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Ore localization, associated torsion and algorithms

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Abstract

Ore localization, associated torsion and algorithms

Ore localization of rings and modules is a generalization of the classical notion of commutative localization to the non-commutative case: given a left Ore set S in a (non-commutative) domain R, we can construct the left Ore localization $S^{-1}R$.

After recalling the theory of Ore localization of domains this work introduces the notion of $\mathrm{LSat}_T(M)$, the left T-closure of a subset M of a left R-module with respect to a quasi-multiplicatively closed subset T of R. We present two immediate applications of this construction. The first one leads to the concept of S-closure of a submodule of a free module and gives insight into the extension-contraction problem for localizations, while the second one results in $\mathrm{LSat}(S) := \mathrm{LSat}_R(S)$ for a left Ore set S, which is a saturated superset of S that reveals the structure of the localized ring and can be seen as the canonical form for S, since the localizations $S^{-1}R$ and $\mathrm{LSat}(S)^{-1}R$ are isomorphic. Furthermore, $\mathrm{LSat}(S)$ gives us a complete characterization of the units in $S^{-1}R$.

Equipped with this notion, we explore the concept of Ore localized modules together with the notions of local torsion and annihilators with a view towards the algebraic systems theory.

We also give an algorithm to compute the S-closure of an ideal in a special case which has applications in the theory of D-modules.

Ore-Lokalisierung, assoziierte Torsion und Algorithmen

Ore-Lokalisierung von Ringen und Moduln ist eine Verallgemeinerung des klassischen Konzepts der kommutativen Lokalisierung auf den nicht-kommutativen Fall: Aus einer gegebenen Links-Ore-Menge S in einem (nicht-kommutativen) Bereich können wir die Links-Ore-Lokalisierung $S^{-1}R$ konstruieren.

Nach einer Auffrischung der Theorie der Ore-Lokalisierung von Integritätsbereichen stellen wir $\mathrm{LSat}_T(M)$ vor, den Links-T-Abschluss einer Teilmenge M eines Links-R-Moduls bezüglich einer quasi-multiplikativ abgeschlossenen Teilmenge T von R. Wir geben zwei direkte Anwendungen dieser Konstruktion. Die Erste führt zum Konzept des S-Abschlusses eines Untermoduls eines freien Moduls und gibt Einblick in das Erweiterungs-Kontraktions-Problem für Lokalisierungen. Die Zweite ergibt $\mathrm{LSat}(S) := \mathrm{LSat}_R(S)$ für eine Links-Ore-Menge S, eine saturierte Obermenge von S die die Struktur des lokalisierten Ringes enthüllt und als eine Standardform für S betrachtet werden kann, da die Lokalisierungen $S^{-1}R$ und $\mathrm{LSat}(S)^{-1}R$ isomorph sind. Weiterhin liefert uns $\mathrm{LSat}(S)$ eine vollständige Beschreibung der Einheiten in $\mathrm{LSat}(S)^{-1}R$.

Mit diesen Werkzeugen ausgestattet widmen wir uns dem Konzept der Ore-lokalisierten Moduln zusammen mit den Begriffen der lokalen Torsion sowie der Annihilatoren und geben einen Ausblick zur algebraischen Systemtheorie.

Abschließend stellen wir einen Algorithmus vor, um den S-Abschluss eines Ideals in einem Spezialfall zu berechnen, der Anwendungen in der D-Modul-Theorie hat.

Contents

Introduction

In the commutative world, localizing a domain R is straight-forward: take a subset S of R that is multiplicatively closed (meaning $1 \in S$ and $st \in S$ for all $s, t \in S$) and introduce a specific equivalence relation on the tuples of $S \times R$, then the localization $S^{-1}R$ is $S \times R$ modulo the equivalence relation. If R is a non-commutative domain we can salvage this process to obtain a *left Ore localization* by additionally requiring S to be a *left Ore set*, that is, for any pair $(s, r) \in S \times R$ there is a pair $(\tilde{s}, \tilde{r}) \in S \times R$ such that $\tilde{s}r = \tilde{r}s$.

After recalling basic algebraic structures and ring-theoretic concepts in Chapter 1, in Chapter 2 we give an overview of construction and properties of Ore localized domains with digressions to the commutative case as well as to *graded localizations*, rounded off with a view towards special cases of localizations. Chapter 3 deals with the question under which assumptions certain properties related to localization are preserved under ring homomorphisms.

The "reverse" property to multiplicative closedness is the concept of a *saturated* set, where $st \in S$ implies $s \in S$ and $t \in S$. One of the starting points for this thesis was the following problem: given a left Ore set S in a domain R, does there exist a saturated superset T of S that is left Ore and satisfies $S^{-1}R \cong T^{-1}R$? In Chapter 4 we give a positive answer to this question by introducing the notion of $\mathrm{LSat}(S)$, the left saturation closure of S in R, which has the desired properties and additionally gives us a complete characterization of the units in the localization $S^{-1}R$: a left fraction (s, r) is a unit in $S^{-1}R$ if and only if $r \in \mathrm{LSat}(S)$.

As it turns out, $\mathrm{LSat}(S)$ is just a special case of a more general construction that also encompasses the notion of S-closure of a submodule which is related to the extension-contraction problem: given a left ideal I in R, what is the preimage of $(S^{-1}R)I$ under the embedding $R \to S^{-1}R$, which maps r to $(1, r)$?

Analogously to the commutative case we can define *Ore localized modules* as the tensor product $S^{-1}R \otimes_R M$ of the Ore localization $S^{-1}R$ with an R-module M over the base ring R. In Chapter 5 we see that $S^{-1}R \otimes_R \cdot$ is an exact covariant functor which is compatible with finite presentation.

Given a domain R and an R-module M let $r \in R \setminus \{0\}$ and $m \in M$ such that $rm = 0$. Viewed from the module M this effect is called torsion. In this thesis, we consider *local torsion*, that is torsion of M with respect to an arbitrary non-empty subset of R. In particular, torsion with respect to left Ore sets enjoys many of the properties well-known from the classical notion of torsion. While torsion occurs in modules, the same phenomenon viewed from R is called annihilation, which gives rise to the concept of *(pre-)annihilators*. Here, our main focus is the compatibility of localizing and taking annihilators. As an application we consider the Ore localization of finitely presented modules from the viewpoint of algebraic systems theory.

Lastly, Chapter 6 recalls the basics of the theory of *Gröbner bases* in G-algebras, the induced Gröbner basis theory in rational Ore localized G-algebras and the concept of *central saturation*. With the help of these ingredients, we present an algorithm to compute the S-closure of an ideal in a G-algebra A, where $S \cup \{0\}$ is a commutative polynomial ring contained in the center of A, a setting that is of importance in the theory of D-modules.

To illustrate the discussed concepts we regularly turn to a main example in the first Weyl algebra that accompanies most of this thesis.

1. Basics and notation

Definition 1.1. For $i, j \in \mathbb{N}$, define the *Kronecker delta*

$$\delta_{i,j} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

1.1. Algebraic structures with one operation

Definition 1.2. A *magma* is a non-empty set M together with a binary operation $* : M \times M \to M$, $(m_1, m_2) \to m_1 * m_2$. We denote it as $(M, *)$ or just by M. An element $m \in M$ is called

- *left-cancellative* (resp. *right-cancellative*), if $m * a = m * b$ (resp. $a * m = b * m$) implies $a = b$ for all $a, b \in M$.
- *cancellative*, if m is both left- and right-cancellative.
- a *neutral element*, if $m * a = a = a * m$ for all $a \in M$.

The magma $(M, *)$ (resp. its operation $*$) is called

- *associative*, if $a * (b * c) = (a * b) * c$ holds for all $a, b, c \in M$.
- *commutative*, if $a * b = b * a$ holds for all $a, b \in M$.
- *left-cancellative*/*right-cancellative*/*cancellative*, if the respective property holds for all elements of M.

A non-empty subset $N \subseteq M$ is called a *submagma* of M, if N is closed under $*$, that is, if $a * b \in N$ for all $a, b \in N$.

Remark 1.3. A magma $(M, *)$ has at most one neutral element. To see this, assume that $e_1, e_2 \in M$ are neutral elements, then we have $e_1 = e_1 * e_2 = e_2$.

Definition 1.4. A *semigroup* is an associative magma. A *subsemigroup* of a semigroup M is a submagma of M.

Definition 1.5. A *monoid* is a semigroup $(M, *)$ with a neutral element $e \in M$. We sometimes write $(M, *, e)$ to highlight the neutral element. An element $m \in M$ is called

- *invertible*, if there exists $n \in M$ such that $m * n = e = n * m$. We call n an *inverse* of m.
- *absorbing*, if $m * n = m = n * m$ holds for all $n \in M$.

A *submonoid* of M is a subsemigroup of M that contains the neutral element of M.
The submonoid *generated* by $N \subseteq M$ is the smallest submonoid of M with respect to inclusion that contains N (by convention, we set $[\emptyset] = \{e\}$). We denote it by $[N]$.

Remark 1.6. Let $(M, *, e)$ be a monoid.

- An element m in a monoid $(M, *, e)$ has at most one inverse. To see this, assume that $a, b \in M$ are inverse to m, then $a = a * e = a * (m * b) = (a * m) * b = e * b = b$.

- Similarly, M has at most one absorbing element: let $a, b \in M$ be absorbing, then $a = a * b = b$.
- If $N \subseteq M$ is a subset, then

$$[N] = \bigcup_{n \in \mathbb{N}_0} \{s_1 \cdots s_n \mid s_i \in N \text{ for all } 1 \leq i \leq n\}.$$

  By convention, the empty product ($n = 0$) is equal to e.
- Let J be a non-empty index set such that $M_j \subseteq M$ for all $j \in J$. Then

$$\left[\bigcup_{j \in J} M_j\right] = \left[\bigcup_{j \in J} [M_j]\right].$$

Definition 1.7. A *group* is a monoid in which every element is invertible. An *Abelian group* is a commutative group. A *subgroup* of a group G is a submonoid of G that is closed under taking inverses.

1.2. Algebraic structures with two operations

Definition 1.8. A *ring* is a non-empty set R together with two binary operations $+, \cdot : R \times R \to R$, such that

- $(R, +)$ is an Abelian group with neutral element $0 \in R$,
- $(R, \cdot)$ is a monoid with neutral element $1 \in R$, and
- the *distributivity laws* hold: for all $a, b, c \in R$ we have $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.

We write $(R, +, \cdot, 0, 1)$, $(R, +, \cdot)$ or just R. If there is more than one ring in play, we sometimes write $+_R$, $\cdot_R$, 0_R and 1_R to distinguish the operations respectively elements from those of other rings.
A *subring* of a ring $(R, +, \cdot, 0, 1)$ is a subset $S \subseteq R$ such that $(S, +, \cdot, 0, 1)$ is a ring.
Given two rings R and S, a *ring homomorphism* from R to S is a mapping $\varphi : R \to S$ such that

- $\varphi(1_R) = 1_S$,
- $\varphi(a +_R b) = \varphi(a) +_S \varphi(b)$ for all $a, b \in R$, and
- $\varphi(a \cdot_R b) = \varphi(a) \cdot_S \varphi(b)$ for all $a, b \in R$.

A ring *monomorphism*/*epimorphism*/*isomorphism* is an injective/surjective/bijective ring homomorphism.

Definition 1.9. Let $(R, +, \cdot)$ be a ring. We call R

- *commutative* or a *commutative ring*, if $(R, \cdot)$ is a commutative monoid.

- a *skew field* or a *division ring*, if $R \neq \{0\}$ and any element of the monoid $(R, \cdot)$ except 0 is invertible.
- a *(commutative) field*, if R is both commutative and a skew field.

Convention 1.10. In the following, by "ring" we always mean a ring which is not the zero ring.

1.3. General ring-theoretic concepts

Definition 1.11. Let R be a ring.

- An element $u \in R$ is called *invertible* or a *unit* in R, if there exists $v \in R$ such that $uv = 1 = vu$. The set of all units of R is denoted by $U(R)$.
- An element $r \in R$ is called a *zero-divisor* if there exists $a \in R$ such that $ar = 0$ or $ra = 0$. We call r *regular* if r is not a zero-divisor.

Remark 1.12. For any ring R, the set $U(R)$ is in fact a (possibly non-commutative) group, as both the product of two units as well as the inverse of a unit is again a unit.

Definition 1.13. Let R be a ring. Then R is called

- *Dedekind-finite* if for all $a, b \in R$, $ab = 1$ implies $ba = 1$.
- a *domain* if $ab = 0$ implies $a = 0$ or $b = 0$ for all $a, b \in R$.
- *left Noetherian* (resp. *right Noetherian*) if every left ideal (resp. right ideal) of R is finitely generated over R.

Lemma 1.14. *Any domain R is Dedekind-finite.*

Proof: Let $a, b \in R$ such that $ab = 1$, then in particular, we have $b \neq 0$. Now

$$0 = b(ab - 1) = bab - b = (ba - 1)b,$$

and since $b \neq 0$ and R is a domain, we conclude $ba - 1 = 0$, that is, $ba = 1$. □

Definition 1.15. Let R be a domain.

- An element $r \in R \setminus \{0\}$ is called *reducible*, if it can be written as the product of two non-units in R, that is, there exist $p, q \in R \setminus U(R)$ such that $r = pq$. We call r *irreducible*, if r is not reducible.
- We call R a *factorization ring* or *factorization domain*, if every element of $R \setminus \{0\}$ has at least one factorization into finitely many irreducible elements.

Definition 1.16. Let R be a ring. The *center* of R is defined as

$$Z(R) := \{r \in R \mid wr = rw \text{ for all } w \in R\}.$$

The elements of $Z(R)$ are called *central* elements. A *central unit* of R is an element in $U(R) \cap Z(R)$. We denote the set of all central units of R by $U_Z(R)$.

Remark 1.17. The center of a ring R is a commutative subring of R.

Lemma 1.18. *Let R be a ring. Then $U_Z(R)$ is a commutative subgroup of $U(R)$.*

Proof: Since $1 \in U_Z(R)$, we have $U_Z(R) \neq \emptyset$. Let $u, v \in U_Z(R)$, then for all $r \in R$ we have $uvr = urv = ruv$, thus $uv \in U_Z(R)$. Furthermore, $u^{-1}r = u^{-1}ruu^{-1} = u^{-1}uru^{-1} = ru^{-1}$ implies $u^{-1} \in U_Z(R)$. Therefore, $U_Z(R)$ is a subgroup of $U(R)$. Lastly, since $U_Z(R) \subseteq Z(R)$, $U_Z(R)$ is commutative. □

1.4. Graded rings

Definition 1.19. Let $(G, \cdot, e)$ be a monoid.

- A ring R is called *G-graded* if there exists a family $\{R_g\}_{g\in G}$ of Abelian subgroups of R with respect to addition such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$.
- An element $r \in R_g$ is called *homogeneous* of *degree* g and we write $\deg(r) = g$. The set of all homogeneous elements of R is $h(R) := \bigcup_{g\in G} R_g \subseteq R$. A set $M \subseteq R$ is called *homogeneous* if $M \subseteq h(R)$.
- Any $r \in R$ has a unique representation $r = \sum_{g\in G} a_g$ in its homogeneous parts, where $a_g \in R_g$. The homogeneous part of r corresponding to $g \in G$ is denoted as $r_g := a_g$. Furthermore, the *graded length* of r is $\mathrm{gl}(r) := |\{g \in G \mid r_g \neq 0\}| \in \mathbb{N}_0$ (note that $r_g = 0$ for almost all $g \in G$).

Lemma 1.20. *Let G be a monoid, R a G-graded domain, $a \in h(R) \setminus \{0\}$ and $r \in R$.*

(a) If G is right-cancellative and $ra \in h(R)$, then $r \in h(R)$.

(b) If G is left-cancellative and $ar \in h(R)$, then $r \in h(R)$.

Proof: Let $h \in G$ such that $a \in R_h$ and let $r = \sum_{i=1}^n r_{g_i}$, where $r_{g_i} \in R_{g_i} \setminus \{0\}$ and $g_i \neq g_j$ for all $i \neq j$. Then $ra = \sum_{i=1}^n r_{g_i}a$. Since R is a domain we have $r_{g_i}a \neq 0$ for all i. Furthermore, since G is right-cancellative, $g_i h = g_j h$ implies $g_i = g_j$ for all i and j, which by assumption means $i = j$. Thus $\mathrm{gl}(r) = \mathrm{gl}(ra) = 1$ and therefore $r \in h(R)$. The second result follows analogously. □

Definition 1.21. An *ordered semigroup* is a semigroup $(G, \cdot)$ together with a total order $\leq$ on G that is compatible with the semigroup operation: for all $x, y, z \in G$, $x \leq y$ implies $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$. An *ordered monoid* is a monoid that is an ordered semigroup.

Example 1.22. The most common ordered monoids used for grading are $\mathbb{N}_0^n$ and $\mathbb{Z}^n$ with $n \in \mathbb{N}$.

