Midterm Solution

Question 1.

(1) We obtain $A^{-1} = \frac{1}{4}\begin{pmatrix} -1 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & -1 \end{pmatrix}$.

(2) As row echelon form of A we obtain e.g. $\begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & i-1 \\ 0 & 0 & 0 \end{pmatrix}$.
Hence A is not invertible, and its rank is $\operatorname{rk} A = 2$.

Question 2.

(1) We obtain

$$\int_2^3 \frac{x^2+1}{x^2-1}\,\mathrm{d}x \;=\; \int_2^3 \left(1 + \frac{1}{x-1} - \frac{1}{x+1}\right)\mathrm{d}x \;=\; [x + \ln|x-1| - \ln|x+1|]_2^3 \;=\; 1 - \ln 2 + \ln 3\,.$$

(2) We obtain

$$\int \frac{\mathrm{d}x}{(x^2+2)^3} \;=\; \frac{x}{8(x^2+2)^2} + \frac{3}{8}\int\frac{\mathrm{d}x}{(x^2+2)^2} \;=\; \frac{x}{8(x^2+2)^2} + \frac{3x}{32(x^2+2)} + \frac{3}{32}\frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + \text{const.}$$

Question 3.

(1) We apply Gauss transformations to $(A|b)$ to obtain e.g. the row echelon form

$$\left(\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore,

$$\{x \in \mathbf{R}^5 \;:\; Ax = b\} \;=\; \{\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \mid \lambda, \mu \in \mathbf{R}\}\,.$$

(2) We have $\operatorname{rk} A = 3$. As a linear independent tuple of column vectors, we may take e.g.

$$\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)\,.$$

Justification: a row echelon form of the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

is e.g. given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\,.$$

Therefore, the only linear combination of these three columns that yields the zero vector is trivial.

Question 4.

(1) The general solution is $y = x^2 + Cx$ for $C \in \mathbf{R}$. Now $y(1) = 2$ yields $C = 1$. Hence the solution satisfying this initial condition is given by

$$y \;=\; x^2 + x\,.$$

(2) The general solution is, besides the constant solution $y = 0$, given by $y = (C - x^3/3)^{-1}$ for $C \in \mathbf{R}$. Now $y(1) = 2$ yields $C = 5/6$. Hence the solution satisfying this initial condition is given by

$$y \;=\; (5/6 - x^3/3)^{-1} \,.$$

Question 5.

(1) We obtain

$$\mathrm{V}(f, 1, \infty) \;=\; \pi \int_1^\infty x^{-2}\,\mathrm{d}x \;=\; \pi \,.$$

(2) We obtain

$$\mathrm{S}(f, 1, \infty) \;=\; 2\pi \int_1^\infty x^{-1}\sqrt{1 + x^{-4}}\,\mathrm{d}x \,.$$

Now $x^{-1}\sqrt{1 + x^{-4}} \geq x^{-1}$ for $x \in [1, \infty)$, and $\int_1^\infty x^{-1}\,\mathrm{d}x = \infty$. Hence the integral for the surface area diverges, and we have

$$\mathrm{S}(f, 1, \infty) \;=\; \infty \,.$$