Solution 12

Problem 57.

(1) Since $\det A = \det\left(\begin{smallmatrix} 1 & 3 \\ -2 & t \end{smallmatrix}\right) = 1\cdot t - 3\cdot(-2) = t+6$, the matrix A is invertible for $t \in \mathbf{R}$ with $t \neq -6$ (also written as: it is invertible for $t \in \mathbf{R} \smallsetminus \{-6\}$).

(2) Using Gauß transformations, we obtain

$$\det A \;=\; \det\begin{pmatrix} 89 & 90 & 91 \\ 70 & 71 & 72 \\ 17 & 18 & 19 \end{pmatrix} \;=\; \det\begin{pmatrix} 72 & 72 & 72 \\ 53 & 53 & 53 \\ 17 & 18 & 19 \end{pmatrix} \;=\; 72\cdot 53\cdot\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 17 & 18 & 19 \end{pmatrix} \;=\; 72\cdot 53\cdot\det\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 17 & 18 & 19 \end{pmatrix} \;=\; 0\,.$$

So A is not invertible.

(3) Using Gauß transformations followed by a development by the first column, we obtain

$$\det\begin{pmatrix} i & 1 & 0 \\ -1 & i & t \\ 1 & t & 1 \end{pmatrix} \;=\; \det\begin{pmatrix} 0 & 1-it & -i \\ 0 & t+i & t+1 \\ 1 & t & 1 \end{pmatrix} \;=\; +1\cdot\det\begin{pmatrix} 1-it & -i \\ t+i & t+1 \end{pmatrix} \;=\; (1-it)(t+1)-(-i)(t+i) \;=\; t(1-it)\,.$$

So A is invertible whenever $t(1-it) \neq 0$, that is, whenever $t \neq 0$ and $t \neq -i$ (also written as: it is invertible for $t \in \mathbf{C} \smallsetminus \{0,-i\}$).

(4) Using Gauß transformations together with Laplace development, we can calculate e.g. as follows.

$$\det A \;=\; \det\begin{pmatrix} -1 & 1 & 1 & t \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ -1 & t & -1 & 1 \end{pmatrix} \;=\; \det\begin{pmatrix} -1 & 1 & 1 & t \\ 0 & -2 & 0 & 1-t \\ 0 & -2 & -2 & 1-t \\ 0 & t-1 & -2 & 1-t \end{pmatrix} \;=\; -\det\begin{pmatrix} -2 & 0 & 1-t \\ -2 & -2 & 1-t \\ t-1 & -2 & 1-t \end{pmatrix} \;=\; -\det\begin{pmatrix} -2 & 0 & 1-t \\ -2 & -2 & 1-t \\ t+1 & 0 & 0 \end{pmatrix}$$

$$=\; -(t+1)\det\begin{pmatrix} 0 & 1-t \\ -2 & 1-t \end{pmatrix} \;=\; -(t+1)(-(-2)(1-t)) \;=\; 2(t^2-1)\,.$$

So A is invertible whenever $t \neq 1$ and $t \neq -1$ (also written as: it is invertible for $t \in \mathbf{R} \smallsetminus \{-1,+1\}$).

(5) For instance, we can calculate

$$\det A \;=\; \det\begin{pmatrix} 1 & i & i & i & 1 \\ i & 1 & 1 & i & 1 \\ 1 & i & 1 & i & 1 \\ t & i & t & i & t \\ i & t & t & i & 1 \end{pmatrix} \;=\; i\det\begin{pmatrix} 1 & i & i & 1 & 1 \\ i & 1 & 1 & 1 & 1 \\ 1 & i & 1 & 1 & 1 \\ t & i & t & 1 & t \\ i & t & t & 1 & 1 \end{pmatrix} \;=\; i\det\begin{pmatrix} 1 & i & i & 1 & 0 \\ i & 1 & 1 & 1 & 0 \\ 1 & i & 1 & 1 & 0 \\ t & i & t & 1 & t-1 \\ i & t & t & 1 & 0 \end{pmatrix} \;=\; i\cdot(-(t-1))\det\begin{pmatrix} 1 & i & i & 1 \\ i & 1 & 1 & 1 \\ 1 & i & 1 & 1 \\ i & t & t & 1 \end{pmatrix}$$

$$=\; i(1-t)\det\begin{pmatrix} 1 & i & i & 1 \\ 0 & 2 & 2 & 1-i \\ 0 & 0 & 1-i & 0 \\ 0 & t+1 & t+1 & 1-i \end{pmatrix} \;=\; i(1-t)\det\begin{pmatrix} 2 & 2 & 1-i \\ 0 & 1-i & 0 \\ t+1 & t+1 & 1-i \end{pmatrix} \;=\; i(1-t)\det\begin{pmatrix} 2 & 2 & 1-i \\ 0 & 1-i & 0 \\ t-1 & t-1 & 0 \end{pmatrix}$$

$$=\; i(1-t)\cdot(1-i)\det\begin{pmatrix} 0 & 1-i \\ t-1 & t-1 \end{pmatrix} \;=\; i(1-t)(1-i)(-(t-1)(1-i)) \;=\; 2(t-1)^2\,.$$

So A is invertible whenever $t \neq 1$ (also written as: it is invertible for $t \in \mathbf{C} \smallsetminus \{1\}$).

