

# A genus question for orders

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## General

$R$  : principal ideal domain

$p \in R$  : prime

$\Lambda \subseteq R \times \cdots \times R =: \Gamma$  : orders over  $R$  such that

- $\Lambda_{(p)}$  is local
- $p^m \Gamma \subseteq \Lambda \subseteq \Gamma$  for some  $m \geq 0$

Construct

$$\begin{array}{ccc} \alpha = (\alpha_i)_i & \Gamma_{(p)}^* = \overbrace{R_{(p)}^* \times \cdots \times R_{(p)}^*}^{\text{normalised idèles}} & \\ \downarrow & \downarrow & \\ \bar{\alpha} & \Gamma_{(p)}^* / \Lambda_{(p)}^* \Gamma^* =: \underbrace{\text{Cl}(\Lambda)}_{\text{idèle class group}} & \end{array}$$

To  $\alpha \in \Gamma_{(p)}^*$ , we attach the projective indecomposable  $\Lambda$ -module

$$\Lambda(\alpha) := \{ (x_i)_i \in \Gamma : (x_i \alpha_i^{-1})_i \in \Lambda_{(p)} \}.$$

- All projective indecomposable  $\Lambda$ -modules are of this form.
- $\Lambda(\alpha) \simeq \Lambda(\beta) \Leftrightarrow \bar{\alpha} = \bar{\beta}$
- $\text{Hom}_{\Lambda}(\Lambda(\alpha), \Lambda(\beta)) \simeq \Lambda(\alpha^{-1}\beta)$
- As  $\Lambda_{(p)}$ -modules, we have  $\Lambda(\alpha)_{(p)} \simeq \Lambda_{(p)}$ .

A ring Morita-equivalent to  $\Lambda$  is of the form

$$\Xi_{\alpha_1, \dots, \alpha_t} := \text{End}_{\Lambda}(\Lambda(\alpha_1) \oplus \cdots \oplus \Lambda(\alpha_t))$$

for some  $t \geq 1$  and some  $\alpha_i \in \Gamma_{(p)}^*$  for  $i \in [1, t]$ .

E.g.  $\Xi_{1, \dots, 1} \simeq \Lambda^{t \times t}$ . Note that

$$(\Xi_{\alpha_1, \dots, \alpha_t})_{(p)} \simeq (\Xi_{1, \dots, 1})_{(p)}$$

## Example

$R := \mathbf{Z}$

$p := 5$

$$\Lambda := \underbrace{\{ (z, z') \in \mathbf{Z} \times \mathbf{Z} : z \equiv_5 z' \}}_{\text{“ } \mathbf{Z}_{-5} - \mathbf{Z} \text{”}} \subseteq \mathbf{Z} \times \mathbf{Z} =: \Gamma$$

Calculate

$$\begin{aligned} \text{Cl}(\Lambda) &= \frac{\mathbf{Z}_{(5)}^* \times \mathbf{Z}_{(5)}^*}{\{ (z, z') \in \mathbf{Z}_{(5)}^* \times \mathbf{Z}_{(5)}^* : z \equiv_5 z' \} \cdot (\{\pm 1\} \times \{\pm 1\})} \\ &= \langle \overline{(2, 1)} \rangle \simeq \mathbf{C}_2 \not\cong 1, \end{aligned}$$

whence Krull-Schmidt fails in  $\Lambda$ -proj. Therefore,  $\Lambda$  being a certain endomorphism ring, it also fails in  $\mathbf{ZS}_5$ -mod – as is well-known.

To  $\alpha := (2, 1)$ ,  $1 = (1, 1) \in \mathbf{Z}_{(5)}^* \times \mathbf{Z}_{(5)}^* = \Gamma_{(p)}^*$ , we attach

$$\Lambda(\alpha) = \{ (z, z') \in \mathbf{Z} \times \mathbf{Z} : -2z \equiv_5 z' \}$$

$$\Lambda(1) = \{ (z, z') \in \mathbf{Z} \times \mathbf{Z} : z \equiv_5 z' \} = \Lambda.$$

