

## Solution 1

We will give brief solutions to the homework problems, usually omitting the pictures.

**Problem 1.** Let  $f(x) = x(x-1)(x+1) = x^3 - x$ . We have

$$\begin{aligned} \int_{-1}^0 (x^3 - x) dx &= \left[ \frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^0 \\ &= \frac{1}{4}, \end{aligned}$$

and

$$\begin{aligned} \int_0^1 (x^3 - x) dx &= \left[ \frac{x^4}{4} - \frac{x^2}{2} \right]_0^1 \\ &= -\frac{1}{4}. \end{aligned}$$

The sign switch from  $1/4$  to  $-1/4$  is not a coincidence – note that  $f(-x) = -f(x)$ , and that the area below the  $x$ -axis counts negatively.

The area of the triangle with vertices  $(-1, 0)$ ,  $(0, 0)$  and  $(-1/2, f(-1/2) = 3/8)$  is given by  $1 \cdot (3/8)/2 = 3/16$ . The value of this triangle area approximates the integral  $\int_{-1}^0 f(x) dx$  from below, and indeed  $3/16 \leq 1/4$ .

**Problem 2.** Let  $f(x) = \sin x$ . We have  $\int_0^\pi \sin x dx = [-\cos x]_0^\pi = 2$ . The area calculated by the integral is circumscribed by the rectangle with vertices  $(0, 0)$ ,  $(0, 1)$ ,  $(\pi, 0)$  and  $(\pi, 1)$ , whose area has value  $\pi$ . Hence  $2 \leq \pi$  (but this is of course only a rough estimate, as we already see from the picture.)

**Problem 3.** Let  $f(x) = 1/x$ . We have  $\int_1^s (1/x) dx = [\ln x]_1^s = \ln s$ . In particular,

$$\lim_{s \rightarrow \infty} \int_1^s \frac{dx}{x} = \lim_{s \rightarrow \infty} \ln s = \infty.$$

Geometrically, this means that the area between the graph of  $1/x$  and the  $x$ -axis, from  $x = 1$  on to the right, is infinite.

Likewise, we have  $\int_s^1 (1/x) dx = [\ln x]_s^1 = -\ln s$ . In particular,

$$\lim_{s \rightarrow 0^+} \int_s^1 \frac{dx}{x} = \lim_{s \rightarrow 0^+} (-\ln s) = \infty.$$

Geometrically, this means that the area between the graph of  $1/x$  and the  $x$ -axis, and between  $x = 0$  and  $x = 1$ , is infinite (as one could have also derived from mirroring the result above at the graph of  $y = x$ ).

**Problem 4.** Let  $f(x) = 1/x^2$ . We have  $\int_1^s (1/x^2) dx = [-1/x]_1^s = 1 - 1/s$ . In particular,

$$\lim_{s \rightarrow \infty} \int_1^s \frac{dx}{x^2} = \lim_{s \rightarrow \infty} 1 - 1/s = 1.$$

Geometrically, this means that the area between the graph of  $1/x^2$  and the  $x$ -axis, from  $x = 1$  on to the right, is **finite**.

Similarly, we have  $\int_s^1 (1/x^2) dx = [-1/x]_s^1 = 1/s - 1$ . In particular,

$$\lim_{s \rightarrow 0^+} \int_s^1 \frac{dx}{x^2} = \lim_{s \rightarrow 0^+} (1/s - 1) = \infty.$$

Geometrically, this means that the area between the graph of  $1/x^2$  and the  $x$ -axis, and between  $x = 0$  and  $x = 1$ , is infinite.

**Problem 5.** We consider a car that is constantly accelerating by  $1 \frac{\text{m}}{\text{s}^2}$ , at that at the time  $t_0 = 0$  s is driving with velocity  $20 \frac{\text{m}}{\text{s}}$ .

Letting  $s(t_0) = 0$  m be the starting point, at which it drives with velocity  $v(t_0) = \dot{s}(t_0) = 20 \frac{\text{m}}{\text{s}}$ . It constantly accelerates with  $a(t) = \ddot{s}(t) = 1 \frac{\text{m}}{\text{s}^2}$  for all  $t$  (where the dot denotes the derivative by  $t$ ). Hence, by the Fundamental Theorem,

$$\dot{s}(t) - \dot{s}(t_0) = \int_{t_0}^{t_1} \ddot{s}(t) dt = t \cdot 1 \frac{\text{m}}{\text{s}},$$

i.e.  $v(t) = t \frac{\text{m}}{\text{s}^2} + \dot{s}(t_0) = t \frac{\text{m}}{\text{s}^2} + 20 \frac{\text{m}}{\text{s}}$ . Hence, again invoking the Fundamental Theorem,

$$\begin{aligned} s(t_1) &= s(t_1) - s(t_0) \\ &= \int_{t_0}^{t_1} \dot{s}(t) dt \\ &= \left[ \frac{t^2}{2} \frac{\text{m}}{\text{s}^2} + t \cdot 20 \frac{\text{m}}{\text{s}} \right]_{t_0}^{t_1} \\ &= \frac{(60 \text{ s})^2}{2} \frac{\text{m}}{\text{s}^2} + 60 \text{ s} \cdot 20 \frac{\text{m}}{\text{s}} \\ &= 3000 \text{ m} . \end{aligned}$$

So apparently, it is a sports car.