Lemma 1.23. *Let G be an ordered monoid with respect to $\preceq$ and R a G-graded domain. Then $h(R) \setminus \{0\}$ is saturated.*

Proof: Let $a, b \in R$ with $ab \in R_d \setminus \{0\}$ for some $d \in G$. Write $a = \sum_{i=1}^n a_{x_i}$ and $b = \sum_{j=1}^m b_{y_j}$, where $x_i, y_j \in G$, $a_{x_i} \in R_{x_i} \setminus \{0\}$, $b_{y_j} \in R_{y_j} \setminus \{0\}$, $x_i \prec x_{i+1}$ and $y_j \prec y_{j+1}$ for all i, j. Now $(ab)_{x_1+y_1} = a_{x_1}b_{y_1} \neq 0$ and $(ab)_{x_n+y_m} = a_{x_n}b_{y_m} \neq 0$, thus $x_1 + y_1 = d = x_n + y_m$. This implies $n = m = 1$, therefore $a, b \in h(R) \setminus \{0\}$. □

Remark 1.24. Lemma 1.23 does not hold for arbitrary G-gradings: consider $R = \mathbb{Z}[x]$ and $G = \mathbb{Z}/2\mathbb{Z}$, where $R_0 = \bigoplus_{i\in\mathbb{N}_0} \mathbb{Z}x^{2i}$ and $R_1 = \bigoplus_{i\in\mathbb{N}_0} \mathbb{Z}x^{2i+1}$. Then $(x+1)(x-1) = x^2 - 1 \in R_0$, but neither $x + 1$ nor $x - 1$ is homogeneous.

1.5. Multiplicatively closed subsets and saturated sets

Definition 1.25. Let R be a ring and $S \subseteq R$ a subset. We call S

- *quasi-multiplicatively closed* if $1_R \in S$ and for all $s, t \in S$ we also have $st \in S$.
- *multiplicatively closed* if S is quasi-multiplicatively closed and $0_R \notin S$.
- *left saturated* (resp. *right saturated*) if, for all $s, t \in R$, $st \in S$ implies $t \in S$ (resp. $s \in S$).
- *saturated* if S is both left and right saturated.

Some authors allow multiplicatively closed sets to contain 0_R.

Remark 1.26. Let R be a ring. Since $ab \neq 0$ implies $a \neq 0 \neq b$ for all $a, b \in R$, we have that $R \setminus \{0\}$ is saturated. If R is a domain, then $R \setminus \{0\}$ is also multiplicatively closed.

Remark 1.27. Let S be a multiplicatively closed saturated subset of a ring R and $x, y \in R$. Then we have $xy \in S$ if and only if $x \in S$ and $y \in S$.

Lemma 1.28. *Let S be a quasi-multiplicatively closed subset of a domain R. Then $S \setminus \{0\}$ is multiplicatively closed.*

Proof: By assumption, we clearly have $1 \in S \setminus \{0\}$ and $0 \notin S \setminus \{0\}$. Let $a, b \in S \setminus \{0\}$. Then $ab \in S$, since S is quasi-multiplicatively closed, and $ab \in R \setminus \{0\}$, since R is a domain. This implies $ab \in S \cap (R \setminus \{0\}) = S \setminus \{0\}$. □

Lemma 1.29. *Let R be a ring and $\{S_j\}_{j\in J}$ be a family of quasi-multiplicatively closed subsets of R. Then $T := \bigcap_{j\in J} S_j$ is a quasi-multiplicatively closed subset of R. If S_k is a multiplicatively closed set for some $k \in J$, then T is multiplicatively closed as well.*

Proof: We have $1 \in S_j$ for all $j \in J$, which implies $1 \in T$. Now let $a, b \in T$, then $a, b \in S_j$ for all $j \in J$. Since S_j is multiplicatively closed we have $ab \in S_j$ for all $j \in J$, which implies $ab \in T$. Lastly, if $0 \notin S_k$ for some $k \in J$, then $0 \notin T$. □

Lemma 1.30. *Let R be a ring. Then the following holds:*

(a) *$U(R)$ is multiplicatively closed.*

(b) *If R is Dedekind-finite, then $U(R)$ is saturated.*

(c) *$Z(R)$ is quasi-multiplicatively closed, but not multiplicatively closed.*

(d) *If R is a domain, then $Z(R) \setminus \{0\}$ is multiplicatively closed.*

(e) *$U_Z(R)$ is multiplicatively closed.*

Proof: *(a)* Since $R \neq \{0\}$, we have $0 \notin U(S^{-1}R)$. Since $U(R)$ is a group, it is closed under multiplication and contains 1.

(b) Let $a, b \in R$ with $ab \in U(R)$. Then there exists $u \in U(R)$ such that $(ua)b = u(ab) = 1 = (ab)u = a(bu)$. If R is Dedekind-finite, this implies $a, b \in U(R)$.

(c) As a subring of R, $Z(R)$ is closed under multiplication and contains 1 and 0.

(d) If R is a domain, then $Z(R) \setminus \{0\}$ is multiplicatively closed by Lemma 1.28.

(e) As the intersection of a multiplicatively closed set and a quasi-multiplicatively closed set, $U_Z(R)$ is multiplicatively closed by Lemma 1.29.

□

Lemma 1.31. *Let R be a domain and M a submonoid of $R \setminus \{0\}$ with respect to the ring multiplication. Then M is cancellative.*

Proof: Let $a, b, c \in M$ such that $ac = bc$, which is equivalent to $(a - b)c = 0$. Since R is a domain and $c \neq 0$, we have $a = b$. Analogously, $ca = cb$ implies $a = b$ as well. □

Proposition 1.32. *Let R be a ring and $S \subseteq R$. The following are equivalent:*

(1) S is a multiplicatively closed subset of R.

(2) S is a submonoid of $R \setminus \{0\}$ with respect to the ring multiplication.

Proof: By definition, S is a multiplicatively closed subset of R if and only if $1 \in S$, $0 \notin S$, and S is closed under the ring multiplication, which in turn is equivalent to S being a submonoid of $R \setminus \{0\}$. □

1.6. G-algebras

The G-algebras are a class of non-commutative Noetherian domains that are "close enough" to commutative polynomial rings in the sense that many concepts like monomials or Gröbner bases can be salvaged.

Definition 1.33. Let K be a field and A a K-algebra generated by $x_1, \ldots, x_n$. Then A has a *Poincaré-Birkhoff-Witt basis* (or *PBW basis* for short), if the set of *monomials*

$$\mathrm{Mon}(A) := \{x^\alpha \mid \alpha \in \mathbb{N}_0^n\} := \{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}_0\}$$

is a K-basis of A.

Definition 1.34 ([Lev05])**.** Let K be a field. Consider a K-algebra

$$A := K\langle x_1, \ldots, x_n \mid \{x_j x_i = c_{i,j} \cdot x_i x_j + d_{i,j}\} \text{ for } 1 \leq i < j \leq n\rangle,$$

where $c_{i,j} \in K \setminus \{0\}$ and $d_{i,j}$ are polynomials in A whose terms do not contain expressions of the form $x_k x_l$ for $l < k$. We call A a *G-algebra* if the following two conditions are met:

Ordering condition: There exists a global monomial ordering $<$ on $K[x_1, \ldots, x_n]$ such that

$$\mathrm{lm}(d_{i,j}) < x_i x_j \quad \text{for all} \quad 1 \leq i < j \leq n \quad \text{where} \quad d_{i,j} \neq 0.$$

Non-degeneracy condition: For all $1 \leq i < j < k \leq n$ we have

$$\mathcal{NDC}_{i,j,k} := c_{i,k} c_{j,k} \cdot d_{i,j} x_k - x_k d_{i,j} + c_{j,k} \cdot x_j d_{i,k} - c_{i,j} \cdot d_{i,k} x_j + d_{j,k} x_i - c_{i,j} c_{i,k} \cdot x_i d_{j,k} = 0.$$

Definition 1.35. Let $A = K\langle x_1, \ldots, x_n \mid \{x_j x_i = c_{i,j} \cdot x_i x_j + d_{i,j}\} \text{ for } 1 \leq i < j \leq n\rangle$ be a G-algebra.

- If all $d_{i,j} = 0$, we call A *quasi-commutative.*
- If all $c_{i,j} = 1$, we call A an *algebra of Lie type.*

Remark 1.36. A quasi-commutative G-algebra of Lie type in n variables over K is exactly the commutative polynomial ring in n variables over K.

Theorem 1.37 (Theorem 4.7 in [Lev05])**.** *Let A be a G-algebra.*

(a) A has a PBW basis.

(b) A is (left and right) Noetherian.

(c) A is a domain.

A G-algebra in one variable is just a univariate (commutative) polynomial ring. If we admit two variables, there are still only five possible variations, which all can be extended to G-algebras in $2n$ variables in a straight-forward manner:

Theorem 1.38 (Theorem 1 in [LKM11])**.** *Let K be a field, $q \in K \setminus \{0\}$ and $\alpha, \beta, \gamma \in K$. Consider the K-algebra*

$$A(q, \alpha, \beta, \gamma) := K\langle x, y \mid yx = qxy + \alpha x + \beta y + \gamma\rangle.$$

Up to isomorphism, $A(q, \alpha, \beta, \gamma)$ is exactly one of the so-called model algebras*:*

(1) The commutative algebra $K[x, y] = K\langle x, y \mid yx = xy\rangle$.

(2) The first Weyl algebra $K\langle x, \partial \mid \partial \cdot x = x \cdot \partial + 1\rangle$.

(3) The first shift algebra $K\langle x, s \mid s \cdot x = x \cdot s + s = (x + 1) \cdot s\rangle$.

(4) The first q-shift algebra $K\langle x, s \mid s \cdot x = q \cdot x \cdot s\rangle$.

(5) The first q-Weyl algebra $K\langle x, \partial \mid \partial \cdot x = q \cdot x \cdot \partial + 1\rangle$.

The Weyl algebras are perhaps the best-known G-algebras. They are used to model the action of (partial) derivatives.

Definition 1.39. Let K be a field and $n \in \mathbb{N}$. The *n-th Weyl algebra* over K is the K-algebra

$$W_n := K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid Q\rangle$$

with the set of relations

$$Q := \{x_j x_i = x_i x_j, \partial_j \partial_i = \partial_i \partial_j, \partial_i x_j = x_j \partial_i + \delta_{i,j} \mid 1 \leq i, j \leq n\}.$$

Main example, part 1

Convention 1.40. In the main example, we denote the first Weyl algebra W_1 by $\mathcal{D}$.

Remark 1.41. The first Weyl algebra can be $\mathbb{Z}$-graded by assigning the degree -1 to x and 1 to ∂. For $z \in \mathbb{Z}$ we have

$$\mathcal{D}_z = \left\{ \sum_{(a,b)\in\mathbb{N}_0^2} c_{a,b} x^a \partial^b \in \mathcal{D} \mid (b - a = z \vee c_{a,b} \neq 0)\ \forall (a,b) \in \mathbb{N}_0^2 \right\}.$$

Definition 1.42. The element $\theta := x\partial \in \mathcal{D}$ is called the *Euler operator*.

Lemma 1.43. *Let $z \in \mathbb{Z}$ and $m, n \in \mathbb{N}$. In $\mathcal{D}$, we have $(\theta + z)^m x^n = x^n(\theta + z + n)^m$ and $\partial^n(\theta + z)^m = (\theta + z + n)^m \partial^n$.*

Proof: We have

$$(\theta + z)x = \theta x + zx = x\partial x + zx = x(x\partial + 1) + xz = x(x\partial + z + 1) = x(\theta + z + 1)$$

as well as

$$\partial(\theta + z) = \partial\theta + \partial z = \partial x\partial + \partial z = (x\partial + 1)\partial + z\partial = (x\partial + z + 1)\partial = (\theta + z + 1)\partial.$$

The full statement follows by induction on n and m. □

Corollary 1.44. *Let $f \in K[\theta] \subseteq \mathcal{D}$ and $n \in \mathbb{N}$. Then $f(\theta)x^n = x^n f(\theta + n)$ and $\partial^n f(\theta) = f(\theta + n)\partial^n$.*

Proof: Let $f = \sum_{i=0}^k f_i \theta^i$ with $f_i \in K$, then by Lemma 1.43 we have

$$f(\theta)x^n = \sum_{i=0}^k f_i \theta^i x^n = \sum_{i=0}^k f_i x^n (\theta + n)^i = x^n \sum_{i=0}^k f_i (\theta + n)^i = x^n f(\theta + n)$$

and

$$\partial^n f(\theta) = \partial^n \sum_{i=0}^k f_i \theta^i = \sum_{i=0}^k f_i \partial^n \theta^i = \sum_{i=0}^k f_i (\theta + n)^i \partial^n = f(\theta + n)\partial^n.$$

□

Lemma 1.45. *Let $z, k \in \mathbb{Z}$ and $r \in \mathcal{D}_z$. Then $(\theta + z + k)r = r(\theta + z)$.*

Proof: Since $r \in \mathcal{D}_k$, we have a representation $r = \sum_{\substack{(a,b)\in\mathbb{N}_0^2 \\ b-a=k}} c_{a,b} x^a \partial^b$. Then

$$\begin{aligned}(\theta + z + k)r &= \sum_{\substack{(a,b)\in\mathbb{N}_0^2 \\ b-a=k}} c_{a,b}(\theta + z + k)x^a \partial^b = \sum_{\substack{(a,b)\in\mathbb{N}_0^2 \\ b-a=k}} c_{a,b} x^a (\theta + z + k + a)\partial^b \\ &= \sum_{\substack{(a,b)\in\mathbb{N}_0^2 \\ b-a=k}} c_{a,b} x^a \partial^b (\theta + z + k + a - b) = \sum_{\substack{(a,b)\in\mathbb{N}_0^2 \\ b-a=k}} c_{a,b} x^a \partial^b (\theta + z) = r(\theta + z).\end{aligned}$$

□

Definition 1.46. In $\mathcal{D}$, define the set $\Theta := [\theta + \mathbb{Z}] = [\{\theta + z \mid z \in \mathbb{Z}\}]$.

Remark 1.47. The multiplicatively closed set Θ is homogeneous, since it is generated as a monoid by homogeneous elements of degree zero. In particular, $(\theta + z_1)(\theta + z_2) = (\theta + z_2)(\theta + z_1)$ for all $z_1, z_2 \in \mathbb{Z}$ by Lemma 1.45.

2. Ore localization of domains

Working in a non-commutative setting, one almost always has to distinguish between "left" and "right" structures and properties. To avoid duplication we only state the "left" version of the theory developed in the following. Nevertheless, the "right" analoga of the results hold as well.

2.1. Construction and basic properties

Most of the first part of this section is taken directly from our previous work in [Hof14], which also includes the proofs omitted here for brevity. Other sources for details on the topic of Ore localizations are [MR01] and [Š06].

Definition 2.1. Let S be a subset of a ring R. We say that S satisfies the *left Ore condition* in R if for all $s \in S$ and $r \in R$ there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$.

Remark 2.2. By iteration, the left Ore condition on a set S also guarantees that any finite selection of elements from S has a common left multiple in S.

Definition 2.3. Let S be a multiplicatively closed subset of a domain R. We call S a *left Ore set* in R if it satisfies the left Ore condition in R.

Definition 2.4. Let S be a left Ore set in a domain R. A ring R_S together with a monomorphism $\varphi : R \to R_S$ is called a *left Ore localization* of R at S if:

(1) For all $s \in S$, $\varphi(s)$ is a unit in R_S.

(2) For all $x \in R_S$, we have $x = \varphi(s)^{-1}\varphi(r)$ for some $s \in S$ and $r \in R$.

We mostly write $S^{-1}R$ instead of R_S.

Theorem 2.5. *Let S be a left Ore set in a domain R. The left Ore localization of R at S can be constructed as follows: Let $S^{-1}R := S \times R/\sim$, where $\sim$ is the equivalence relation*

$$(s_1, r_1) \sim (s_2, r_2) \quad \Leftrightarrow \quad \exists\ \tilde{s} \in S, \exists\ \tilde{r} \in R : \tilde{s}s_2 = \tilde{r}s_1 \text{ and } \tilde{s}r_2 = \tilde{r}r_1.$$

Together with the operations

$$+ : S^{-1}R \times S^{-1}R \to S^{-1}R, \quad (s_1, r_1) + (s_2, r_2) := (\tilde{s}s_1, \tilde{s}r_1 + \tilde{r}r_2),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}s_1 = \tilde{r}s_2$, and

$$\cdot : S^{-1}R \times S^{-1}R \to S^{-1}R, \quad (s_1, r_1) \cdot (s_2, r_2) := (\tilde{s}s_1, \tilde{r}r_2),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}r_1 = \tilde{r}s_2$, $(S^{-1}R, +, \cdot)$ becomes a ring.

Remark 2.6. For $s \in S$ and $r \in R$, we denote the elements of $S^{-1}R$ by $s^{-1}r$ or, by abuse of notation, by (s, r).

Remark 2.7. The construction in Theorem 2.5 works perfectly fine for quasi-multiplicatively closed sets S. We restrict ourselves to multiplicatively closed sets in this work to avoid trivial situations: if $0 \in S$, then 0 becomes invertible in $S^{-1}R$, thus $S^{-1}R = \{0\}$.

Remark 2.8. The left Ore condition guarantees that (at least formally) we can rewrite any right fraction rs^{-1} into a left fraction $\tilde{s}^{-1}\tilde{r}$ by taking denominators in the equation $\tilde{s}r = \tilde{r}s$, but not necessarily the other way around: to this end, we additionally need the right Ore condition.

Remark 2.9. Let S be a multiplicatively closed subset of an arbitrary ring R (not necessarily a domain). Even in this more general case we can define a left Ore localization of R at S by imposing additional restraints on S. We still need S to satisfy the left Ore condition, but this is not sufficient to deal with the presence of zero-divisors. For this, we require S to only consist of regular elements. Furthermore, we need S to be *left reversible*: for all $r \in R$ and $s \in S$ such that $rs = 0$ there exists $t \in S$ such that $tr = 0$ (in a domain, every left Ore set is also left reversible). A left reversible left Ore set of regular elements is also called a *left denominator set*. Under this conditions the construction given in Theorem 2.5 yields the left Ore localization of a ring at a left denominator set (see also [Š06] and [MR01] for details and proofs).