We obtain

$$\begin{aligned} \Xi_{\alpha, 1} &= \text{End}_{\Lambda}(\Lambda(\alpha) \oplus \Lambda(1)) \\ &\simeq \{ ((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), (\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix})) \in \mathbf{Z}^{2 \times 2} \times \mathbf{Z}^{2 \times 2} : \begin{array}{l} a \equiv_5 a', \quad 2b \equiv_5 b', \\ -2c \equiv_5 c', \quad d \equiv_5 d' \end{array} \} \\ \Xi_{1, 1} &= \text{End}_{\Lambda}(\Lambda(1) \oplus \Lambda(1)) \\ &\simeq \{ ((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), (\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix})) \in \mathbf{Z}^{2 \times 2} \times \mathbf{Z}^{2 \times 2} : \begin{array}{l} a \equiv_5 a', \quad b \equiv_5 b', \\ c \equiv_5 c', \quad d \equiv_5 d' \end{array} \}. \end{aligned}$$

## General (cont.)

Write  $\text{Cl}(\Lambda)^t := \{ \bar{\alpha}^t : \bar{\alpha} \in \text{Cl}(\Lambda) \} \leq \text{Cl}(\Lambda)$ .

Suppose given  $t \geq 1$  and  $\alpha_i, \beta_i \in \Gamma_{(p)}^*$  for  $i \in [1, t]$ .

*Proposition.* We have

$$\Xi_{\alpha_1, \dots, \alpha_t} \simeq \Xi_{\beta_1, \dots, \beta_t}$$

as  $\Lambda$ -algebras if and only if

$$(\bar{\alpha}_1 \cdots \bar{\alpha}_t)(\bar{\beta}_1 \cdots \bar{\beta}_t)^{-1} \in \text{Cl}(\Lambda)^t.$$

Thus we partition the Morita-equivalence class of  $\Lambda$  into isoclasses.

Main ingredient :

Jacobinski's Cancellation Theorem.

In this context it states that given  $P, Q$  finitely generated projective  $\Lambda$ -modules, we have the implication

$$P \oplus \Lambda \simeq Q \oplus \Lambda \implies P \simeq Q.$$

(Without Jacobinski, it is possible to give a complicated proof for the case  $t = 2$ .)

## Example (cont.)

Since  $(\bar{\alpha} \cdot \bar{1})(\bar{1} \cdot \bar{1})^{-1} = \bar{\alpha} \notin \underbrace{\text{Cl}(\Lambda)^2}_{\simeq 1} \leq \underbrace{\text{Cl}(\Lambda)}_{= \langle \bar{\alpha} \rangle \simeq C_2}$ , we have

$$\Xi_{\alpha,1} \not\simeq \Xi_{1,1}.$$

Since  $\text{Cl}(\Lambda)^3 = \text{Cl}(\Lambda)$ , we have

$$\Xi_{\alpha,1,1} \simeq \Xi_{1,1,1}.$$

Since  $\text{Cl}(\Lambda)^4 \simeq 1$ , we have

$$\Xi_{\alpha,1,1,1} \not\simeq \Xi_{1,1,1,1}.$$

Representatives of the isoclasses within the Morita-equivalence class of  $\Lambda$  are given by

$$\Xi_1 (= \Lambda), \Xi_{1,1}, \Xi_{\alpha,1}, \Xi_{1,1,1}, \Xi_{1,1,1,1}, \Xi_{\alpha,1,1,1}, \dots$$



5. Using Specht lattices and their duals for the Wedderburn isomorphisms, the answers were all affirmative for  $(\mathbf{ZS}_n)_{[q]}$  for  $n \in [1, 6]$ ,  $q \in \pi(\mathbf{S}_n)$ .

To illustrate, there exists an endomorphism ring of an indecomposable projective  $(\mathbf{ZS}_6)_{[2]}$ -module isomorphic to

$$\Lambda := \{ (a, b, c, d, e, f) \in \mathbf{Z}^{\times 6} : a \equiv_2 e, a + d - 2f \equiv_{16} b + c - 2e \equiv_8 0, e - f \equiv_4 c - d \equiv_2 0 \},$$

having  $\text{Cl}(\Lambda) \simeq \mathbf{C}_4$ .