Problem 58.

(1) We obtain

$$f'(x,y) \;=\; (2x+y \quad x+2y)\,.$$

Solving the system of linear equations $f'(x,y) = 0$ yields $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ as the only critical point of f.

Now the Hesse matrix becomes

$$\mathrm{H}_f(x,y) \;=\; \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\,,$$

in particular, $\mathrm{H}_f(0,0) \;=\; \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. The principal minors in $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ are

$$\det(2) \;=\; +2\,, \qquad \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \;=\; 3\,.$$

(2) We obtain

$$f'(x,y) \;=\; (-4y^3+4x^3 \quad 5y^4-12xy^2)\,.$$

So for $f'(x,y) = 0$ we need $4x^3 - 4y^3 = 0$, that is, $x = y$. Plugging this into the second equation $5y^4 - 12xy^2$, we get $5x^4 = 12x^3$, that is, either $x = 0$ or $x = \frac{12}{5}$. So the critical points of f are given by $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ and by $\left(\begin{smallmatrix} 12/5 \\ 12/5 \end{smallmatrix}\right)$.

Now the Hesse matrix becomes

$$\mathrm{H}_f(x,y) \;=\; \begin{pmatrix} 12x^2 & -12y^2 \\ -12y^2 & 20y^3-24xy \end{pmatrix}\,.$$

So at the critical point $\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$, we get $\mathrm{H}_f(0,0) = \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$. Thus there, the principal minors are both zero, and we can not decide whether it is a maximum or a minimum.

At the critical point $\left(\begin{smallmatrix}12/5\\12/5\end{smallmatrix}\right)$, we get $\mathrm{H}_f(12/5, 12/5) = \left(\begin{smallmatrix}12^3/5^2 & -12^3/5^2\\ -12^3/5^2 & 2\cdot 12^3/5^2\end{smallmatrix}\right)$. Therefore, the principal minors are

$$\det(12^3/5^2) = +12^3/5^2\,, \qquad \det\left(\begin{smallmatrix}12^3/5^2 & -12^3/5^2\\ -12^3/5^2 & 2\cdot 12^3/5^2\end{smallmatrix}\right) = \frac{12^6}{5^4}\det\left(\begin{smallmatrix}1&-1\\-1&2\end{smallmatrix}\right) = +\frac{12^6}{5^4}\,.$$

Since both of them are positive, we have a local minimum at $\left(\begin{smallmatrix}12/5\\12/5\end{smallmatrix}\right)$.

(3) We obtain

$$f'(x,y,z) = (z-2x-10 \quad z-4y \quad x+y-2z-10)\,.$$

The system of linear equations $f'(x,y,z)=0$ has the unique solution $\left(\begin{smallmatrix}-11\\-3\\-12\end{smallmatrix}\right)$, which is thus the only critical point of f.

Now the Hesse matrix becomes

$$\mathrm{H}_f(x,y,z) = \left(\begin{smallmatrix}-2&0&1\\0&-4&1\\1&1&-2\end{smallmatrix}\right)\,,$$

so in particular $\mathrm{H}_f(-11,-3,-12) = \left(\begin{smallmatrix}-2&0&1\\0&-4&1\\1&1&-2\end{smallmatrix}\right)$. Its principal minors are

$$\det(-2) = -2\,, \qquad \det\left(\begin{smallmatrix}-2&0\\0&-4\end{smallmatrix}\right) = +8\,, \qquad \det\left(\begin{smallmatrix}-2&0&1\\0&-4&1\\1&1&-2\end{smallmatrix}\right) = -10\,.$$

So f has a local maximum in $\left(\begin{smallmatrix}-11\\-3\\-12\end{smallmatrix}\right)$.

(4) We obtain

$$f'(x,y,z) = \Big((1-(x+y+z)yz)e^{-xyz} \quad ((1-(x+y+z)xz)e^{-xyz} \quad ((1-(x+y+z)xy)e^{-xyz}\Big)\,.$$

So $f'(x,y,z)=0$ requires first of all that neither x, y, z, nor $(x+y+z)$ vanishes. From

$$(x+y+z)yz = (x+y+z)xz = (x+y+z)xy = 1$$

we may therefore conclude that $x=y=z$, and thus that $x = 3^{-1/3}$. So the only critical point of f is $\left(\begin{smallmatrix}3^{-1/3}\\3^{-1/3}\\3^{-1/3}\end{smallmatrix}\right)$.