Definition 2.10. Let S be a left Ore set in a domain R. The *structural homomorphism* of $S^{-1}R$ is the mapping $\rho_{S,R} : R \to S^{-1}R,\ r \mapsto (1, r)$.

Theorem 2.11. *Let S be a left Ore set in a domain R and $(s, r) \in S^{-1}R$.*

(a) We have $(s, r) = 0$ if and only if $r = 0$.

(b) We have $(s, r) = 1$ if and only if $s = r$.

(c) Let $w \in R$ with $ws \in S$, then $(s, r) = (ws, wr)$.

(d) The left Ore localization $S^{-1}R$ is a domain.

(e) The structural homomorphism $\rho_{S,R}$ is a monomorphism of rings.

Remark 2.12. With the last result we can see now that $S^{-1}R$ together with $\rho_{S,R}$ indeed meet the requirements in Definition 2.4.

Definition 2.13. A domain R is called a *left Ore domain* if $R \setminus \{0\}$ is a left Ore set in R. The associated localization $(R \setminus \{0\})^{-1}R$ is called the *left quotient (skew) field* of R and is denoted by $\mathrm{Quot}(R)$.

Lemma 2.14. *Let R be a left Ore domain.*

(a) The localization $\mathrm{Quot}(R)$ *is a skew field.*

(b) Let S be a left Ore set in R. Then $S^{-1}R$ is a left Ore domain.

Proof: *(a)* By construction, $\mathrm{Quot}(R)$ is already a domain. The inverse of an element $(s, r) \in \mathrm{Quot}(R) \setminus \{0\}$ is given by $(r, s) \in \mathrm{Quot}(R) \setminus \{0\}$.

(b) Let $(s_1, r_1) \in S^{-1}R$ and $(s_2, r_2) \in S^{-1}R \setminus \{0\}$, then $r_2 \in R \setminus \{0\}$. Since R is a left Ore domain there exist $x' \in R \setminus \{0\}$ and $y' \in R$ such that $x'r_1 = y'r_2$. Define $x := (1, x') \cdot (1, s_1) \in S^{-1}R \setminus \{0\}$ and $y := (1, y') \cdot (1, s_2) \in S^{-1}R$, then

$$x(s_1, r_1) = (1, x')(1, s_1)(s_1, r_1) = (1, x'r_1) = (1, y', r_2) = (1, y')(1, s_2)(s_2, r_2) = y(s_2, r_2).$$

□

Lemma 2.15. *Let R be a domain and J a non-empty index set such that $S_j \subseteq R$ is a left Ore set in R for every $j \in J$. Then $T := [\bigcup_{j\in J} S_j]$ is a left Ore set in R.*

Proof: The set T is quasi-multiplicatively closed by construction and $0 \notin T$ follows from the assumption that R is a domain. Any element in T can be written as a finite product of elements from the sets S_j. Since the natural numbers are well-ordered, every element $s \in T$ has a representation with a minimal number of factors. Denoting this number by $a(s)$, this gives us the partition $T = \biguplus_{n\in\mathbb{N}} T_n$, where

$$T_n = \{s \in T \mid a(s) = n\}.$$

Now we show the left Ore property on T by induction on n: let $r \in R$ and $s \in T$.

(IB) If $s \in T_1$, then $s \in S_j$ for some $j \in J$. By the left Ore property on S_j there exist $\tilde{s} \in S_j \subseteq T$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$.

(IH) Assume that there is an $n \in \mathbb{N}$ with the following property: for all $t \in T_{\leq n} := \bigcup_{i=1}^{n} T_n$ there exist $\tilde{t} \in T$ and $\tilde{r} \in R$ such that $\tilde{t}r = \tilde{r}t$.

(IS) If $s \in T_{n+1}$, then there is a representation $s = \prod_{i=1}^{n+1} s_i$, where $s_i \in S_{k_i}$ for some $k_i \in J$. Now define $t := \prod_{i=2}^{n+1} s_i$, then we have $s = s_1 t$ and $t \in T_{\leq n}$. By the induction hypothesis, there exist $\tilde{t} \in T$ and $\tilde{r} \in R$ such that $\tilde{t}r = \tilde{r}t$. Furthermore, by the left Ore property on S_{k_1}, there exist $\hat{s} \in S_{k_1}$ and $\hat{r} \in R$ such that $\hat{s}\tilde{r} = \hat{r}s_1$. Define $\mathring{t} := \hat{s}\tilde{t} \in T$ and $\mathring{r} := \hat{r} \in R$, then

$$\mathring{t}r = \hat{s}\tilde{t}r = \hat{s}\tilde{r}t = \hat{r}s_1 t = \hat{r}s = \mathring{r}s$$

concludes the proof.

□

Types of Ore localizations

The following classification has been introduced by V. Levandovskyy and describes the three most common types of Ore localizations:

Definition 2.16. Let K be a field and R a K-algebra and a Noetherian domain.

- Let $S \subseteq R$ be a left Ore set in R that is generated as a monoid by at most countably many elements. Then $S^{-1}R$ is called a *monoidal localization*.
- Let $K[x_1, \dots, x_n] \subseteq R$ and $\mathfrak{p} \subseteq K[x_1, \dots, x_n]$ a prime ideal, then $S := K[x_1, \dots, x_n] \setminus \mathfrak{p}$ is multiplicatively closed. If S is a left Ore set in R, then $S^{-1}R$ is called a *geometric localization*.
- If $T \subseteq R$ is a sub-K-algebra and $S := T \setminus \{0\}$ is a left Ore set in R, then $S^{-1}R$ is called a *(partial) rational localization*.

In our previous work in [Hof14] we have developed an algorithmic framework for basic computations in Ore localized G-algebras (or OLGAs for short) in special "computation-friendly" cases (finitely generated monoidal localizations, geometric localization at maximal ideals, rational localizations where T is generated by a subset of the variables). There also exist a proof-of-concept implementation of the algorithms in the computer algebra system Singular ([DGPS15]).

The algorithm developed in Chapter 6 applies to the following situation:

Definition 2.17 (Rational localization of G-algebras)**.** Let

$$R = K\langle x_1, \dots, x_n \mid \{x_j x_i = c_{i,j} \cdot x_i x_j + d_{i,j}\} \text{ for } 1 \leq i < j \leq n\rangle$$

be a G-algebra and $\{x_1, \dots, x_n\} = X \uplus Y$ a partition of the variables such that $d_{i,j}$ only contains variables from X for all $x_i, x_j \in X$. Then $B := K\langle X \mid \{x_j x_i = c_{i,j} \cdot x_i x_j + d_{i,j}\} \text{ for } x_i, x_j \in X\rangle$ is again a G-algebra. If $S := B \setminus \{0\}$ is a left Ore set in R, then $S^{-1}R$ is called a *rational* OLGA.

2.2. Commutative localization

Lemma 2.18. *Let S be a multiplicatively closed subset of regular elements of a commutative ring R. Then S is a left (and right) denominator set of R.*

Proof: Given $s \in S$ and $r \in R$, by commutativity we have $rs = sr$. For the same reason we have $rs = 0$ if and only if $sr = 0$. □

Lemma 2.19. *Let S be a multiplicatively closed subset of regular elements of a commutative ring R and consider the construction of $S^{-1}R$ in Theorem* 2.5*. Let $(s_1, r_1), (s_2, r_2) \in S \times R$.*

(a) The equivalence relation $\sim$ simplifies to

$$(s_1, r_1) \sim (s_2, r_2) \quad \Leftrightarrow \quad s(s_1 r_2 - s_2 r_1) = 0 \textit{ for some } s \in S.$$

(b) The addition rule simplifies to

$$(s_1, r_1) + (s_2, r_2) = (s_1 s_2, s_2 r_1 + s_1 r_2).$$

(c) The multiplication rule simplifies to

$$(s_1, r_1) \cdot (s_2, r_2) = (s_1 s_2, r_1 r_2).$$

Proof: *(a)* By Theorem 2.5, $(s_1, r_1) \sim (s_2, r_2)$ implies that there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s_1 = \tilde{r}s_2$ and $\tilde{s}r_1 = \tilde{r}r_2$. But then

$$\tilde{s}s_1 r_2 = \tilde{r}s_2 r_2 = s_2 \tilde{r} r_2 = s_2 \tilde{s} r_1 = \tilde{s} s_2 r_1,$$

which is equivalent to $\tilde{s}(s_1 r_2 - s_2 r_1) = 0$.
On the other hand, let $s \in S$ such that $s(s_1 r_2 - s_2 r_1) = 0$. Define $\tilde{s} := s s_2 \in S$ and $\tilde{r} := s s_1 \in R$. Then $\tilde{s}s_1 = s s_2 s_1 = s s_1 s_2 = \tilde{r} s_2$ and $\tilde{s} r_1 = s s_2 r_1 = s s_1 r_2 = \tilde{r} r_2$, which implies $(s_1, r_1) \sim (s_2, r_2)$.

(b) By definition, $(s_1, r_1) + (s_2, r_2) = (\tilde{s}s_1, \tilde{s}r_1 + \tilde{r}r_2)$, where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}s_1 = \tilde{r}s_2$. In the commutative setting, we can choose $\tilde{s} := s_2$ and $\tilde{r} := s_1$, since then $\tilde{s}s_1 = s_2 s_1 = s_1 s_2 = \tilde{r} s_2$. But now

$$(s_1, r_1) + (s_2, r_2) = (\tilde{s}s_1, \tilde{s}r_1 + \tilde{r}r_2) = (s_2 s_1, s_2 r_1 + s_1 r_2) = (s_1 s_2, s_2 r_1 + s_1 r_2).$$

(c) By definition, $(s_1, r_1) \cdot (s_2, r_2) = (\tilde{s}s_1, \tilde{r}r_2)$, where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}r_1 = \tilde{r}s_2$. In the commutative setting, we can choose $\tilde{s} := s_2$ and $\tilde{r} := r_1$, since then $\tilde{s}r_1 = s_2r_1 = r_1s_2 = \tilde{r}s_2$. But now

$$(s_1, r_1) \cdot (s_2, r_2) = (\tilde{s}s_1, \tilde{r}r_2) = (s_2s_1, r_1r_2) = (s_1s_2, r_1r_2).$$

□

From Lemma 2.19 we can see that in the commutative case, Ore localization at regular elements coincides with the classical notion of localizing a commutative ring at a multiplicatively closed subset. Furthermore, the resulting ring $S^{-1}R$ is again commutative:

Corollary 2.20. *Let S be a multiplicatively closed subset of regular elements of a commutative ring R. Then $S^{-1}R$ is a commutative ring.*

Proof: Let $(s_1, r_1), (s_2, r_2) \in S^{-1}R$. With the simplified multiplication from Lemma 2.19, we get

$$(s_1, r_1) \cdot (s_2, r_2) = (s_1s_2, r_1r_2) = (s_2s_1, r_2r_1) = (s_2, r_2) \cdot (s_1, r_1).$$

□

2.3. Induced graded localizations

Lemma 2.21. *Let G be a monoid, R a G-graded domain and S a multiplicatively closed subset of R contained in $h(R)$. Then S is a left Ore set in R if for all $s \in S$ and $r \in h(R)$ there exist $\tilde{s} \in S$ and $\tilde{r} \in h(R)$ such that $\tilde{s}r = \tilde{r}s$. If G is right-cancellative, the converse holds as well.*

Proof: Let $s \in S$ and $r \in R$. We use induction on the graded length n of r:

(IB) If $n = 1$, then $r \in h(R)$. By assumption there exist $\tilde{s} \in S$ and $\tilde{r} \in h(R)$ such that $\tilde{s}r = \tilde{r}s$.

(IH) Assume that for any $s \in S$ and any $r \in R$, where the graded length of r is less than n, there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}r = \tilde{r}s$.

(IS) Let $r = \sum_{i=1}^{n} r_{g_i}$, where $r_{g_i} \in R_{g_i}$. By the induction hypothesis, there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}\sum_{i=1}^{n-1} r_{g_i} = \tilde{r}s$. By the induction base, there exist $\hat{s} \in S$ and $\hat{r} \in R$ such that $\hat{s}r_{g_n} = \hat{r}s$. Again by the induction base, there exist $\bar{s} \in S$ and $\bar{r} \in R$ such that $\bar{s}\tilde{s} = \bar{r}\hat{s}$. Now define $\mathring{s} := \bar{s}\tilde{s} \in S$ and $\mathring{r} := \bar{s}\tilde{r} + \bar{r}\hat{r} \in R$. Then

$$\mathring{s}r = \mathring{s}\sum_{i=1}^{n-1} r_{g_i} + \mathring{s}r_{g_n} = \bar{s}\tilde{s}\sum_{i=1}^{n-1} r_{g_i} + \bar{r}\hat{s}r_{g_n} = \bar{s}\tilde{r}s + \bar{r}\hat{r}s = (\bar{s}\tilde{r} + \bar{r}\hat{r})s = \mathring{r}s.$$

Now assume that G is right-cancellative and let $s \in S$ and $r \in h(R)$. If S is a left Ore set in R, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{r}s = \tilde{s}r \in h(R)$. Since G is right-cancellative, this implies $\tilde{r} \in h(R)$ by Lemma 1.20. □

Proposition 2.22. *Let $(G, \cdot, e)$ be a group, R a G-graded domain and $S \subseteq h(R)$ a left Ore set in R. Then $S^{-1}R$ is a G-graded ring with*

$$(S^{-1}R)_g := \left\{(s, r) \in S^{-1}R \mid r \in h(R) \wedge \deg(s)^{-1} \cdot \deg(r) = g\right\}.$$

Proof: • Let $g \in G$ and $(s_1, r_1), (s_2, r_2) \in (S^{-1}R)_g$, then

$$\deg(s_1)^{-1}\deg(r_1) = g = \deg(s_2)^{-1}\deg(r_2).$$

We have $(s_1, r_1) + (s_2, r_2) = (ss_1, sr_1 + rr_2)$, where $s \in S$ and $r \in R$ such that $ss_1 = rs_2$. Now $s, s_1, s_2 \in h(R)$ implies $r \in h(R)$ and $\deg(s)\deg(s_1) = \deg(r)\deg(s_2)$. Thus,

$$\begin{aligned}\deg(s)\deg(r_1) &= \deg(s)\deg(s_1)\deg(s_1)^{-1}\deg(r_1)\\ &= \deg(r)\deg(s_2)\deg(s_2)^{-1}\deg(r_2)\\ &= \deg(r)\deg(r_2)\end{aligned}$$

shows that $sr_1 + rr_2$ is homogeneous of degree $\deg(s)\deg(r_1)$. Finally,

$$\begin{aligned}\deg(ss_1)^{-1}\deg(sr_1 + rr_2) &= (\deg(s)\deg(s_1))^{-1}\deg(sr_1)\\ &= \deg(s_1)^{-1}\deg(s)^{-1}\deg(s)\deg(r_1)\\ &= \deg(s_1)^{-1}\deg(r_1)\\ &= g\end{aligned}$$

implies that $(S^{-1}R)_g$ is closed under addition and therefore an Abelian subgroup of $S^{-1}R$ with respect to addition.

- Let $g, h \in G$, $(s_g, r_g) \in (S^{-1}R)_g$ and $(s_h, r_h) \in (S^{-1}R)_h$. We have $(s_g, r_g) \cdot (s_h, r_h) = (ss_g, rr_h)$, where $s \in S$ and $r \in h(R)$ such that $sr_g = rs_h$. Then $\deg(s)\deg(r_g) = \deg(r)\deg(s_h)$ and thus

$$\begin{aligned}\deg(ss_g)^{-1}\deg(rr_h) &= \deg(s_g)^{-1}\deg(s)^{-1}\deg(r)\deg(r_h)\\ &= \deg(s_g)^{-1}\deg(s)^{-1}\deg(r)\deg(s_h)\deg(s_h)^{-1}\deg(r_h)\\ &= \deg(s_g)^{-1}\deg(s)^{-1}\deg(s)\deg(r_g)\deg(s_h)^{-1}\deg(r_h)\\ &= \deg(s_g)^{-1}\deg(r_g)\deg(s_h)^{-1}\deg(r_h)\\ &= gh,\end{aligned}$$

which shows $(S^{-1}R)_g(S^{-1}R)_h \subseteq (S^{-1}R)_{gh}$.

- Let $(s, r) \in S^{-1}R$ and let $r = \sum_{g\in G} r_g$, where $r_g \in R_g$. Then

$$(s, r) = (s, \sum_{g\in G} r_g) = \sum_{g\in G}(s, r_g) \in \sum_{g\in G}(S^{-1}R)_{\deg(s)^{-1}g} \subseteq \sum_{g\in G}(S^{-1}R)_g.$$

Now let $g, h \in G$ and $0 \neq (s, r) \in (S^{-1}R)_g \cap (S^{-1}R)_h$, then $g = \deg(s)^{-1}\deg(r) = h$, thus $(S^{-1}R)_g \cap (S^{-1}R)_h = \{0\}$. Therefore, $S^{-1}R = \bigoplus_{g\in G}(S^{-1}R)_g$.

