Solution 9

Problem 42.

(1) We obtain

$$\begin{array}{rcl} f^{(0)}(x) &=& (\cos x)^{-1} \\ f^{(1)}(x) &=& (\sin x)(\cos x)^{-2} \\ f^{(2)}(x) &=& (-(\cos x)^2+2)(\cos x)^{-3} \\ f^{(3)}(x) &=& (\sin x)(-(\cos x)^2+6)(\cos x)^{-4} \\ f^{(4)}(x) &=& ((\cos x)^4-20(\cos x)^2+24)(\cos x)^{-5} \\ f^{(5)}(x) &=& (\sin x)((\cos x)^4-60(\cos x)^2+120)(\cos x)^{-6}\,. \end{array} \qquad \begin{array}{rcl} f^{(0)}(0) &=& 1 \\ f^{(1)}(0) &=& 0 \\ f^{(2)}(0) &=& 1 \\ f^{(3)}(0) &=& 0 \\ f^{(4)}(0) &=& 5 \end{array}$$

Hence

$$(\cos x)^{-1} \;=\; 1+\frac{1}{2}\,x^2+\frac{5}{24}\,x^4+\underbrace{\frac{f^{(5)}(\xi)}{5!}}_{=\,R_5(x)}\,.$$

Moreover, for $|x|\le\pi/4$ we have

$$\begin{array}{rcl} |R_5(x)| &=& |(\sin\xi)((\cos\xi)^4-60(\cos\xi)^2+120)(\cos\xi)^{-6}\cdot x^5/120| \\ &\le& 2^{-1/2}\cdot(1+60+120)\cdot 2^{6/2}\cdot|x|^5/120 \\ &=& 2^{5/2}\cdot\frac{181}{120}\,|x|^5 \\ &\approx& 8.5324\cdot|x|^5\,. \end{array}$$

(2) Using the binomial series, we get

$$(2+x)^{4/5} \;=\; 2^{4/5}(1+(x/2))^{4/5} \;=\; \sum_{n=0}^{\infty} 2^{4/5}\cdot\binom{4/5}{n}\cdot(x/2)^n\,.$$

for $|x|<2$. To calculate $R_4(x)$, however, we have to revert to the direct method.

$$\begin{array}{rcl} f^{(0)}(x) &=& (2+x)^{4/5} \\ f^{(1)}(x) &=& (4/5)(2+x)^{4/5-1} \\ f^{(2)}(x) &=& (4/5)(4/5-1)(2+x)^{4/5-2} \\ f^{(3)}(x) &=& (4/5)(4/5-1)(4/5-2)(2+x)^{4/5-3} \\ f^{(4)}(x) &=& (4/5)(4/5-1)(4/5-2)(4/5-3)(2+x)^{4/5-4}\,. \end{array} \qquad \begin{array}{rcl} f^{(0)}(0) &=& 2^{4/5} \\ f^{(1)}(0) &=& (4/5)\cdot 2^{4/5-1} \\ f^{(2)}(0) &=& (4/5)(4/5-1)\cdot 2^{4/5-2} \\ f^{(3)}(0) &=& (4/5)(4/5-1)(4/5-2)\cdot 2^{4/5-3} \end{array}$$

Hence

$$(2+x)^{4/5} \;=\; 2^{4/5}(1+\frac{2}{5}\,x-\frac{1}{50}\,x^2+\frac{1}{250}\,x^3)+\underbrace{\frac{f^{(4)}(\xi)}{4!}}_{=\,R_4(x)}\,.$$

Moreover, for $x\ge 0$ we have

$$|R_4(x)| \;=\; \left|\frac{(-264/625)(2+\xi)^{-16/5}}{4!}\,x^4\right| \;\le\; \frac{11}{625}\,2^{-16/5}\,x^4 \;\approx\; 0.001915\cdot x^4\,.$$

(3) We obtain

$$\begin{array}{rcl} f^{(0)}(x) &=& (\ln x)^2 \\ f^{(1)}(x) &=& (2\ln x)x^{-1} \\ f^{(2)}(x) &=& (2-2\ln x)x^{-2} \\ f^{(3)}(x) &=& (-6+4\ln x)x^{-3} \\ f^{(4)}(x) &=& (22-12\ln x)x^{-4} \\ f^{(5)}(x) &=& (-100+48\ln x)x^{-5} \\ f^{(6)}(x) &=& (548-240\ln x)x^{-6}\,. \end{array} \qquad \begin{array}{rcl} f^{(0)}(1) &=& 0 \\ f^{(1)}(1) &=& 0 \\ f^{(2)}(1) &=& 2 \\ f^{(3)}(1) &=& -6 \\ f^{(4)}(1) &=& 22 \\ f^{(5)}(1) &=& -100 \end{array}$$

Hence

$$(\ln x)^2 \;=\; (x-1)^2-(x-1)^3+\frac{11}{12}\,(x-1)^4-\frac{5}{6}\,(x-1)^5+\underbrace{\frac{f^{(6)}(\xi)}{6!}\cdot(x-1)^6}_{=\,R_6(x)}\,.$$

Moreover, for $x \geq 1$ we have

$$|R_6(x)| \;=\; |(548 - 240\ln\xi)\xi^{-6}| \cdot \frac{(x-1)^6}{6!} \;\leq\; \frac{548}{6!}\,(x-1)^6 \;=\; \frac{137}{180}\,(x-1)^6 \,.$$

Problem 43.

(1) We get

$$|a_n|^{1/n} \;=\; n^{-2/n} \;=\; e^{(\ln n)(-2/n)} \;\to\; e^0 \;=\; 1 \,.$$

So the radius of convergence is $R = 1$.