Now the Hesse matrix becomes

$$\mathrm{H}_f(x,y,z) = \begin{pmatrix} (-yz-(1-(x+y+z)yz)yz)e^{-xyz} & (-yz-(x+y+z)z-(1-(x+y+z)yz)xz)e^{-xyz} & (-yz-(x+y+z)y-(1-(x+y+z)yz)xy)e^{-xyz} \\ (-xz-(x+y+z)z-(1-(x+y+z)xz)yz)e^{-xyz} & (-xz-(1-(x+y+z)xz)xz)e^{-xyz} & (-xz-(x+y+z)x-(1-(x+y+z)xz)xy)e^{-xyz} \\ (-xy-(x+y+z)y-(1-(x+y+z)xy)yz)e^{-xyz} & (-xy-(x+y+z)x-(1-(x+y+z)xy)xz)e^{-xyz} & (-xy-(1-(x+y+z)xy)xy)e^{-xyz} \end{pmatrix}\,,$$

and in particular, $\mathrm{H}_f(3^{-1/3},3^{-1/3},3^{-1/3}) = 3^{-2/3}e^{-1/3}\cdot\left(\begin{smallmatrix}-1&-4&-4\\-4&-1&-4\\-4&-4&-1\end{smallmatrix}\right)$. Its principal minors are

$$(3^{-2/3}e^{-1/3})^1\det(-1) = -3^{-2/3}e^{-1/3}\,,$$

$$(3^{-2/3}e^{-1/3})^2\det\left(\begin{smallmatrix}-1&-4\\-4&-1\end{smallmatrix}\right) = -(3^{-2/3}e^{-1/3})^2\cdot 15\,,$$

$$(3^{-2/3}e^{-1/3})^3\det\left(\begin{smallmatrix}-1&-4&-4\\-4&-1&-4\\-4&-4&-1\end{smallmatrix}\right) = 3^{-2}e^{-1}\det\left(\begin{smallmatrix}-1&-4&-4\\0&15&12\\0&12&15\end{smallmatrix}\right) = -9e^{-1}\,.$$

So we can not decide whether f has a local maximum or minimum in $\left(\begin{smallmatrix}3^{-1/3}\\3^{-1/3}\\3^{-1/3}\end{smallmatrix}\right)$.

Problem 59. Let $v_1 = \left(\begin{smallmatrix}2\\5\\0\end{smallmatrix}\right)$, $v_2 = \left(\begin{smallmatrix}3\\-2\\0\end{smallmatrix}\right)$ and $v_3 = \left(\begin{smallmatrix}1\\-1\\2\end{smallmatrix}\right)$ be vectors in $\mathbf{R}^3$.

(1) Since $\det\left(\begin{smallmatrix}2&3&1\\5&-2&-1\\0&-1&2\end{smallmatrix}\right) = 2\det\left(\begin{smallmatrix}2&3\\5&-2\end{smallmatrix}\right) = 2(-4-15) = -38 \neq 0$, the columns of the matrix (v_1, v_2, v_3) form a linearly independent tuple.

(2) For instance,

$$u \overset{\text{def}}{=} v_1\times(v_1\times v_2) = \left(\begin{smallmatrix}2\\5\\0\end{smallmatrix}\right)\times\left(\left(\begin{smallmatrix}2\\5\\0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}3\\-2\\0\end{smallmatrix}\right)\right) = \left(\begin{smallmatrix}2\\5\\0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0\\0\\-19\end{smallmatrix}\right) = \left(\begin{smallmatrix}-95\\38\\0\end{smallmatrix}\right)$$

is, by construction, orthogonal to both v_1 and $v_1 \times v_2$.

(3) To find such a vector, we consider the equation for λ_1 and λ_2 resulting from

$$(\lambda_1 v_1 + \lambda_2 v_2) \cdot v_3 \;=\; 0\,,$$

namely

$$\lambda_1 \cdot (-3) + \lambda_2 \cdot 5 \;=\; 0\,.$$

So for instance we can take $\lambda_1 = 5$ and $\lambda_2 = 3$, so that $w \overset{\text{def}}{=} 5v_1 + 3v_2 = \left(\begin{smallmatrix} 19 \\ 19 \\ 0 \end{smallmatrix}\right)$ actually is orthogonal to v_3.

Problem 60.

(1) The assertion is not true. For instance, consider $f(x, y) = x^4 + y^4$. We have $f'(x, y) = (4x^3 \;\; 4y^3)$, so that $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ is a critical point of f. It is a local minimum of $f(x, y)$, in fact, $f(0, 0) = 0 \leq f(x, y)$ even for all $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbf{R}^2$. However, the Hesse matrix, given by $\mathrm{H}_f(x, y) = \left(\begin{smallmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{smallmatrix}\right)$, takes the value $\mathrm{H}_f(0, 0) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ at this critical point. In particular, its principal minors both vanish, and are hence not both positive.

Another, but trivial, example is the function $f(x, y) = 0$. Here the Hesse matrix is zero, whereas every point $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbf{R}^2$ is a local minimum (and a local maximum).

(2) The assertion is true. In fact,

$$\det A \;=\; \det A^{\mathrm{t}} \;=\; \det(-A) \;=\; (-1)^n \det A = -\det A\,,$$

and so $\det A = 0$, whence A is not invertible.