□

2.4. Localization at specific Ore sets

As one would expect, localizing at a set of units is rather unspectacular:

Lemma 2.23. *Let R be a domain and U a submonoid of $U(R)$. Then U is a left Ore set in R and $R \cong U^{-1}R$ as rings.*

Proof: As a submonoid of $U(R)$, U is clearly multiplicatively closed. Let $r \in R$ and $u \in U$. Then, since $1 \in U$ and $ru^{-1} \in R$, we have $1 \cdot r = r = ru^{-1} \cdot u$, which shows the left Ore property. Now consider the structural monomorphism $\rho_{U,R} : R \to U^{-1}R,\ r \mapsto (1, r)$. It remains to show surjectivity: let $(u, r) \in S^{-1}R$, then

$$(u, r) = (u^{-1}u, u^{-1}r) = (1, u^{-1}r) = \rho_{U,R}(u^{-1}r) \in \mathrm{im}(\rho_{U,R})$$

□

Remark 2.24. Lemma 2.23 applies in particular to the cases $U = U(R)$, $U = \{1\}$ and $U = U_Z(R)$.

Lemma 2.25. *Let R be a domain and Z a submonoid of $Z(R)$. Then Z is a left Ore set in R.*

Proof: As a submonoid of $Z(R)$, Z is clearly multiplicatively closed. Furthermore, since all elements of Z are central, Z also satisfies the left Ore condition in R. □

Remark 2.26. In the situation of Lemma 2.25, the same simplification steps as in Lemma 2.19 can be applied, as the proof only uses commutativity to permute elements of R with elements of S.

Lemma 2.27. *Let $S \subseteq R$ be a left Ore set in a domain R. Then $T := [S \cup U_Z(R)]$ is a left Ore set in R and $S^{-1}R \cong T^{-1}R$ as rings.*

Proof: As $U_Z(R) \subseteq Z(R)$, the central units commutate with all elements of R, in particular, $U_Z(R)$ is a left Ore set. Then T is a left Ore set in R by Lemma 2.15. Since central units commutate with all elements of R and in particular with all elements in S we have that $T = U_Z(R)S$, that is, every element $t \in T$ has a representation $t = us$, where $u \in U_Z(T)$ and $s \in S$. By the forthcoming Lemma 3.1 the mapping $\varphi : S^{-1}R \to T^{-1}R,\ (s, r) \mapsto (s, r)$ is a ring monomorphism since $S \subseteq T$. To see surjectivity, let $(t, r) \in T^{-1}R$, then $t = us$ for some $u \in U_Z(T)$ and $s \in S$. But then

$$(t, r) = (us, r) = (u^{-1}us, u^{-1}r) = (s, u^{-1}r) = \varphi(s, u^{-1}r) \in \mathrm{im}(\varphi).$$

□

Since enriching a left Ore set S with central units does not change the localization, for theoretical purposes we might assume without loss of generality that S already contains $U_Z(R)$. Furthermore, we will see in Chapter 4 that we actually can assume that all units are contained in S.

Lemma 2.28. *Let R be a factorization domain and a left Ore domain. Define M to be the set of all irreducible elements of R. Then $S := [M]$ is a saturated left Ore set in R and $S^{-1}R = \mathrm{Quot}(R)$.*

Proof: Since R is a factorization domain, any element of $R \setminus \{0\}$ can be written as a product of finitely many irreducible elements (note that units are irreducible by our definition) and is thus contained in S, which implies $S = R \setminus \{0\}$. Since R is a left Ore domain, $R \setminus \{0\}$ is a left Ore set in R and saturated by Remark 1.26. Then $\mathrm{Quot}(R) = S^{-1}R$ by definition. □

Main example, part 2

Lemma 2.29. *Let $z \in \mathbb{Z}$ and $r \in \mathcal{D}$. Then there exist $\tilde{s} \in \Theta = [\theta + \mathbb{Z}]$ and $\tilde{r} \in \mathcal{D}$ such that $\tilde{r}(\theta + z) = \tilde{s}r$.*

Proof: Consider again the $\mathbb{Z}$-grading on $\mathcal{D}$ introduced in Remark 1.41. Let $n := \mathrm{gl}(r)$ and $r = \sum_{i=1}^{n} r_{k_i}$ the decomposition of r into its homogeneous parts, where $k_i \in \mathbb{Z}$ and $r_{k_i} \in \mathcal{D}_{k_i}$ for all $i \in \{1, \ldots, n\}$. Define

$$\tilde{s} := \prod_{i=1}^{n} (\theta + z + k_i) \in \Theta \quad \text{and} \quad \tilde{r} := \sum_{i=1}^{n} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} (\theta + z + k_j) \right) r_{k_i} \in \mathcal{D}.$$

Then by Lemma 1.45 we have

$$\begin{aligned}
\tilde{s}r &= \sum_{i=1}^{n} \tilde{s} r_{k_i} \\
&= \sum_{i=1}^{n} \left(\prod_{j=1}^{n} (\theta + z + k_j) \right) r_{k_i} \\
&= \sum_{i=1}^{n} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} (\theta + z + k_j) \right) (\theta + z + k_i) r_{k_i} \\
&= \sum_{i=1}^{n} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} (\theta + z + k_j) \right) r_{k_i} (\theta + z) \\
&= \tilde{r}(\theta + z).
\end{aligned}$$

□

Lemma 2.30. *The set $\Theta = [\theta + \mathbb{Z}]$ is a left Ore set in $\mathcal{D}$, but not saturated.*

Proof: Let $r \in \mathcal{D}$, $z_1, z_2 \in \mathbb{Z}$ and consider $s := (\theta + z_1)(\theta + z_2) \in \Theta$. By Lemma 2.29, there exist $s_2 \in \Theta$ and $r_2 \in \mathcal{D}$ such that $s_2 r = r_2(\theta + z_2)$. Again by Lemma 2.29, there exist $s_1 \in \Theta$ and $r_1 \in \mathcal{D}$ such that $s_1 r_2 = r_1(\theta + z_1)$. Define $\tilde{s} := s_1 s_2 \in \Theta$ and $\tilde{r} := r_1 \in \mathcal{D}$, then

$$\tilde{r}s = r_1(\theta + z_1)(\theta + z_2) = s_1 r_2 (\theta + z_2) = s_1 s_2 r = \tilde{s}r.$$

As every element of Θ has the form $\prod_{i=1}^{n} (\theta + z_i)$ for some $n \in \mathbb{N}_0$ and $z_i \in \mathbb{Z}$, the statement follows by induction on n.
To see that Θ is not saturated, consider that $\theta = x\partial \in \Theta$, but $x \notin \Theta$ as well as $\partial \notin \Theta$. □

Remark 2.31. The localization $\Theta^{-1}\mathcal{D}$ is a monoidal localization.

3. Properties under homomorphisms

3.1. Embedding of localizations

Lemma 3.1. *Let $S, T \subseteq R$ be Ore sets in R with $S \subseteq T$. Then the mapping*

$$\varphi : S^{-1}R \to T^{-1}R, \quad (s, r) \mapsto (s, r),$$

is a ring monomorphism.

Proof: Let $a := (s_1, r_1), b := (s_2, r_2) \in S^{-1}R$.

Well-definedness: Let $a = b$ in $S^{-1}R$, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s}s_1 = \tilde{r}s_2$ and $\tilde{s}r_1 = \tilde{r}r_2$. As $S \subseteq T$, this implies that $\varphi(a) = (s_1, r_1) = (s_2, r_2) = \varphi(b)$ in $T^{-1}R$.

Additivity: We have

$$\varphi(a) + \varphi(b) = (s_1, r_1) + (s_2, r_2) = (\hat{t}s_1, \hat{t}r_1 + \hat{r}r_2),$$

where $\hat{t} \in T$ and $\hat{r} \in R$ satisfy $\hat{t}s_1 = \hat{r}s_2$. On the other hand, we have

$$\varphi(a + b) = \varphi(\tilde{s}s_1, \tilde{s}r_1 + \tilde{r}r_2) = (\tilde{s}s_1, \tilde{s}r_1 + \tilde{r}r_2),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}s_1 = \tilde{r}s_2$. Now let $\mathring{t} \in T$ and $\mathring{r} \in R$ such that $\mathring{t}\hat{t}s_1 = \mathring{r}\tilde{s}s_1$, then, as $s_1 \neq 0$, we have $\mathring{t}\hat{t} = \mathring{r}\tilde{s}$. Furthermore, we have

$$(\mathring{t}\hat{r} - \mathring{r}\tilde{r})s_2 = \mathring{t}\hat{r}s_2 - \mathring{r}\tilde{r}s_2 = \mathring{t}\hat{t}s_1 - \mathring{r}\tilde{s}s_1 = (\mathring{t}\hat{t} - \mathring{r}\tilde{s})s_1 = 0,$$

which implies $\mathring{t}\hat{r} = \mathring{r}\tilde{r}$ as $s_2 \neq 0$. But then

$$\mathring{t}(\hat{t}r_1 + \hat{r}r_2) - \mathring{r}(\tilde{s}r_1 + \tilde{r}r_2) = (\mathring{t}\hat{t} - \mathring{r}\tilde{s})r_1 + (\mathring{t}\hat{r} - \mathring{r}\tilde{r})r_2 = 0$$

proves $\varphi(a) + \varphi(b) = \varphi(a + b)$ in $T^{-1}R$.

Multiplicativity: We have

$$\varphi(a) \cdot \varphi(b) = (s_1, r_1) \cdot (s_2, r_2) = (\hat{t}s_1, \hat{r}r_2),$$

where $\hat{t} \in T$ and $\hat{r} \in R$ satisfy $\hat{t}r_1 = \hat{r}s_2$. On the other hand, we have

$$\varphi(a \cdot b) = \varphi(\tilde{s}s_1, \tilde{r}r_2) = (\tilde{s}s_1, \tilde{r}r_2),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R$ satisfy $\tilde{s}r_1 = \tilde{r}s_2$. Now let $\mathring{t} \in T$ and $\mathring{r} \in R$ such that $\mathring{t}\hat{t}s_1 = \mathring{r}\tilde{s}s_1$, then, as $s_1 \neq 0$, we have $\mathring{t}\hat{t} = \mathring{r}\tilde{s}$. Furthermore, we have

$$(\mathring{t}\hat{r} - \mathring{r}\tilde{r})s_2 = \mathring{t}\hat{r}s_2 - \mathring{r}\tilde{r}s_2 = \mathring{t}\hat{t}r_1 - \mathring{r}\tilde{s}r_1 = (\mathring{t}\hat{t} - \mathring{r}\tilde{s})r_1 = 0,$$

which implies $\mathring{t}\hat{r} = \mathring{r}\tilde{r}$ as $s_2 \neq 0$. But then

$$\mathring{t}\hat{r}r_2 - \mathring{r}\tilde{r}r_2 = (\mathring{t}\hat{r} - \mathring{r}\tilde{r})r_2 = 0$$

proves $\varphi(a) \cdot \varphi(b) = \varphi(a \cdot b)$ in $T^{-1}R$.

Injectivity: Let $\varphi(s, r) = (s, r) = 0$ in $T^{-1}R$. Then we have $r = 0$, and thus $(s, r) = (s, 0) = 0$ in $S^{-1}R$ as well.

□

Remark 3.2. The structural homomorphism $\rho_{T,R}$ is a special case of φ in Lemma 3.1, where $S = \{1\}$.

3.2. Lifting of homomorphisms to localizations

Lemma 3.3. *Let R_1, R_2 be domains, $\phi : R_1 \to R_2$ a homomorphism of rings and $S \subseteq R_1$ a left Ore set in R_1 such that $\phi(S) \subseteq R_2$ is a left Ore set in R_2. Consider*

$$\Phi : S^{-1}R_1 \to \phi(S)^{-1}R_2, \quad (s, r) \mapsto (\phi(s), \phi(r)).$$

(a) The map Φ is a homomorphism of rings.

(b) The map Φ is injective if and only if ϕ is injective.

(c) If ϕ is surjective, so is Φ.

Proof: *(a)* Let $a := (s_1, r_1), b := (s_2, r_2) \in S^{-1}R_1$.

Well-definedness: Let $a = b$ in $S^{-1}R_1$, then there exist $\tilde{s} \in S$ and $\tilde{r} \in R_1$ such that $\tilde{s}s_1 = \tilde{r}s_2$ and $\tilde{s}r_1 = \tilde{r}r_2$. Then we have

$$\phi(\tilde{s})\phi(s_1) = \phi(\tilde{s}s_1) = \phi(\tilde{r}s_2) = \phi(\tilde{r})\phi(s_2)$$

and

$$\phi(\tilde{s})\phi(r_1) = \phi(\tilde{s}r_1) = \phi(\tilde{r}r_2) = \phi(\tilde{r})\phi(r_2),$$

which implies $\Phi(a) = (\phi(s_1), \phi(r_1)) = (\phi(s_2), \phi(r_2)) = \Phi(b)$ in $\phi(S)^{-1}R_2$, since $\phi(\tilde{s}) \in \phi(S)$ and $\phi(\tilde{r}) \in R_2$.

Additivity: We have

$$\Phi(a + b) = \Phi((\hat{s}s_1, \hat{s}r_1 + \hat{r}r_2)) = (\phi(\hat{s}s_1), \phi(\hat{s}r_1 + \hat{r}r_2)),$$

where $\hat{s} \in S$ and $\hat{r} \in R_1$ satisfy $\hat{s}s_1 = \hat{r}s_2$. On the other hand, we have

$$\Phi(a) + \Phi(b) = (\phi(s_1), \phi(r_1)) + (\phi(s_2), \phi(r_2)) = (\phi(\tilde{s}s_1), \phi(\tilde{s})\phi(r_1) + \tilde{r}\phi(r_2)),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R_2$ satisfy $\phi(\tilde{s})\phi(s_1) = \tilde{r}\phi(s_2)$. Now let $\mathring{s} \in S$ and $\mathring{r} \in R_2$ such that

$$\phi(\mathring{s}\hat{s})\phi(s_1) = \phi(\mathring{s})\phi(\hat{s}s_1) = \mathring{r}\phi(\tilde{s}s_1) = \mathring{r}\phi(\tilde{s}\phi(s_1)).$$

Since $\phi(s_1) \neq 0$ and R_2 is a domain, we have $\phi(\mathring{s}\hat{s}) = \mathring{r}\phi(\tilde{s})$. Furthermore, we have

$$\begin{aligned}
(\phi(\mathring{s})\phi(\hat{r}) - \mathring{r}\tilde{r})\phi(s_2) &= \phi(\mathring{s})\phi(\hat{r}s_2) - \mathring{r}\tilde{r}\phi(s_2) \\
&= \phi(\mathring{s})\phi(\hat{r}s_2) - \mathring{r}\phi(\tilde{s})\phi(s_1) \\
&= \phi(\mathring{s})\phi(\hat{s}s_1) - \mathring{r}\phi(\tilde{s})\phi(s_1) \\
&= (\phi(\mathring{s})\phi(\hat{s}) - \mathring{r}\phi(\tilde{s}))\phi(s_1) \\
&= 0,
\end{aligned}$$

which implies $\phi(\mathring{s})\phi(\hat{r}) = \mathring{r}\tilde{r}$, since $\phi(s_1) \neq 0$. But then

$$\begin{aligned}
\phi(\mathring{s})\phi(\hat{s}r_1 + \hat{r}r_2) - \mathring{r}(\phi(\tilde{s})\phi(r_1) + \tilde{r}\phi(r_2)) &= (\phi(\mathring{s}\hat{s}) - \mathring{r}\phi(\tilde{s}))\phi(r_1) \\
&\quad + (\phi(\mathring{s})\phi(\hat{r}) - \mathring{r}\tilde{r})\phi(r_2) \\
&= 0 \cdot \phi(r_1) + 0 \cdot \phi(r_2) = 0
\end{aligned}$$

proves $\Phi(a + b) = \Phi(a) + \Phi(b)$ in $\phi(S)^{-1}R_2$.

Multiplicativity: We have

$$\Phi(a \cdot b) = \Phi((s_1, r_1) \cdot (s_2, r_2)) = \Phi((\hat{s}s_1, \hat{r}r_2)) = (\phi(\hat{s}s_1), \phi(\hat{r}r_2)),$$

where $\hat{s} \in S$ and $\hat{r} \in R_1$ satisfy $\hat{s}r_1 = \hat{r}s_2$. On the other hand, we have

$$\Phi(a) \cdot \Phi(b) = (\phi(s_1), \phi(r_1)) \cdot (\phi(s_2), \phi(r_2)) = (\phi(\tilde{s})\phi(s_1), \tilde{r}\phi(r_2)) = (\phi(\tilde{s}s_1), \tilde{r}\phi(r_2)),$$

where $\tilde{s} \in S$ and $\tilde{r} \in R_2$ satisfy $\phi(\tilde{s}r_1) = \phi(\tilde{s})\phi(r_1) = \tilde{r}\phi(s_2)$. Now let $\mathring{s} \in S$ and $\mathring{r} \in R_2$ such that

$$\phi(\mathring{s}\hat{s})\phi(s_1) = \phi(\mathring{s})\phi(\hat{s}s_1) = \mathring{r}\phi(\tilde{s}s_1) = \mathring{r}\phi(\tilde{s})\phi(s_1).$$

Since $\phi(s_1) \neq 0$ and R_2 is a domain, we have $\phi(\mathring{s}\hat{s}) = \mathring{r}\phi(\tilde{s})$. Furthermore, we have

$$\begin{aligned}(\phi(\mathring{s})\phi(\hat{r}) - \mathring{r}\tilde{r})\phi(s_2) &= \phi(\mathring{s})\phi(\hat{r})\phi(s_2) - \mathring{r}\tilde{r}\phi(s_2) \\ &= \phi(\mathring{s})\phi(\hat{r}s_2) - \mathring{r}\phi(\tilde{s}r_1) \\ &= \phi(\mathring{s})\phi(\hat{s}r_1) - \mathring{r}\phi(\tilde{s}r_1) \\ &= (\phi(\mathring{s})\phi(\hat{s}) - \mathring{r}\phi(\tilde{s}))\phi(r_1) \\ &= 0,\end{aligned}$$

which implies $\phi(\mathring{s}\hat{r}) = \mathring{r}\tilde{r}$, since $\phi(s_2) \neq 0$. But then

$$\phi(\mathring{s})\phi(\hat{r}r_2) - \mathring{r}\tilde{r}\phi(r_2) = (\phi(\mathring{s}\hat{r}) - \mathring{r}\tilde{r})\phi(r_2) = 0$$

proves $\Phi(a \cdot b) = \Phi(a) \cdot \Phi(b)$ in $\phi(S)^{-1}R_2$.

