & -2 & -1 & -1 \\ 2 & 1 & 1 & -2 & 0 & -1 \\ 1 & 1 & 0 & -4 & -1 & -2 \\ 1 & 0 & 0 & 1 & 1 & -1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2/3 & -1 \\ 0 & 1 & 0 & 0 & -1/3 & -1 \\ 0 & 0 & 1 & 0 & -1/3 & 2 \\ 0 & 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

to get the general solution

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \frac{s}{3} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \\ -1 \\ 3 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} , \qquad s,\ t \in \mathbf{R} ,$$

yielding e.g. the basis

$$\left(\begin{pmatrix} -2 \\ 1 \\ -2 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}\right)$$

for $U_1 \cap U_2$.

Finally, we calculate

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) = 3 + 3 - 2 = 4 .$$

Question 2.

We have

$$\mathrm{V}(f, 0, 1) = \pi \int_0^1 \frac{x^4}{(x^2+1)^4} \,\mathrm{d}x .$$

Partial fraction decomposition yields

$$\frac{x^4}{(x^2+1)^4} = \frac{1}{(1+x^2)^4} - \frac{2}{(1+x^2)^3} + \frac{1}{(1+x^2)^2} .$$

Hence

$$\begin{aligned}
\int_0^1 \tfrac{x^4}{(x^2+1)^4}\,\mathrm{d}x &= \int_0^1 \tfrac{\mathrm{d}x}{(x^2+1)^4} - 2\int_0^1 \tfrac{\mathrm{d}x}{(x^2+1)^3} + \int_0^1 \tfrac{\mathrm{d}x}{(x^2+1)^2} \\
&= \left[\tfrac{x}{6(x^2+1)^3}\right]_0^1 - \tfrac{7}{6}\int_0^1 \tfrac{\mathrm{d}x}{(x^2+1)^3} + \int_0^1 \tfrac{\mathrm{d}x}{(x^2+1)^2} \\
&= \tfrac{1}{48} - \tfrac{7}{6}\left[\tfrac{x}{4(x^2+1)^2}\right]_0^1 + \tfrac{1}{8}\int_0^1 \tfrac{\mathrm{d}x}{(x^2+1)^2} \\
&= -\tfrac{5}{96} + \tfrac{1}{8}\left[\tfrac{x}{2(x^2+1)}\right]_0^1 + \tfrac{1}{16}\left[\arctan x\right]_0^1 \\
&= \tfrac{\pi}{64} - \tfrac{1}{48} .
\end{aligned}$$

Finally, $\mathrm{V}(f,0,1) = \frac{\pi^2}{64} - \frac{\pi}{48}$.

Question 3.

(1) We get

$$\begin{aligned} c_m &= \tfrac{1}{2\pi}\int_0^{2\pi} e^x(\sin x)e^{-imx}\,\mathrm{d}x \\ &= \tfrac{1}{4\pi i}\int_0^{2\pi}(e^{(1+i-im)x} - e^{(1-i-im)x}\,\mathrm{d}x \\ &= \tfrac{1}{4\pi i}\left[\tfrac{1}{1+i-im}e^{(1+i-im)x} - \tfrac{1}{1-i-im}e^{(1-i-im)x}\right]_0^{2\pi} \\ &= \tfrac{e^{2\pi}-1}{2\pi(m^2-2+2im)}\,. \end{aligned}$$

Since f is continuous, it exactly equals its Fourier series, and we get

$$f(x) \;=\; \frac{e^{2\pi}-1}{2\pi}\sum_{m=-\infty}^{+\infty}\frac{e^{imx}}{m^2-2+2im}\,.$$

(2) Parseval's lemma gives

$$\frac{(e^{2\pi}-1)^2}{4\pi^2}\left(\frac{1}{4}+2\sum_{m=1}^{\infty}\frac{1}{m^4+4}\right) \;=\; \frac{1}{2\pi}\int_0^{2\pi} e^{2x}(\sin x)^2\,\mathrm{d}x \;=\; \frac{e^{4\pi}-1}{16\pi}\,,$$

hence

$$\sum_{m=1}^{\infty}\frac{1}{m^4+4} \;=\; \frac{\pi}{8}\frac{e^{2\pi}+1}{e^{2\pi}-1} - \frac{1}{8}\,.$$

(3) Equating $f(x)$ and its Fourier series at $x=0$, we obtain

$$0 \;=\; \frac{(e^{2\pi}-1)}{2\pi}\left(\sum_{m=-\infty}^{+\infty}\frac{1}{m^2-2+2im}\right) \;=\; \frac{(e^{2\pi}-1)}{2\pi}\left(-\frac{1}{2}+2\sum_{m=1}^{\infty}\frac{m^2-2}{m^4+4}\right)\,,$$

hence

$$\begin{aligned} \sum_{m=1}^{\infty}\frac{m^2}{m^4+4} &= \frac{1}{4}+2\sum_{m=1}^{\infty}\frac{1}{m^4+4} \\ &= \frac{\pi}{4}\frac{e^{2\pi}+1}{e^{2\pi}-1}\,. \end{aligned}$$

Question 4.

The constant solution $y=0$ of the differential equation does not satisfy $y(1)=\ln 2$.