(2) We get

$$|a_n|^{1/n} \;=\; 3n(n!)^{-1/n} \,.$$

Now by weak Stirling, we have

$$3ne^{-\ln(n+1)+1} \;\leq\; 3n(n!)^{-1/n} \;\leq\; 3ne^{-\ln n - 1/n + 1} \,.$$

Both the lower bound and the upper bound tend to $3e$, hence the radius of convergence is $R = (3e)^{-1}$.

(3) We get

$$|a_n|^{1/n} \;=\; n \;\to\; +\infty \,.$$

So the radius of convergence is $R = 0$.

Problem 44.

(1) By weak Stirling, we get

$$e^{(n\ln n + 1 - n)/n}n^{-1} \;\leq\; (n!)^{1/n}\,n^{-1} \;\leq\; e^{((n+1)\ln(n+1)-n)/n}n^{-1}$$

The lower estimate is equal to $e^{1/n-1}$, thus tends to e^{-1}. The upper estimate is equal to $e^{\ln(1+1/n)+\ln(n+1)/n-1}$, which likewise tends to e^{-1}. Hence

$$\lim_{n\to\infty}(n!)^{1/n}\,n^{-1} \;=\; 1/e \,.$$

(2) By weak Stirling, we get

$$\binom{3n}{n}^{1/n} \;\geq\; = \; e^{(3\ln(3n)+1/n-3)-((2+1/n)\ln(2n+1)-2)-((1+1/n)\ln(n+1)-1)} \,,$$

and this lower estimate tends to $27/4$.

We get the upper estimate

$$\binom{3n}{n}^{1/n} \;\leq\; = \; e^{((3+1/n)\ln(3n+1)-3)-(2\ln(2n)+1/n-2)-(\ln n+1/n-1)} \,,$$

and this upper estimate tends to $27/4$, too. So

$$\lim_{n\to\infty}\binom{3n}{n}^{1/n} \;=\; \frac{27}{4} \,.$$

Problem 45.

We have

$$\begin{aligned}
e^z e^w \;&=\; \left(\textstyle\sum_{k=0}^{\infty}\frac{z^k}{k!}\right)\left(\textstyle\sum_{l=0}^{\infty}\frac{w^l}{l!}\right)\\
&=\; \textstyle\sum_{s=0}^{\infty}\sum_{k+l=s}\frac{z^k}{k!}\cdot\frac{w^l}{l!}\\
&\overset{l=s-k}{=}\; \textstyle\sum_{s=0}^{\infty}\frac{1}{s!}\sum_{k=0}^{s}\frac{s!}{k!(s-k)!}z^k w^{s-k}\\
&=\; \textstyle\sum_{s=0}^{\infty}\frac{1}{s!}\sum_{k=0}^{s}\binom{s}{k}z^k w^{s-k}\\
&\overset{\text{binomial theorem}}{=}\; \textstyle\sum_{s=0}^{\infty}\frac{1}{s!}(z+w)^s\\
&=\; e^{z+w} \,.
\end{aligned}$$

Problem 46.

Using Euler's formula, we get

$$\begin{aligned}
\int e^x(\sin x)^3\,\mathrm{d}x &= \tfrac{i}{8}\int e^x(e^{ix}-e^{-ix})^3\,\mathrm{d}x \\
&= \tfrac{i}{8}\int(e^{(1+3i)x}-3e^{(1+i)x}+3e^{(1-i)x}-e^{(1-3i)x})\,\mathrm{d}x \\
&= \tfrac{i}{8}(\tfrac{1}{1+3i}e^{(1+3i)x}-\tfrac{3}{1+i}e^{(1+i)x}+\tfrac{3}{1-i}e^{(1-i)x}-\tfrac{1}{1-3i}e^{(1-3i)x})+\text{const.} \\
&= \tfrac{i}{8}(\tfrac{1-3i}{10}e^{(1+3i)x}-\tfrac{3-3i}{2}e^{(1+i)x}+\tfrac{3+3i}{2}e^{(1-i)x}-\tfrac{1+3i}{10}e^{(1-3i)x})+\text{const.} \\
&= \tfrac{i+3}{80}e^{(1+3i)x}+\tfrac{-3i-3}{16}e^{(1+i)x}+\tfrac{3i-3}{16}e^{(1-i)x}+\tfrac{-i+3}{80}e^{(1-3i)x}+\text{const.} \\
&= -\tfrac{1}{40}e^x\sin(3x)+\tfrac{3}{40}e^x\cos(3x)+\tfrac{3}{8}e^x\sin x-\tfrac{3}{8}e^x\cos x+\text{const.}
\end{aligned}$$

Problem 47.

(1) The assertion is false. For instance, take $f(x) \overset{\text{def}}{=} e^{-1/x^2}$, where $f(0) \overset{\text{def}}{=} 0$. For $n \geq 0$, we have $f^{(n)}(x)$ being equal to e^{-1/x^2}, times some polynomial in $1/x$, which becomes zero as x tends to 0, and which is again differentiable in 0. So the Taylor series of $f(x)$ is equal to

$$0 + 0\cdot x + 0\cdot x^2 + \cdots\ ,$$

which converges for all $x \in \mathbf{R}$ to 0. However, for $x \neq 0$, we have $f(x) \neq 0$.

(The problem with this function is that for $x \neq 0$, the error term $R_{n+1}(x)$ does not tend to 0 as n tends to ∞.)

(2) The assertion is true (since we always have one true assertion). In fact, if the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is > 1, then in particular the series $\sum_{n=0}^{\infty} a_{2n} i^n$ converges. Now $\sum_{n=0}^{\infty}(-1)^n a_{2n}$ is the real part of this series, so it converges as well.