(b) First, let ϕ be injective and $(s, r_1) \in S^{-1}R_1$ such that $0 = \Phi((s, r_1)) = (\phi(s), \phi(r_1))$. Then $\phi(r_1) = 0$, which implies $r_1 = 0$ by injectivity of ϕ. Now we have $(s, r_1) = 0$ in $S^{-1}R_1$, therefore Φ is injective.
Now, let Φ be injective and $r_1 \in R_1$ such that $\phi(r_1) = 0$, then $\Phi(1, r_1) = (\phi(1), \phi(r_1)) = 0$ in $\phi(S)^{-1}R_2$. By injectivity of Φ we have $(1, r_1) = 0$ in $S^{-1}R_1$. This implies $r_1 = 0$, thus ϕ is injective.

(c) Let $(\phi(s), r_2) \in \phi(S)^{-1}R_2$. By surjectivity of ϕ we have $r_2 = \phi(r_1)$ for some $r_1 \in R_1$. But then $(\phi(s), r_2) = (\phi(s), \phi(r_1)) = \Phi((s, r_1)) \in \mathrm{im}(\Phi)$, hence Φ is surjective.

□

3.3. Multiplicative closedness

Lemma 3.4. *Let $\varphi : R_1 \to R_2$ be a homomorphism of rings.*

(a) If $S \subseteq R_1$ is multiplicatively closed, then $\varphi(S)$ is quasi-multiplicatively closed. Furthermore, $\varphi(S)$ is multiplicatively closed if and only if $S \cap \ker(\varphi) = \emptyset$.

(b) If $S \subseteq R_2$ is multiplicatively closed, then $\varphi^{-1}(S)$ is multiplicatively closed.

Proof: *(a)* As $1 \in S$, we have $1 = \varphi(1) \in \varphi(S)$. Now consider $\varphi(s_1), \varphi(s_2) \in \varphi(S)$. Then $\varphi(s_1) \cdot \varphi(s_2) = \varphi(s_1 s_2) \in \varphi(S)$, as $s_1 s_2 \in S$. Thus, $\varphi(S)$ is quasi-multiplicatively closed.
If $\varphi(S)$ is multiplicatively closed, then $0 \notin \varphi(S)$, which implies $S \cap \ker(\varphi) = \emptyset$.
On the other hand, if $S \cap \ker(\varphi) = \emptyset$, then there is no $s \in S$ such that $\varphi(s) = 0$, therefore $0 \notin \varphi(S)$ and thus $\varphi(S)$ is multiplicatively closed.

(b) As $\varphi(1) = 1 \in S$, we have $1 \in \varphi^{-1}(S)$. As $\varphi(0) = 0 \notin S$, we have $0 \notin \varphi^{-1}(S)$. Now consider $a_1, a_2 \in \varphi^{-1}(S)$, then there exist $s_1, s_2 \in S$ such that $\varphi(a_1) = s_1$ and $\varphi(a_2) = s_2$. But now $\varphi(a_1 \cdot a_2) = \varphi(a_1) \cdot \varphi(a_2) = s_1 \cdot s_2 \in S$ and therefore $a_1 a_2 \in \varphi^{-1}(S)$. Thus, $\varphi^{-1}(S)$ is multiplicatively closed.

$\square$

Corollary 3.5. *Let $\varphi : R_1 \to R_2$ be a monomorphism of rings and $S \subseteq R_1$ multiplicatively closed. Then $\varphi(S)$ is multiplicatively closed.*

Proof: By assumption, $0_R \notin S$. Therefore, we have $S \cap \ker(\varphi) = S \cap \{0\} = \emptyset$, as φ is injective. By Lemma 3.4, we have that $\varphi(S)$ is multiplicatively closed. $\square$

3.4. Left Ore condition

In contrast to multiplicative closedness, the left Ore condition has strong requirements to be preserved under homomorphisms.

Lemma 3.6. *Let $\varphi : R_1 \to R_2$ be a homomorphism of rings.*

(a) If $S \subseteq R_1$ satisfies the Ore condition in R_1 and φ is surjective, then $\varphi(S)$ satisfies the Ore condition in R_2.

(b) If $S \subseteq R_2$ satisfies the Ore condition in R_2 and φ is bijective, then $\varphi^{-1}(S)$ satisfies the Ore condition in R_1.

Proof: *(a)* Let $\varphi(s) \in \varphi(S)$ and $r_2 \in R_2$. Since φ is surjective, there exists $r_1 \in R_1$ such that $\varphi(r_1) = r_2$. By the Ore condition on S in R_1, there exist $\tilde{s} \in S$ and $\tilde{r}_1 \in R_1$ such that $\tilde{s} r_1 = \tilde{r}_1 s$. Let $\tilde{r}_2 := \varphi(\tilde{r}_1) \in R_2$. But then we have

$$\varphi(\tilde{s}) r_2 = \varphi(\tilde{s})\varphi(r_1) = \varphi(\tilde{s} r_1) = \varphi(\tilde{r}_1 s) = \varphi(\tilde{r}_1)\varphi(s) = \tilde{r}_2 \varphi(s),$$

thus $\varphi(S)$ satisfies the Ore condition in R_2.

(b) Let $a \in \varphi^{-1}(S)$ and $r_1 \in R_1$. By the Ore condition on S in R_2, there exist $s \in S$ and $r_2 \in R_2$ such that $s\varphi(r_1) = r_2 \varphi(a)$. Since φ is surjective, there exist $\tilde{a} \in \varphi^{-1}(s) \subseteq \varphi^{-1}(S)$ and $\tilde{r}_1 \in R_1$ such that $\varphi(\tilde{r}_1) = r_2$. Then we have

$$\varphi(\tilde{a} r_1) = \varphi(\tilde{a})\varphi(r_1) = s\varphi(r_1) = r_2 \varphi(a) = \varphi(\tilde{r}_1)\varphi(a) = \varphi(\tilde{r}_1 a)$$

and by injectivity of φ we get $\tilde{a} r_1 = \tilde{r}_1 a$. Thus $\varphi^{-1}(S)$ satisfies the Ore condition in R_1.

$\square$

Remark 3.7. In the second part of Lemma 3.6 we can weaken the requirements: instead of φ being surjective, let $\varphi(R_1)$ be right saturated and $S \subseteq \varphi(R_1)$. Then, given $a \in \varphi^{-1}(S)$ and $r_1 \in R_1$ and after acquiring $s \in S$ and $r_2 \in R_2$ such that $s\varphi(r_1) = r_2\varphi(a)$, we can proceed as follows:
Since $S \subseteq \varphi(R_1)$, there exists $\tilde{a} \in \varphi^{-1}(s) \subseteq R_1$. Then $r_2\varphi(a) = s\varphi(r_1) = \varphi(\tilde{a} r_1) \in \varphi(R_1)$. Since $\varphi(R_1)$ is right saturated, we have $r_2 \in \varphi(R_1)$ and thus there exists $\tilde{r}_1 \in R_1$ such that $\varphi(\tilde{r}_1) = r_2$. From here on, the rest of the proof is identical.

Proposition 3.8. *Let $\varphi : R_1 \to R_2$ be an isomorphism of rings and $S \subseteq R_1$. Then S is a left Ore set in R_1 if and only if $\varphi(S)$ is a left Ore set in R_2.*

Proof: If S is an Ore set in R_1, then $\varphi(S)$ is an Ore set in R_2 by Corollary 3.5 and Lemma 3.6. If $\varphi(S)$ is an Ore set in R_2, then $S = \varphi^{-1}(\varphi(S))$ is an Ore set in R_1 by Lemma 3.4 and Lemma 3.6. $\square$

3.5. Isomorphisms of tensor products of Ore localizations

Lemma 3.9. *Let S_1 and S_2 be two left Ore sets in a domain R such that $S_1 \subseteq S_2$. Then*

$$\psi : S_2^{-1}R \otimes_R S_1^{-1}R \to S_2^{-1}R, \quad (s_2, r_2) \otimes (s_1, r_1) \mapsto (s_2, r_2) \cdot (s_1, r_1),$$

is a isomorphism of left $(S_2^{-1}R)$-modules (and thus in particular of left R-modules).

Proof: Let $a := a_2 \otimes a_1 := (s_2, r_2) \otimes (s_1, r_1), b := (t_2, q_2) \otimes (t_1, q_1) \in S_2^{-1}R \otimes_R S_1^{-1}R$ and $\lambda \in S_2^{-1}R$.

Additivity: Let $\tilde{s} \in S_2$ and $\tilde{r} \in R$ such that $\tilde{s}s_2 = \tilde{r}t_2$. Then we have

$$\begin{aligned}
\psi(a+b) &= \psi(((s_2, r_2) \otimes (s_1, r_1)) + ((t_2, q_2) \otimes (t_1, q_1))) \\
&= \psi(((\tilde{s}s_2, \tilde{s}r_2) \otimes (s_1, r_1)) + ((\tilde{r}t_2, \tilde{r}q_2) \otimes (t_1, q_1))) \\
&= \psi(((\tilde{s}s_2, 1) \otimes ((1, \tilde{s}r_2) \cdot (s_1, r_1))) + ((\tilde{s}s_2, 1) \otimes ((1, \tilde{r}q_2) \cdot (t_1, q_1)))) \\
&= \psi((\tilde{s}s_2, 1) \otimes ((1, \tilde{s}r_2) \cdot (s_1, r_1) + (1, \tilde{r}q_2) \cdot (t_1, q_1))) \\
&= (\tilde{s}s_2, 1) \cdot ((1, \tilde{s}r_2) \cdot (s_1, r_1) + (1, \tilde{r}q_2) \cdot (t_1, q_1)) \\
&= (\tilde{s}s_2, 1) \cdot (1, \tilde{s}r_2) \cdot (s_1, r_1) + (\tilde{r}t_2, 1) \cdot (1, \tilde{r}q_2) \cdot (t_1, q_1) \\
&= (\tilde{s}s_2, \tilde{s}r_2) \cdot (s_1, r_1) + (\tilde{r}t_2, \tilde{r}q_2) \cdot (t_1, q_1) \\
&= (s_2, r_2) \cdot (s_1, r_1) + (t_2, q_2) \cdot (t_1, q_1) \\
&= \psi((s_2, r_2) \otimes (s_1, r_1)) + \psi((t_2, q_2) \cdot (t_1, q_1)) \\
&= \psi(a) + \psi(b).
\end{aligned}$$

Scalar multiplicativity: We have

$$\psi(\lambda \cdot a) = \psi(\lambda \cdot (a_2 \otimes a_1)) = \psi((\lambda \cdot a_2) \otimes a_1) = \lambda \cdot a_2 \cdot a_1 = \lambda \cdot \psi(a_2 \otimes a_1) = \lambda \cdot \psi(a).$$

Surjectivity: Let $x \in S_2^{-1}R$, then $x = x \cdot 1 = \psi(x \otimes 1) \in \operatorname{im}(\psi)$.

Injectivity: Let $0 = \psi(a) = \psi(a_2 \otimes a_1) = a_2 \cdot a_1$. Since $S_2^{-1}R$ is a domain we have $a_2 = 0$ or $a_1 = 0$, but both cases imply $a = a_2 \otimes a_1 = 0$.

□

Lemma 3.10. *Let S_1 and S_2 be two left Ore sets in a domain R such that $S_1 \subseteq S_2$. Then*

$$\psi : S_1^{-1}R \otimes_R S_2^{-1}R \to S_2^{-1}R, \quad (s_1, r_1) \otimes (s_2, r_2) \mapsto (s_1, r_1) \cdot (s_2, r_2),$$

is a isomorphism of left $(S_1^{-1}R)$-modules (and thus in particular of left R-modules).

Proof: Analogously to Lemma 3.9. □

4. Saturation closure

This chapter contains the main contribution of this thesis: $\mathrm{LSat}_T(M)$, the notion of left T-closure or left T-saturation of M. Here, T is a subset of a ring R and M is a subset of a left R-module N.

4.1. The general construction

We start with the general case, where we only require non-emptiness of the parameters:

Definition 4.1. Let $T \subseteq R$ be a non-empty subset of a ring R, N a left R-module and $\emptyset \neq M \subseteq N$. Then

$$\mathrm{LSat}_T(M) := \{m \in N \mid tm \in M \text{ for some } t \in T\}.$$

Lemma 4.2. *Let $T \subseteq R$ be a non-empty subset of a ring R, N a left R-module and $\emptyset \neq M \subseteq N$.*

(a) If $P \subseteq M$ is non-empty, then $\mathrm{LSat}_T(P) \subseteq \mathrm{LSat}_T(M)$.

(b) If $S \subseteq T$ is non-empty, then $\mathrm{LSat}_S(M) \subseteq \mathrm{LSat}_T(M)$.

(c) If $1 \in T$, then $M \subseteq \mathrm{LSat}_T(M)$.

(d) We have $0 \in M$ if and only if $0 \in \mathrm{LSat}_T(M)$.

(e) If $0 \in T$, then the following are equivalent:

(1) $\mathrm{LSat}_T(M) = N$.

(2) $0 \in \mathrm{LSat}_T(M)$.

(3) $0 \in M$.

(f) If $0 \notin M$, then $\mathrm{LSat}_T(M) = \begin{cases} \mathrm{LSat}_{T\setminus\{0\}}(M), & \text{if } T \neq \{0\}, \\ \emptyset, & \text{if } T = \{0\}. \end{cases}$

Proof: *(a)* Let $m \in \mathrm{LSat}_T(P)$, then $tm \in P \subseteq M$ for some $t \in T$. Thus, $m \in \mathrm{LSat}_T(M)$.

(b) Let $m \in \mathrm{LSat}_S(M)$, then $sm \in M$ for some $s \in S \subseteq T$. Thus, $m \in \mathrm{LSat}_T(M)$.

(c) Let $1 \in T$ and $m \in M$, then $1 \cdot m = m \in M$. Thus, $m \in \mathrm{LSat}_T(M)$.

(d) If $0 \in M$, then $t \cdot 0 = 0 \in M$ for any $t \in T$, thus $0 \in \mathrm{LSat}_T(M)$.
If $0 \in \mathrm{LSat}_T(M)$, then $0 = t \cdot 0 \in M$ for some $t \in T$, thus $0 \in M$.

(e) The equivalence of *(2)* and *(3)* is stated above, the implication from *(1)* to *(2)* is obvious. Let $0 \in M$, then for all $m \in N$ we have $0 \cdot m = 0 \in M$ since $0 \in T$, thus $m \in \mathrm{LSat}_T(M)$.

(f) If $T = \{0\}$, then $\mathrm{LSat}_T(M) = \mathrm{LSat}_{\{0\}}(M) = \{m \in N \mid 0 = 0 \cdot m \in M\} = \emptyset$, since $0 \notin M$. If $\emptyset \neq T \setminus \{0\} \subseteq T$, then we have $\mathrm{LSat}_{T\setminus\{0\}}(M) \subseteq \mathrm{LSat}_T(M)$. Let $m \in \mathrm{LSat}_T(M)$, then there exists $t \in T$ such that $tm \in M$. In particular, $tm \neq 0$, which implies $t \neq 0$. Thus, $m \in \mathrm{LSat}_{T\setminus\{0\}}(M)$ and $\mathrm{LSat}_T(M) \subseteq \mathrm{LSat}_{T\setminus\{0\}}(M)$.

□

4.2. Restriction to quasi-multiplicatively closed T

Definition 4.3. Let $T \subseteq R$ be a quasi-multiplicatively closed subset of a ring R, N a left R-module and $\emptyset \neq M \subseteq N$. We call M

- *left T-closed* if $M = \mathrm{LSat}_T(M)$.
- *left T-saturated* if $tm \in M$ implies $m \in M$ for all $t \in T$ and all $m \in N$.

Lemma 4.4. *Let $T \subseteq R$ be a quasi-multiplicatively closed subset of a ring R, N a left R-module and $\emptyset \neq M \subseteq N$. Then we have:*

(a) $M \subseteq \mathrm{LSat}_T(M)$.

(b) $\mathrm{LSat}_T(M)$ *is left T-saturated.*

(c) M is left T-saturated if and only if M is T-closed.

(d) $\mathrm{LSat}_T(M)$ *is the smallest left T-saturated superset of M in the sense that if $\emptyset \neq P \subseteq N$ is a left T-saturated set with $M \subseteq P \subseteq \mathrm{LSat}_T(M)$, then $P = \mathrm{LSat}_T(M)$.*

Proof: *(a)* Follows from Lemma 4.2, since T is quasi-multiplicatively closed and thus $1 \in T$.

(b) Let $t \in T$ and $m \in N$ such that $tm \in \mathrm{LSat}_T(M)$. Then there exists $\tilde{t} \in T$ such that $\tilde{t}tm \in M$. Since $\tilde{t}t \in T$, we have $m \in \mathrm{LSat}_T(M)$.

(c) Let M be left T-saturated and $m \in \mathrm{LSat}_T(M)$, then there exists $t \in T$ such that $tm \in M$. Since M is left T-saturated, we have $m \in M$ and therefore $\mathrm{LSat}_T(M) = M$.
Now let M be T-closed, then $M = \mathrm{LSat}_T(M)$, which is left T-saturated.