To get the general solution, we separate and integrate, thus obtaining

$$\int\frac{e^y+e^{-y}}{e^y-e^{-y}}\,y'\,\mathrm{d}x \;=\; \int\frac{\mathrm{d}x}{x}\,,$$

whence

$$\ln(e^y-e^{-y}) \;=\; \ln x + C_0\,,$$

with a constant C_0, and so

$$e^y-e^{-y} \;=\; Cx\,,$$

with a constant C. We solve

$$(e^y)^2 - Cx(e^y) - 1 \;=\; 0$$

to get the general solution

$$y \;=\; \ln\left(\frac{Cx}{2}+\sqrt{\frac{C^2x^2}{4}+1}\right)\,.$$

Now

$$y(1) \;=\; \ln\left(\frac{C}{2}+\sqrt{\frac{C^2}{4}+1}\right) \;\overset{!}{=}\; \ln 2$$

yields $\frac{C^2}{4}+1=\left(2-\frac{C}{2}\right)^2$, hence $C=\frac{3}{2}$. So

$$y \;=\; \ln\left(\frac{3x}{4}+\sqrt{\frac{9x^2}{16}+1}\right)$$

is the solution of the differential equation satisfying $y(1)=\ln 2$.

Question 5.

We obtain

$$f(x,y) \;=\; \det\begin{pmatrix} 1 & x & 0 & 0 \\ y & 1 & x & 0 \\ 0 & y & 1 & x \\ 0 & 0 & y & 1 \end{pmatrix} \;=\; \det\begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1-xy & x & 0 \\ 0 & y & 1-xy & 0 \\ 0 & 0 & y & 1 \end{pmatrix} \;=\; (1-xy)^2-xy \;=\; 1-3xy+x^2y^2\;,$$

hence $f'(x,y)=(2xy^2-3y \quad 2x^2y-3x)$.

So $\binom{0}{0}$ is a critical point.

If $x\neq 0$, then, at a critical point, also $y\neq 0$, and it remains to satisfy $2xy=3$.

Altogether, the set of critical points of f is given by

$$\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} s \\ \frac{3}{2s} \end{pmatrix} \,\middle|\, s\in \mathbf{R}\smallsetminus\{0\}\right\}\,.$$

Since

$$\mathrm{H}_f(x,y) \;=\; \begin{pmatrix} 2y^2 & 4xy-3 \\ 4xy-3 & 2x^2 \end{pmatrix}\,,$$

we have $\mathrm{H}_f(0,0)=\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ with principal minors 0 and -9, and $\mathrm{H}_f(s,\frac{3}{2s})=\begin{pmatrix} \frac{9}{2s^2} & 3 \\ 3 & 2s^2 \end{pmatrix}$ with principal minors $\frac{9}{2s^2}$ and 0.

In all cases it is impossible to decide whether the critical point under consideration is an extremum.

Question 6.

(1) We have $f'(x,y)=\begin{pmatrix} 2x & -2y \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $g'(u,v,w)=\left(\frac{w}{v} \quad -\frac{uw}{v^2} \quad \frac{u}{v}\right)$, hence

$$\begin{aligned}(g\circ f)'(x,y) &= g'(f(x,y))\cdot f'(x,y)\\ &= \left(\frac{x-y}{x+y} \quad -\frac{(x-y)^2}{x+y} \quad x-y\right)\cdot\begin{pmatrix} 2x & -2y \\ 1 & 1 \\ 1 & -1 \end{pmatrix}\\ &= (2(x-y) \quad 2(y-x))\,.\end{aligned}$$

(2) Since $(g\circ f)(x,y)=\frac{(x^2-y^2)(x-y)}{(x+y)}=(x-y)^2$, we can calculate the derivative directly to obtain

$$(g\circ f)'(x,y) \;=\; (2(x-y) \quad 2(y-x))\,.$$

Question 7.

We have $|a_n|^{1/n}=\frac{\sqrt{2}}{n}\cdot(n!)^{1/n}$. The weak form of Stirling's formula gives

$$\sqrt{2}\cdot e^{\frac{1}{n}-n} \;=\; \frac{\sqrt{2}}{n}\cdot(n^n e^{1-n})^{1/n} \;\le\; \frac{\sqrt{2}}{n}\cdot(n!)^{1/n} \;\le\; \frac{\sqrt{2}}{n}\cdot((n+1)^{(n+1)}e^{-n})^{1/n} \;=\; \sqrt{2}\cdot e^{\ln\left(\frac{n+1}{n}\right)+\frac{\ln(n+1)}{n}-1}\,.$$

Both the upper bound and the lower bound tend to $e^{-1}\sqrt{2}$, hence so does $|a_n|^{1/n}$ itself. Therefore, $R=e/\sqrt{2}$

Question 8.

(1) We have

$$\begin{aligned} f^{(0)}(x) &= x^2 e^x \\ f^{(1)}(x) &= (x^2+2x)e^x \\ f^{(2)}(x) &= (x^2+4x+2)e^x \\ f^{(3)}(x) &= (x^2+6x+6)e^x \\ f^{(4)}(x) &= (x^2+8x+12)e^x \\ f^{(5)}(x) &= (x^2+10x+20)e^x \ , \end{aligned}$$

and so on, leading to

$$f^{(n)}(x) = (x^2+2nx+n(n-1))e^x \ ,$$

since then

$$\begin{aligned} f^{(n+1)}(x) &= ((x^2+2nx+n(n-1))e^x)' \\ &= (x^2+2nx+n(n-1)+2x+2n)e^x \\ &= (x^2+2(n+1)x+(n+1)((n+1)-1))e^x \ , \end{aligned}$$

as it should be.

(2) The Taylor series of $f(x)$ in $x_0 = 0$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n = \sum_{n=0}^{\infty} \frac{n(n-1)}{n!}\, x^n = \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!} \ .$$

(3) Alternatively, we get

$$x^2 e^x = x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} = \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!} \ ,$$

as expected.