(d) Let $\emptyset \neq P \subseteq N$ be a left T-saturated set with $M \subseteq P \subseteq \mathrm{LSat}_T(M)$. Let $m \in \mathrm{LSat}_T(M)$, then there exists $t \in T$ such that $tm \in M \subseteq P$. Since P is left T-saturated, we have $m \in P$ and therefore $\mathrm{LSat}_T(M) = P$.

□

Remark 4.5. Due to Lemma 4.4, we can interpret $\mathrm{LSat}_T(M)$ for quasi-multiplicatively closed T as the *left T-saturation closure* of M in N.

Corollary 4.6. *Let $T \subseteq R$ be a quasi-multiplicatively closed subset of a ring R and N a left R-module. Furthermore, let $M_1, M_2 \subseteq N$ be non-empty subsets of N such that $M_1 \subseteq M_2 \subseteq \mathrm{LSat}_T(M_1)$. Then* $\mathrm{LSat}_T(M_2) = \mathrm{LSat}_T(M_1)$.

Proof: Since $M_1 \subseteq M_2$ we have $\mathrm{LSat}_T(M_1) \subseteq \mathrm{LSat}_T(M_2)$. Now

$$M_2 \subseteq \mathrm{LSat}_T(M_1) \subseteq \mathrm{LSat}_T(M_2)$$

implies that $\mathrm{LSat}_T(M_1)$ is a left T-saturated superset of M_2 that is contained in $\mathrm{LSat}_T(M_2)$. From Lemma 4.4 we get $\mathrm{LSat}_T(M_2) = \mathrm{LSat}_T(M_1)$. □

From here on, the theory diverges and we consider two cases.

4.3. S-closure of submodules

Definition 4.7. Let $S \subseteq R$ be a left Ore set in a domain R and $P \subseteq N$ a left submodule of a left R-module N. The S-*closure* of P is defined as

$$P^S := \mathrm{LSat}_S(P) = \{m \in N \mid sm \in P \text{ for some } s \in S\} \supseteq P.$$

Furthermore, P is called *left S-closed* if $P = P^S$.

Lemma 4.8. *Let $S \subseteq R$ be a left Ore set in a domain R and $P \subseteq N$ a left submodule of a left R-module N. Then P^S is a submodule of N.*

Proof: Let $p, p_1, p_2 \in P^S$ and $r \in R$. Then there exist $s, s_1, s_2 \in S$ such that $sp, s_1p_1, s_2p_2 \in P$.

- By the Ore condition on S there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $x := \tilde{s}s_1 = \tilde{r}s_2 \in S$. Then $x(p_1 + p_2) = xp_1 + xp_2 = \tilde{s}s_1p_1 + \tilde{r}s_2p_2 \in P$, thus $p_1 + p_2 \in P^S$.
- By the Ore condition on S there exist $\hat{s} \in S$ and $\hat{r} \in R$ such that $\hat{s}r = \hat{r}s$. Then $\hat{s}rp = \hat{r}sp \in P$, thus $rp \in P^S$.

□

Now we show the connection between S-closure and the extension-contraction problem:

Definition 4.9. Let $\varphi : R \to T$ be a homomorphism of rings.

- Let I be a left ideal in R. The *extension* of I with respect to φ is the left ideal $I^e := T\varphi(I)$ in T.
- Let J be a left ideal in T. The *contraction* of J with respect to φ is the left ideal $J^c := \varphi^{-1}(J)$ in R.

Lemma 4.10. *In the situation of Definition 4.9 we have $I \subseteq (I^e)^c$ and $(J^c)^e \subseteq J$.*

Proof: We have

$$I \subseteq \varphi^{-1}(\varphi(I)) \subseteq \varphi^{-1}(T\varphi(I)) = \varphi^{-1}(I^e) = (I^e)^c$$

as well as

$$(J^c)^e = T\varphi(\varphi^{-1}(J)) \subseteq TJ = J.$$

□

Lemma 4.11. *Let S be a left Ore set in a domain R and J a left ideal in $S^{-1}R$. We have $(J^c)^e = J$ with respect to $\rho := \rho_{S,R}$.*

Proof: Let $(s, r) \in J$, then $\rho(r) = (1, r) = (1, s) \cdot (s, r) \in J$, thus $r \in J^c$. Now $\rho(r) \in \rho(J^c)$ and therefore $(s, r) = (s, 1) \cdot (1, r) = (s, 1) \cdot \rho(r) \in (J^c)^e$. □

Proposition 4.12. *Let S be a left Ore set in a domain R and I a left ideal in R. Then $(I^e)^c = \mathrm{LSat}_S(I)$ with respect to $\rho := \rho_{S,R}$.*

Proof: Let $r \in (I^e)^c$, then there exist $s \in S$ and $a \in I$ such that

$$r \in \rho^{-1}((s, 1) \cdot \rho(a)) = \rho^{-1}((s, a))$$

which implies $(1, r) = \rho(r) = (s, a)$. Then there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{s} \cdot 1 = \tilde{r}s$ and $\tilde{s}r = \tilde{r}a \in I$, thus $r \in \mathrm{LSat}_S(I)$.
On the other hand, let $r \in \mathrm{LSat}_S(I)$, then there exists $s \in S$ such that $sr \in I$. Now

$$r \in \rho^{-1}((1, r)) = \rho^{-1}((s, 1) \cdot (1, sr)) = \rho^{-1}((s, 1) \cdot \rho(sr)) \subseteq \rho^{-1}(S^{-1}R\rho(I)) = (I^e)^c.$$

□

4.4. Left saturation with respect to R

If we set $N = T = R$, then “left R-saturated” simply means “left saturated”.

Definition 4.13. Let $M \subseteq R$ be a non-empty subset of a ring R. The *left saturation* of M in R is defined as

$$\mathrm{LSat}(M) := \mathrm{LSat}_R(M) = \{r \in R \mid wr \in M \text{ for some } w \in R\} \supseteq M.$$

Lemma 4.14. *Let $M \subseteq R$ be a non-empty subset of a ring R. Then we have:*

(a) $U(R) \subseteq \mathrm{LSat}(M)$.

(b) The following are equivalent:

  (1) $\mathrm{LSat}(M) = R$.

  (2) $0 \in \mathrm{LSat}(M)$.

  (3) $0 \in M$.

(c) If $0 \notin M$, *then* $\mathrm{LSat}(M) = \mathrm{LSat}_{R\setminus\{0\}}(M) = \{r \in R \mid wr \in M$ *for some* $w \in R \setminus \{0\}\}$.

(d) $\mathrm{LSat}(M)$ *is left saturated.*

(e) M *is left saturated if and only if* $M = \mathrm{LSat}(M)$.

(f) $\mathrm{LSat}(M)$ *is the smallest left saturated superset of* M *in the sense that if* $N \subseteq R$ *is a left saturated set with* $M \subseteq N \subseteq \mathrm{LSat}(M)$, *then* $N = \mathrm{LSat}(M)$.

Proof: The only thing to show is *(a)*: Let $u \in U(R)$ and $m \in M$. Then $m \cdot u^{-1} \cdot u = m \in M$ and $m \cdot u^{-1} \in R$, thus $u \in \mathrm{LSat}(M)$.
Parts *(b)* and *(c)* follow from Lemma 4.2, since $0 \in R$ and $R \neq \{0\}$. The remaining parts *(d)* to *(f)* follow from Lemma 4.4. $\square$

Lemma 4.15. *Let R be a Dedekind-finite ring. Then* $\mathrm{LSat}(\{1\}) = U(R)$. *Furthermore,* $\mathrm{LSat}(U) = U(R)$ *for any* $U \subseteq U(R)$ *with* $1 \in U$.

Proof: By Lemma 4.14 we have $U(R) \subseteq \mathrm{LSat}(\{1\})$. Now let $x \in \mathrm{LSat}(\{1\})$, then there exists $w \in R \setminus \{0\}$ such that $wx = 1$. Since R is Dedekind-finite, this implies $x \in U(R)$.
Furthermore, $\{1\} \subseteq U \subseteq U(R) = \mathrm{LSat}(\{1\})$ implies $\mathrm{LSat}(U) = U(R)$ by Corollary 4.6. $\square$

4.5. Characterization of units

Proposition 4.16. *Let $S \subseteq R$ be an Ore set in a domain R and $(s, r) \in S^{-1}R$. The following are equivalent:*

(1) $(s, r) \in U(S^{-1}R)$.

(2) $(1, r) \in U(S^{-1}R)$.

(3) $r \in \mathrm{LSat}(S)$.

Proof: We always have $(s, r) = (s, 1) \cdot (1, r)$, where $(s, 1) \in U(S^{-1}R)$ with inverse $(1, s)$.

(1)⇒(2): Let $a \in S^{-1}R$ be the inverse of (s, r). But then $a \cdot (s, 1) \in S^{-1}R$ is the inverse of $(1, r)$, as
$$a \cdot (s, 1) \cdot (1, r) = a \cdot (s, r) = 1.$$

(2)⇒(1): Let $a \in S^{-1}R$ be the inverse of $(1, r)$. But then $a \cdot (1, s) \in S^{-1}R$ is the inverse of (s, r), as
$$(s, r) \cdot a \cdot (1, s) = (s, 1) \cdot (1, r) \cdot a \cdot (1, s) = (s, 1) \cdot (1, s) = 1.$$

(2)⇒(3): Let $(1, r) \in U(S^{-1}R)$. Then there exists $(s, w) \in S^{-1}R$ such that $(1, 1) = (s, w) \cdot (1, r) = (s, wr)$, which implies $wr = s \in S$ and thus $r \in \mathrm{LSat}(S)$.

(3)⇒(2): Let $r \in \mathrm{LSat}(S)$ with $w \in R$ such that $wr \in S$. Then $(wr, w) \in S^{-1}R$ satisfies $(wr, w) \cdot (1, r) = (wr, wr) = (1, 1)$ and thus $(1, r) \in U(S^{-1}R)$.

□

Corollary 4.17. *Let $S \subseteq R$ be a left saturated Ore set in a domain R and $(s, r) \in S^{-1}R$. Then $(s, r) \in U(S^{-1}R)$ if and only if $r \in S$.*

Proof: As S is left saturated, we have $\mathrm{LSat}(S) = S$ by Lemma 4.14. By Proposition 4.16, we have $(s, r) \in U(S^{-1}R)$ if and only if $r \in \mathrm{LSat}(S) = S$. □

Given an arbitrary non-empty subset M of R, $\mathrm{LSat}(M)$ is not (right-)saturated in general, which we will see later in the main example. But in the case where S is a left Ore set, we can show that $\mathrm{LSat}(S)$ is indeed saturated via a little trick that involves the localization $S^{-1}R$:

Proposition 4.18. *Let $S \subseteq R$ be a left Ore set in a domain R. Then $\mathrm{LSat}(S)$ is saturated.*

Proof: Let $p, q \in R$ such that $r := pq \in \mathrm{LSat}(S)$. By Proposition 4.16, we have that
$$(1, p) \cdot (1, q) = (1, pq) = (1, r) \in U(S^{-1}R).$$
Since $U(S^{-1}R)$ is saturated by Lemma 1.30, we have $(1, p), (1, q) \in U(S^{-1}R)$. But then $p, q \in \mathrm{LSat}(S)$ by Proposition 4.16. □

As an application, this gives us a criterion to decide whether the extension of an ideal to a localization is proper or not:

Lemma 4.19. *Let $S \subseteq R$ be a left Ore set in a domain R and L a left ideal in R. Then $S^{-1}R = L^e$ with respect to $\rho := \rho_{S,R}$ if and only if $L \cap \mathrm{LSat}(S) \neq \emptyset$.*

Proof: If $x \in L \cap \mathrm{LSat}(S)$, then by Proposition 4.16 $(1, x) = \rho(x) \in \rho(L) \subseteq L^e$ is a unit in $S^{-1}R$ that is contained in the left ideal L^e, which implies $L^e = S^{-1}R$.
Now let $L^e = S^{-1}R$, then $(1, 1) \in L^e$. Therefore there exist $s \in S$, $r \in R$ and $l \in L$ such that $(1, 1) = (s, r) \cdot \rho(l) = (s, r) \cdot (1, l)$. Since the unit group $U(S^{-1}R)$ is saturated by Lemma 1.30, we have $(1, l) \in U(S^{-1}R)$ and thus $l \in L \cap \mathrm{LSat}(S)$ by Proposition 4.16. □

4.6. Localization at left saturation

We have already seen that for a left Ore set S, $\mathrm{LSat}(S)$ is saturated. It remains to show that $\mathrm{LSat}(S)$ itself is a left Ore set and that the localizations $S^{-1}R$ and $\mathrm{LSat}(S)^{-1}R$ are isomorphic.

Lemma 4.20. *Let S be a left Ore set in a domain R. Then $\mathrm{LSat}(S)$ is a left Ore set in R.*

Proof:

- As $0 \notin S$, we have $0 \notin \mathrm{LSat}(S)$. Furthermore, we have $1 \in S \subseteq \mathrm{LSat}(S)$.
- Let $x, y \in \mathrm{LSat}(S)$, then there exist $a, b \in R \setminus \{0\}$ such that $ax \in S$ and $by \in S$. By the Ore condition on S, there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{r}ax = \tilde{s}b$. Then we have $\tilde{r}axy = \tilde{s}by \in S$ and therefore $xy \in \mathrm{LSat}(S)$, thus $\mathrm{LSat}(S)$ is multiplicatively closed.
- Let $x \in \mathrm{LSat}(S)$ and $r \in R$, then there exists $w \in R \setminus \{0\}$ such that $wx \in S$. By the Ore condition on S, there exist $\tilde{s} \in S$ and $\tilde{r} \in R$ such that $\tilde{r}wx = \tilde{s}r$. As $\tilde{r}w \in R$ and $\tilde{s} \in S \subseteq \mathrm{LSat}(S)$, we have that $\mathrm{LSat}(S)$ satisfies the Ore condition in R.

Thus, $\mathrm{LSat}(S)$ is a left Ore set in R. □

Proposition 4.21. *Let $S \subseteq R$ be a left Ore set in a domain R. We have $S^{-1}R \cong \mathrm{LSat}(S)^{-1}R$ as rings.*

Proof: As $S \subseteq \mathrm{LSat}(S)$ is an inclusion of left Ore sets in R, the mapping

$$\varphi : S^{-1}R \to \mathrm{LSat}(S)^{-1}R, \quad (s, r) \mapsto (s, r),$$

is a ring monomorphism by Lemma 3.1. To see surjectivity, consider $(x, r) \in \mathrm{LSat}(S)^{-1}R$, then there exists $w \in R$ such that $wx \in S$. But now we have

$$(x, r) = (wx, wr) = \varphi(wx, wr) \in \mathrm{im}(\varphi).$$

□

At this point we can see that, for every left Ore set S, $\mathrm{LSat}(S)$ gives us a saturated left Ore set that describes essentially the same localization. For theoretical purposes, this allows us to assume without loss of generality that any given left Ore set is already saturated.

Corollary 4.22. *Let $S_1, S_2 \subseteq R$ be left Ore sets in a domain R. If $\mathrm{LSat}(S_1) = \mathrm{LSat}(S_2)$, then $S_1^{-1}R \cong S_2^{-1}R$ as rings.*

Proof: From Proposition 4.21 we get $S_1^{-1}R \cong \mathrm{LSat}(S_1)^{-1}R = \mathrm{LSat}(S_2)^{-1}R \cong S_2^{-1}R$. □

Corollary 4.23. *Let G be an ordered monoid with respect to $\preceq$, R a G-graded domain and S a left Ore set in R. If $S \subseteq h(R)$, then $\mathrm{LSat}(S) \subseteq h(R) \setminus \{0\}$.*

Proof: Let $x \in \mathrm{LSat}(S)$, then $x \neq 0$ and there exists a $w \in R \setminus \{0\}$ such that $wx \in S \subseteq h(R) \setminus \{0\}$. Since $h(R) \setminus \{0\}$ is saturated by Lemma 1.23, we have $x \in h(R) \setminus \{0\}$. □

Remark 4.24. Clearly, if S is not homogeneous, then $\mathrm{LSat}(S)$ is not homogeneous either since it contains S. Thus, in the situation of Corollary 4.23, S is homogeneous if and only if $\mathrm{LSat}(S)$ is homogeneous.

Proposition 4.25. *Let $S \subseteq R$ be an Ore set in a domain R, $p, q \in R$ and $r = pq$. If $(1, r)$ is irreducible in $S^{-1}R$, but not a unit, then $|\{p, q\} \cap \mathrm{LSat}(S)| = 1$.*

Proof: We always have the induced factorization $(1,r)=(1,p)\cdot(1,q)$ in $S^{-1}R$.

- If $|\{p,q\}\cap \mathrm{LSat}(S)|=2$, then $(1,p),(1,q)\in U(S^{-1}R)$ by Proposition 4.16. Thus, $(1,r)\in U(S^{-1}R)$ as a product of two units.
- If $|\{p,q\}\cap \mathrm{LSat}(S)|=0$, then $(1,p),(1,q)\notin U(S^{-1}R)$ by Proposition 4.16. But then $(1,r)$ is reducible.

□

Main example, part 3

Lemma 4.26. *In $\mathcal{D}$, $T:=\mathrm{LSat}(\{x\partial+1\})$ is not (right-)saturated.*

Proof: Assume that T is saturated, then $\partial\in T$ since $\partial x=x\partial+1$. Since T is the left saturation of $x\partial+1$ there exists a $w\in\mathcal{D}$ such that $w\partial=x\partial+1$, or equivalently, $(w-x)\partial=1$. This implies that ∂ is a unit in $\mathcal{D}$, which is a contradiction. □

Lemma 4.27. *In $\mathcal{D}$ (over the field K), we have*

$$\mathrm{LSat}(\Theta)=[\Theta\cup\{x,\partial\}\cup U(K)]=[(\theta+\mathbb{Z})\cup\{x,\partial\}\cup(K\setminus\{0\})].$$

Proof: Let $S:=[(\theta+\mathbb{Z})\cup\{x,\partial\}\cup U(K)]$. Since $\theta=x\partial\in\Theta$, we clearly have $\{x,\partial\}\subseteq\mathrm{LSat}(\Theta)$. and therefore $S\subseteq\mathrm{LSat}(\Theta)$ (note that $U(K)=K\setminus\{0\}=U(\mathcal{D})$ is always contained in $\mathrm{LSat}(\Theta)$ by Lemma 4.14).
To see the other inclusion, consider that by Lemma 1.43, every element $s\in S$ can be written in the form $s=ty^n$, where $y\in\{x,\partial\}$, $n\in\mathbb{N}_0$ and $t\in\Theta$. Then [HL16] implies that every other non-trivial factorization of s can be derived by using the commutation rules given in Lemma 1.43 and rewriting θ respectively $\theta+1$ as $x\partial$ respectively ∂x. But all occurring factors are already contained in S, thus $\mathrm{LSat}(\Theta)\subseteq S$ (the non-trivial factorizations correspond to scattering units between the factors). □

Remark 4.28. Consider the left Ore set $S:=[\Theta\cup\{\partial-1\}]=[(\theta+\mathbb{Z})\cup\{\partial-1\}]$. Makar-Limanov shows in [ML83] that the skew field of fractions of the first Weyl algebra, which is the localization $(\mathcal{D}\setminus\{0\})^{-1}\mathcal{D}$, contains a free subalgebra generated by the elements $(\partial x,1)$ and $(\partial x,1)\cdot(1-\partial,1)$. These two elements can also be found in the (smaller) localization $S^{-1}\mathcal{D}$.
In contrast to $\mathrm{LSat}(\Theta)$, $\mathrm{LSat}(S)$ is inhomogeneous and thus much harder to describe, for example, for all $i\in\mathbb{Z}$, $\mathrm{LSat}(S)$ contains the (irreducible) element $x\partial^2-x\partial+(i+2)\partial-i$, since

$$(x\partial+i+1)(x\partial^2-x\partial+(i+2)\partial-i)=(\partial-1)(x\partial+i)(x\partial+i+1)\in S.$$

5. Ore localization of modules and local torsion

5.1. Ore localization of modules

We define the Ore localization of modules via a tensor product. Analogously to the commutative case one can also use an elementary definition via an equivalence relation. Details can be found in Section 7 of [Š06].

Definition 5.1. Let S be a left Ore set in a domain R and M a left R-module. The *left Ore localization* of M is the left $S^{-1}R$-module $S^{-1}M := S^{-1}R \otimes_R M$.

Formally, tensor products consist of finite sums of elementary tensors. The Ore condition allows us to find a "common denominator", which allows us to express all elements of the localization as elementary tensors.

Lemma 5.2. *Let S be a left Ore set in a domain R and M a left R-module. Every element of $S^{-1}M$ can be presented in the form $s^{-1}m := (s,1) \otimes m$ for some $s \in S$ and $m \in M$.*

Proof: Let $\sum_{i=1}^{n}(s_i,r_i)\otimes m_i \in S^{-1}M$. We have $(s_i,r_i)\otimes m_i = ((s_i,1)\cdot r_i)\otimes m_i = (s_i,1)\otimes r_i m_i$ for all i, so we can assume $r_i = 1$ without loss of generality. Now assume $n \geq 2$. By the Ore condition on S there exist $s \in S$ and $r \in R$ such that $ss_1 = rs_2$. Then

$$\begin{aligned}(s_1,1)\otimes m_1 + (s_2,1)\otimes m_2 &= (ss_1,s)\otimes m_1 + (rs_2,r)\otimes m_2 \\ &= (ss_1,1)\otimes sm_1 + (rs_2,1)\otimes rm_2 \\ &= (ss_1,1)\otimes (sm_1 + rm_2).\end{aligned}$$

The rest follows by induction on n. □

Remark 5.3. Let S be a left Ore set in a domain R and L a left ideal of R. Let $s^{-1}l \in S^{-1}L$, then $s^{-1}l = (s,1)\otimes l = (s,l)\otimes 1$, which we can identify with $(s,l) = (s,1)\cdot(1,l) \in S^{-1}R(\rho_{S,R}(L))$. Thus

$$S^{-1}L \cong S^{-1}R(\rho_{S,R}(L)) = \big\{(s,l) \in S^{-1}R \mid s \in S, l \in L\big\}$$

as left $S^{-1}R$-modules.

Remark 5.4. Let S be a left Ore set in a domain R, M and N two left R-modules and $\varphi : M \to N$ a morphism. Then $S^{-1}\cdot$ becomes a covariant functor from R-mod to $S^{-1}R$-mod via $S^{-1}\varphi := \mathrm{id}\otimes\varphi : S^{-1}M \to S^{-1}N$, which is sometimes called the *localization functor*. From the properties of tensor products we get that $S^{-1}\cdot$ is right-exact.

Proposition 5.5. *Let S be a left Ore set in a domain R. The functor $S^{-1}\cdot = S^{-1}R \otimes_R \cdot$ is exact, in other words, $S^{-1}R$ is flat as a right R-module.*

Proof: Let $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be an exact sequence of R-modules. We have to show the exactness of the sequence $S^{-1}M_1 \xrightarrow{S^{-1}f} S^{-1}M_2 \xrightarrow{S^{-1}g} S^{-1}M_3$, which is equivalent to $\mathrm{im}(S^{-1}f) = \ker(S^{-1}g)$.

- By assumption, we have $g \circ f = 0$, which implies $(S^{-1}g)\cdot(S^{-1}f) = S^{-1}(g\circ f) = S^{-1}0 = 0$ and thus $\mathrm{im}(S^{-1}f) \subseteq \ker(S^{-1}g)$.

- Let $s^{-1}m_2 \in \ker(S^{-1}g) \subseteq S^{-1}M_2$. We have $0 = (S^{-1}g)(s^{-1}m_2) = s^{-1}g(m_2)$, then there exists $r \in R$ with $rs \in S$ and $g(rm_2) = rg(m_2) = 0$. Now $rm_2 \in \ker(g) = \mathrm{im}(f)$, so $rm_2 = f(m_1)$ for some $m_1 \in M_1$. Now we can conclude $\ker(S^{-1}g) \subseteq \mathrm{im}(S^{-1}f)$ from

$$s^{-1}m_2 = (rs)^{-1}rm_2 = (rs)^{-1}f(m_1) = (S^{-1}f)((rs)^{-1}m_1) \in \mathrm{im}(S^{-1}f).$$

□

Localization is also perfectly compatible with finite presentation:

Proposition 5.6. *Let S be a left Ore set in a domain R and $P \in R^{m\times n}$ a presentation matrix of the finitely presented R-Module $M \cong R^{1\times n}/R^{1\times m}P$. Then we have*

$$S^{-1}M \cong (S^{-1}R)^{1\times n}/(S^{-1}R)^{1\times m}P.$$

Proof: The sequence $R^{1\times m} \xrightarrow{\cdot P} R^{1\times n} \longrightarrow M \to 0$ is exact. Exactness of $S^{-1}\cdot$ induces the exactness of

$$S^{-1}R \otimes_R R^{1\times m} \xrightarrow{\cdot P} S^{-1}R \otimes_R R^{1\times n} \longrightarrow S^{-1}R \otimes_R M \to 0.$$

By the homomorphism theorem we have

$$S^{-1}M = S^{-1}R \otimes_R M \cong (S^{-1}R \otimes_R R^{1\times n})/(S^{-1}R \otimes_R R^{1\times m})P.$$

Now the proposition follows since $S^{-1}R \otimes R^{1\times k} \cong (S^{-1}R)^{1\times k}$ for all $k \in \mathbb{N}$. □

5.2. Local torsion

Definition 5.7. Let $\Lambda \subseteq R$ be a non-empty subset of a domain R and M a left R-module.

- An element $m \in M$ is called Λ-*torsion element* if $\lambda m = 0$ for some $\lambda \in \Lambda \setminus \{0\}$.
- The set of all Λ-torsion elements $t_\Lambda(M)$ is called the Λ-*torsion subset* of M.
- An element $m \in M \setminus t_\Lambda(M)$ is called Λ-*regular*.
- The module M is called Λ-*torsion* or Λ-*torsion module* if $t_\Lambda(M) = M$.
- The module M is called Λ-*torsion-free* if $t_\Lambda(M) = \{0\}$.

In the case $\Lambda \in \{R, R \setminus \{0\}\}$ we omit Λ in the notation.

Remark 5.8. If $\Lambda \setminus \{0\}$ is non-empty, we have $t_\Lambda(M) = t_{\Lambda\setminus\{0\}}(M)$. If $\Lambda = \{0\}$, then we set $t_\Lambda(M) = \{0\}$.

Corollary 5.9. *Let R be a domain, $S, T \subseteq R$ two non-empty subsets and M, N two left R-modules such that $S \subseteq T$ and $M \subseteq N$. Then the following holds:*

(a) $t_S(M) \subseteq M$ *and* $t_S(t_S(M)) = t_S(M)$.

(b) $t_S(M) \subseteq t_T(M)$ *and* $t_S(M) \subseteq t_S(N)$.

Local torsion with respect to S is also called S-torsion. In the situation where S is a left Ore set we retain most of the properties of classical R-torsion:

Lemma 5.10 (Structural theorem of local torsion)**.** *Let S be a left Ore set in a domain R and M a left R-module. Consider the mapping*

$$\varepsilon : M \to S^{-1}R \otimes_R M, \quad m \mapsto 1 \otimes m.$$

(a) The mapping ε is a homomorphism of R-modules.

(b) We have $\ker(\varepsilon) = t_S(M)$. In particular, $t_S(M)$ is an R-submodule of M.

(c) We have $t_S(M/t_S(M)) = \{0\}$.

(d) We have $S^{-1}t_S(M) = \{0\}$.

(e) The induced mapping $\varepsilon_t : M/t_S(M) \to S^{-1}R \otimes_R (M/t_S(M))$, $m \mapsto 1 \otimes m$ is injective.

(f) We have $S^{-1}M \cong S^{-1}(M/t_S(M))$.

Proof: *(a)* For $m, n \in M$ and $r \in R$ we have $\varepsilon(rm) = 1 \otimes rm = r \otimes m = r \cdot (1 \otimes m) = r \cdot \varepsilon(m)$ and $\varepsilon(m + n) = 1 \otimes (m + n) = (1 \otimes m) + (1 \otimes n) = \varepsilon(m) + \varepsilon(n)$.

(b) For all $m \in M$ and $s \in S$ we have $1 \otimes m = (s, 1) \cdot (1 \otimes sm)$, therefore $1 \otimes m = 0$ if and only if there exists $s \in S$ such that $sm = 0$. But then

$$\ker(\varepsilon) = \{m \in M \mid 1 \otimes m = 0\} = \{m \in M \mid sm = 0 \text{ for some } s \in S\} = t_S(M).$$

(c) Let $m \in M$ and consider $m + t_S(M) \in t_S(M/t_S(M))$. There exists $s \in S$ such that

$$0 = s(m + t_S(M)) = sm + st_S(M)$$

in $M/t_S(M)$. Now $st_S(M) \subseteq t_S(M)$ implies $sm \in t_S(M)$. But then there exists $\tilde{s} \in S$ such that $(\tilde{s}s)m = \tilde{s}(sm) = 0$. Since $\tilde{s}s \in S$ we have $m \in t_S(M)$, thus $m = 0$ in $M/t_S(M)$.

(d) Let $(s, 1) \otimes m \in S^{-1}t_S(M)$. Then there exists $\tilde{s} \in S$ such that $\tilde{s}m = 0$. Now

$$(s, 1) \otimes m = (\tilde{s}s, \tilde{s}) \otimes m = ((\tilde{s}s, 1) \cdot \tilde{s}) \otimes m = (\tilde{s}s, 1) \otimes \tilde{s}m = (\tilde{s}s, 1) \otimes 0 = 0.$$

(e) Combining *(b)* and *(c)*, we have $\ker(\varepsilon_t) = t_S(M/t_S(M)) = \{0\}$, thus ε_t is injective.

(f) The canonical sequence $0 \to t_S(M) \to M \to M/t_S(M) \to 0$ is exact. Since $S^{-1}\cdot$ is exact by Proposition 5.5, so is

$$0 = S^{-1}t_S(M) \to S^{-1}M \to S^{-1}(M/t_S(M)) \to 0.$$

But then $S^{-1}M \cong S^{-1}(M/t_S(M))$.

□

Local torsion with respect to a left Ore set S is the same as $\mathrm{LSat}(S)$-torsion:

Lemma 5.11. *Let $S \subseteq R$ be a left Ore set in a domain R and M a left R-module. Then $t_S(M) = t_{\mathrm{LSat}(S)}(M)$.*

Proof: As $S \subseteq \mathrm{LSat}(S)$ it only remains to show that $t_{\mathrm{LSat}(S)}(M) \subseteq t_S(M)$. To this end, let $m \in t_{\mathrm{LSat}(S)}(M)$, then there exists $x \in \mathrm{LSat}(S)$ such that $xm = 0$. As $x \in \mathrm{LSat}(S)$, there also exists $r \in R \setminus \{0\}$ such that $rx \in S$. But then $(rx)m = r(xm) = r \cdot 0 = 0$, thus $m \in t_S(M)$. □

Remark 5.12. Let S be a left Ore set in a domain R and $M \neq \{0\}$ a left R-module. Then M falls into exactly one of the following categories:

(i) M is an S-torsion module, which is equivalent to $S^{-1}M = \{0\}$.

(ii) M is an S-torsion-free module.

(iii) M is "generic" in the sense that it is neither S-torsion nor S-free, thus $\{0\} \subsetneq t_S(M) \subsetneq M$. Then we have the exact sequence $0 \to t_S(M) \to M \to M/t_S(M) \to 0$, where $t_S(M)$ is S-torsion and $M/t_S(M)$ is S-torsion-free.

Proposition 5.13. *Let S be a left Ore set in a domain R. Then $t_S(\cdot)$ is a covariant left-exact functor from the category of R-modules to the category of S-torsion R-modules.*

Proof: Let $\varphi : M \to N$ be a morphism of left R-modules. Then $t_S(\cdot)$ becomes a covariant functor via $t_S(\varphi) : t_S(M) \to t_S(N),\ m \mapsto \varphi(m)$, since for $m \in t_S(M)$ with $sm = 0$ for some $s \in S$ we have $s \cdot \varphi(m) = \varphi(sm) = \varphi(0) = 0$, which shows $\mathrm{im}(t_S(\varphi)) \subseteq t_S(N)$.
Now let $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be an exact sequence of left R-modules, we have to show exactness of the induced sequence

$$0 \to t_S(M_1) \xrightarrow{t_S(f)} t_S(M_2) \xrightarrow{t_S(g)} t_S(M_3),$$

which is equivalent to $\ker(t_S(f)) = \{0\}$ and $\mathrm{im}(t_S(f)) = \ker(t_S(g))$.

- By construction, we have $\ker(t_S(f)) = \ker(f) = \{0\}$.
- By assumption, we have $g \circ f = 0$, which implies $t_S(g) \circ t_S(f) = t_S(g \circ f) = t_S(0) = 0$ and thus $\mathrm{im}(t_S(f)) \subseteq \ker(t_S(g))$.
- Let $m_2 \in \ker(t_S(g)) \subseteq t_S(M_2)$, so $0 = t_S(g)(m_2) = g(m_2)$. Then $m_2 \in \ker(g) = \mathrm{im}(f)$, thus $m_2 = f(m_1)$ for some $m_1 \in M_1$. Since $m_2 \in t_S(M_2)$, there exists $s \in S$ such that $sm_2 = 0$, which implies $0 = sm_2 = s \cdot f(m_1) = f(sm_1)$. Then $sm_1 \in \ker(f)$, by injectivity of f we get $sm_1 = 0$ and thus $m_1 \in t_S(M_1)$. Now we can conclude $\ker(t_S(g)) \subseteq \mathrm{im}(t_S(f))$ from
$$m_2 = f(m_1) = t_S(f)(m_1) \in t_S(f)(t_S(M_1)) = \mathrm{im}(t_S(f)).$$

□

Corollary 5.14. *Let S be a left Ore set over a domain R and $0 \to L \to M \to N$ an exact sequence of left R-modules. If M is S-torsion-free, so is L.*

Proof: Since $t_S(\cdot)$ is left-exact, we get the exact sequence $0 \to t_S(L) \to t_S(M) \to t_S(N)$. Now $t_S(M) = \{0\}$ implies $t_S(L) = \{0\}$. □

Remark 5.15. The functor $t_S(\cdot)$ is not right-exact in general: consider $M = R = \mathbb{Z}$, $n \in \mathbb{N} \setminus \{1\}$, $N = \mathbb{Z}/n\mathbb{Z}$ and the canonical surjection $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. With $S = \{n^i \mid i \in \mathbb{N}_0\}$ we get $n \in S \cap n\mathbb{Z}$. But then $t_S(M) = t_S(\mathbb{Z}) = \{0\}$ and $t_S(N) = t_S(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$, thus $t_S(M) \to t_S(N)$ is not surjective.

Lemma 5.16. *Let S be a left Ore set in a domain R and M a left R-module. Then $t_S(M) \cong \operatorname{Tor}_1^R(S^{-1}R/R, M)$.*

Proof: The sequence

$$0 \to R \to S^{-1}R \to S^{-1}R/R \to 0$$

of right R-modules is exact. This induces the long exact sequence

$$\operatorname{Tor}_1^R(S^{-1}R, M) \to \operatorname{Tor}_1^R(S^{-1}R/R, M) \to M \to S^{-1}M \to (S^{-1}R/R) \otimes_R M \to 0,$$

by Corollary 6.30 in [Rot09]. By Proposition 5.5, $S^{-1}R$ is a flat right R-module, which implies $\operatorname{Tor}_1^R(S^{-1}R, M) = \{0\}$ by Theorem 7.2 in [Rot09]. Thus,

$$0 \to \operatorname{Tor}_1^R(S^{-1}R/R, M) \to M \to S^{-1}M$$

is exact, which implies $\operatorname{Tor}_1^R(S^{-1}R/R, M) \cong t_S(M)$. □

Lemma 5.17. *Let S be a left Ore set in a domain R, L a left R-module and $M, N \subseteq L$ two submodules. Then the following holds:*

(a) $t_S(M) \cap t_S(N) = t_S(M \cap N)$.

(b) $t_S(M) \oplus t_S(N) = t_S(M \oplus N)$.

(c) $t_S(M) + t_S(N) \subseteq t_S(M + N)$.

Proof: *(a)* Let $q \in t_S(M \cap N)$. Then $q \in M \cap N$ and there exists $s \in S$ such that $sq = 0$, which implies $q \in t_S(M) \cap t_S(N)$.
On the other hand, let $q \in t_S(M) \cap t_S(N)$. Then $q \in M \cap N$ and there exist $s_m, s_n \in S$ such that $s_m q = 0 = s_n q$. By the left Ore condition on S there exists a common left multiple $\hat{s} \in S$ of s_m and s_n, which implies $\hat{s}q = 0$ and thus $t_S(M \cap N)$.

(b) Let $m \oplus n \in t_S(M \oplus N)$, then there exists $s \in S$ such that $0 = s(m \oplus n) = sm \oplus sn$, which is equivalent to $sm = 0 = sn$. Now $m \in t_S(M)$ and $n \in t_S(N)$ implies $m \oplus n \in t_S(M) \oplus t_S(N)$.
On the other hand, let $m \oplus n \in t_S(M) \oplus t_S(N)$, then there exist $s_m, s_n \in S$ such that $s_m m = 0 = s_n n$. By the left Ore condition on S there exists a common left multiple $\hat{s} \in S$ of s_m and s_n. Now we have $\hat{s}(m \oplus n) = \hat{s}m \oplus \hat{s}n = 0$, which implies $m \oplus n \in t_S(M \oplus N)$.

(c) Let $m + n \in t_S(M) + t_S(N)$, then there exist $s_m, s_n \in S$ such that $s_m m = 0 = s_n n$. By the left Ore condition on S there exists a common left multiple $\hat{s} \in S$ of s_m and s_n. Now we have $\hat{s}(m + n) = \hat{s}m + \hat{s}n = 0$, which implies $m + n \in t_S(M \oplus N)$.

□

The next result shows some absorption properties of local torsion and Ore localization:

Proposition 5.18. *Let S_1 and S_2 be two left Ore sets in a domain R such that $S_1 \subseteq S_2$. Then the following holds:*

(a) $t_{S_2}(t_{S_1}(M)) = t_{S_1}(M) = t_{S_1}(t_{S_2}(M))$.

(b) $S_2^{-1}(S_1^{-1}M) \cong S_2^{-1}M \cong S_1^{-1}(S_2^{-1}M)$.

Proof: *(a)* By Corollary 5.9 we have $t_{S_1}(M) \subseteq t_{S_2}(M) \subseteq M$ and thus

$$t_{S_1}(M) = t_{S_1}(t_{S_1}(M)) \subseteq t_{S_2}(t_{S_1}(M)) \subseteq t_{S_1}(M) = t_{S_1}(t_{S_1}(M)) \subseteq t_{S_1}(t_{S_2}(M)) \subseteq t_{S_1}(M).$$

(b) From the associativity of the tensor product as well as Lemma 3.9 we get

$$\begin{aligned} S_2^{-1}(S_1^{-1}M) &= S_2^{-1}R \otimes_R (S_1^{-1}R \otimes_R M) = (S_2^{-1}R \otimes_R S_1^{-1}R) \otimes_R M \\ &\cong S_2^{-1}R \otimes_R M = S_2^{-1}M, \end{aligned}$$

the second statement follows analogously with Lemma 3.10.

□

Lemma 5.19. *Let S be a quasi-multiplicatively closed subset of a domain R and I, J be left ideals of R such that $I \subseteq J$ and I is left S-closed. Then $t_S(J/I) = \{0\}$.*

Proof: Let $m \in J$ and $m + I \in t_S(J/I)$. Then there exists $s \in S$ such that $s(m + I) \in I$, which implies $sm \in I$. Since I is left S-closed, we have $m \in I$ and thus $m + I = 0$ in J/I. □

Lemma 5.20. *Let S be a left Ore set of a domain R and $\varphi : M \to N$ a homomorphism of left R-modules. If $t_S(N) = \{0\}$, then $t_S(M) \cong t_S(\ker(\varphi))$.*

Proof: Applying the left-exact functor $t_S(\cdot)$ to the exact sequence $0 \to \ker(\varphi) \hookrightarrow M \xrightarrow{\varphi} N$ we get the exact sequence $0 \to t_S(\ker(\varphi)) \to t_S(M) \to t_S(N) = \{0\}$. Thus, $t_S(M) \cong t_S(\ker(\varphi))$. □

5.3. Annihilators in Ore localizations

Definition 5.21. Let R be a ring, M a left R-module and $m \in M$.

- The *left annihilator* of m is $\mathrm{Ann}_R^M(m) := \{r \in R \mid rm = 0\}$.
- The *annihilator* of M is $\mathrm{Ann}_R(M) := \{r \in R \mid \forall m \in M : rm = 0\}$.
- Let M be finitely presented via $M = R^n/P$ for some left submodule $P \subseteq R^n$, then the *(left) pre-annihilator* of M is

$$\mathrm{preAnn}_R(M) := \bigcap_{j=1}^{n} \mathrm{Ann}_R^m(e_j),$$

where $e_1, \ldots, e_n$ denotes the image of the canonical standard basis of R^n in M.

Remark 5.22. For every $m \in M$, $\mathrm{Ann}_R^M(m)$ is a left ideal of R. Furthermore, $\mathrm{Ann}_R(M)$ and $\mathrm{preAnn}_R(M)$ are left ideals as intersections of left ideals, since $\mathrm{Ann}_R(M) = \bigcap_{m \in M} \mathrm{Ann}_R^M(m)$. But $\mathrm{Ann}_R(M)$ is even a two-sided ideal of R: let $r \in R$, $a \in \mathrm{Ann}_R(M)$ and $m \in M$, then $(ar)m = a(rm) = 0$, since $rm \in M$.

Remark 5.23. While $\mathrm{Ann}_R(M)$ is an invariant of M, $\mathrm{preAnn}_R(M)$ does depend on the presentation of M. In the following, when talking about pre-annihilators we implicitly assume that we have fixed a representation of M and consider $\mathrm{preAnn}_R(M)$ with respect to this fixed representation.

Remark 5.24. If M is finitely presented, we always have

$$\operatorname{Ann}_R(M) = \bigcap_{m\in M} \operatorname{Ann}_R^M(m) \subseteq \bigcap_{j=1}^{n} \operatorname{Ann}_R^M(e_j) = \operatorname{preAnn}_R(M).$$

If R is commutative, then $\operatorname{Ann}_R(M) = \operatorname{preAnn}_R(M)$: let $a \in \operatorname{preAnn}_R(M)$ and $m = \sum_{j=1}^{n} c_j e_j$, then

$$am = a\sum_{j=1}^{n} c_j e_j = \sum_{j=1}^{n} a c_j e_j = \sum_{j=1}^{n} c_j a e_j = 0,$$

thus $a \in \operatorname{Ann}_R(M)$.

In a left Ore domain, any finite intersection of non-zero ideals is always non-zero:

Lemma 5.25. *Let R be a left Ore domain and $I_j \neq \{0\}$ a left ideal of R for $j \in \{1,\dots,n\}$. Then $\bigcap_{j=1}^{n} I_j \neq \{0\}$.*

Proof: Let $n \geq 2$ and $f_j \in I_j \setminus \{0\}$ for $j \in \{1, 2\}$. By assumption, $S := R \setminus \{0\}$ is a left Ore set in R. Since $f_1, f_2 \in S$ we have $Rf_1 \cap Sf_2 \neq \emptyset$, which implies $Rf_1 \cap Rf_2 \neq \{0\}$ and thus $\{0\} \subsetneq Rf_1 \cap Rf_2 \subseteq I_1 \cap I_2$. The claim now follows by induction on n. □

A finitely generated module over a commutative domain is a torsion module if and only if its annihilator is non-zero. Note that by Remark 5.24 we have $\operatorname{Ann}_R(M) = \operatorname{preAnn}_R(M)$ in the commutative case, thus we regain the classical result as a special case of the next lemma:

Lemma 5.26. *Let R be a left Ore domain and $M = R^n/P$ a finitely presented left R-module for some left submodule $P \subseteq R^n$. Then M is a torsion module if and only if* $\operatorname{preAnn}_R(M) \neq \{0\}$.

Proof: By Lemma 5.25 we have

$$\begin{aligned} & \{0\} = \operatorname{preAnn}_R(M) = \bigcap_{j=1}^{n} \operatorname{Ann}_R^M(e_j) \\ \Leftrightarrow\quad & \operatorname{Ann}_R^M(e_k) = \{0\} \text{ for some } k \in \{1,\dots,n\} \\ \Leftrightarrow\quad & e_k \in M \setminus t(M) \text{ for some } k \in \{1,\dots,n\} \\ \Leftrightarrow\quad & M \text{ is not a torsion module,} \end{aligned}$$

where the last equivalence is due to the left Ore condition on $R \setminus \{0\}$. □

Taking annihilators of module elements is compatible with Ore localization in an intuitive way: we just localize every parameter.

Proposition 5.27. *Let S be a left Ore set in a domain R, M a left R-module and $m \in M$. Then*

$$S^{-1}R \operatorname{Ann}_R^M(m) = \operatorname{Ann}_{S^{-1}R}^{S^{-1}M}(1^{-1}m)$$

and thus

$$\operatorname{Ann}_R^M(m) \subseteq (\operatorname{Ann}_R^M(m))^S = \rho_{S,R}^{-1}(S^{-1}R \operatorname{Ann}_R^M(m)) = \rho_{S,R}^{-1}(\operatorname{Ann}_{S^{-1}R}^{S^{-1}M}(1^{-1}m)).$$

Proof: First, let $x \in S^{-1}R\,\mathrm{Ann}_R^M(m)$, then there exist $s \in S$, $r \in R$ and $q \in \mathrm{Ann}_R^M(m)$ such that $x = (s,r)\cdot q = (s,rq)$. Thus

$$x \cdot (1^{-1}m) = (s,rq)\cdot(1 \otimes m) = (s,rq)\otimes m = (s,r)\otimes qm = (s,r)\otimes 0 = 0$$

implies $x \in \mathrm{Ann}_{S^{-1}R}^{S^{-1}M}(1^{-1}m)$.
On the other hand, let $y = (s,r) \in \mathrm{Ann}_{S^{-1}R}^{S^{-1}M}(1^{-1}m)$, then

$$0 = y\cdot(1^{-1}m) = (s,r)\cdot(1\otimes m) = (s,r)\otimes m = (s,1)\otimes rm.$$

Multiplying with s we get $0 = 1 \otimes rm$, which implies $\tilde{s}rm = 0$ for some $\tilde{s} \in S$. Now $\tilde{s}r \in \mathrm{Ann}_R^M(m)$ and thus

$$y = (s,q) = (\tilde{s}s, \tilde{s}q) = (\tilde{s}s, 1)\cdot \tilde{s}q \in S^{-1}R\,\mathrm{Ann}_R^M(m).$$

$\square$

A close examination of the second part of the proof of Proposition 5.27 gives us the following result:

Corollary 5.28. *Let S be a left Ore set in a domain R, M a left R-module and $m \in M$. If $1^{-1}m$ is a torsion element in $S^{-1}M$, then m is a torsion element in M.*

In the commutative situation, the compatibility of localization with (direct) sums, Cartesian products and other operations is a well-known fact from commutative algebra. The following result shows the compatibility of intersection and Ore localization of ideals:

Lemma 5.29. *Let S be a left Ore set in a domain R and I_j a left ideal of R for $j \in \{1,\dots,n\}$. Then*

$$S^{-1}R\bigcap_{j=1}^{n} I_j = \bigcap_{j=1}^{n} S^{-1}RI_j.$$

Proof: Without loss of generality let $I_j \neq \{0\}$ for all j, else both sides of the equation are $\{0\}$.
First, let $x \in S^{-1}R\bigcap_{j=1}^{n} I_j$, then $x \in S^{-1}RI_j$ for all j, which implies $x \in \bigcap_{j=1}^{n} S^{-1}RI_j$.
On the other hand, let $x \in (\bigcap_{j=1}^{n} S^{-1}RI_j)\setminus\{0\}$, then $x = (s_j, f_j)$ for some $s_j \in S$ and $f_j \in I_j$. By the left Ore condition on S there exists a common left multiple of the s_j, that is, there exist $s \in S$ and $a_j \in R$ such that $s = a_j s_j$ for all j. Now

$$x = (s_j, f_j) = (a_j s_j, a_j f_j) = (s, a_j f_j)$$

for all j, which implies $y := a_1 f_1 = a_j f_j$ for all j, in particular, we have $y \in (\bigcap_{j=1}^{n} I_j)\setminus\{0\}$. Thus

$$x = (s, a_1 f_1) = (s,y) \in S^{-1}R\bigcap_{j=1}^{n} I_j.$$

$\square$

Now we can expand Proposition 5.27 to show that Ore localization is also compatible with taking pre-annihilators in the same fashion:

Proposition 5.30. *Let S be a left Ore set in a domain R and $M = R^n/P$ a finitely presented left R-module for some left submodule $P \subseteq R^n$. Then*

$$S^{-1}R\,\mathrm{preAnn}_R(M) = \mathrm{preAnn}_{S^{-1}R}(S^{-1}M).$$

Proof: We have

$$\begin{aligned} S^{-1}R\,\mathrm{preAnn}_R(M) &\overset{5.21}{=} S^{-1}R\bigcap_{j=1}^{n}\mathrm{Ann}_R^M(e_j) \\ &\overset{5.29}{=} \bigcap_{j=1}^{n} S^{-1}R\,\mathrm{Ann}_R^M(e_j) \\ &\overset{5.27}{=} \bigcap_{j=1}^{n} \mathrm{Ann}_{S^{-1}R}^{S^{-1}M}(1^{-1}e_j) \\ &\overset{5.6}{=} \mathrm{preAnn}_{S^{-1}R}(S^{-1}M). \end{aligned}$$

□

5.4. Application: Algebraic systems theory

Definition 5.31. Let $\mathcal{D}$ be a ring, $\mathcal{A}$ a left $\mathcal{D}$-module and $R \in \mathcal{D}^{g\times q}$. We define

$$\mathrm{Sol}_{\mathcal{D}}(R,\mathcal{A}) := \{w \in \mathcal{A}^q \mid Rw = 0\}.$$

We recall the following essential result from algebraic systems theory, which can be found in [Sei10]:

Theorem 5.32 (Noether-Malgrange isomorphism)**.** *Let $\mathcal{D}$ be a ring, $\mathcal{A}$ a left $\mathcal{D}$-module, $R \in \mathcal{D}^{g\times q}$, $M := \mathcal{D}^{1\times g}R$ and $\mathcal{M} := \mathcal{D}^{1\times q}/M$. As groups, we have*

$$\mathrm{Sol}_{\mathcal{D}}(R,\mathcal{A}) \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{M},\mathcal{A}).$$

Proposition 5.33. *Let S be a left Ore set in a domain $\mathcal{D}$, M a left $\mathcal{D}$-module and $\mathcal{A}$ a left $S^{-1}\mathcal{D}$-module. Then*

$$\mathrm{Hom}_{\mathcal{D}}(M,\mathcal{A}) \cong \mathrm{Hom}_{S^{-1}\mathcal{D}}(S^{-1}M,\mathcal{A}).$$

Proof: We have

$$\begin{aligned} \mathrm{Hom}_{S^{-1}\mathcal{D}}(S^{-1}M,\mathcal{A}) &= \mathrm{Hom}_{S^{-1}\mathcal{D}}(S^{-1}\mathcal{D}\otimes_{\mathcal{D}} M,\mathcal{A}) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(M,\mathrm{Hom}_{S^{-1}\mathcal{D}}(S^{-1}\mathcal{D},\mathcal{A})) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(M,\mathcal{A}), \end{aligned}$$

where the first isomorphism is the tensor-hom adjunction (see e.g. [Rot09] for details). □

This shows that if we are looking for solutions of a $\mathcal{D}$-module in a solution space $\mathcal{A}$, which is not only a $\mathcal{D}$- but also a $S^{-1}\mathcal{D}$-module, then these solutions come from $M/t_S(M)$, i.e. the S-torsion-free part of $\mathcal{M}$ (since $S^{-1}M \cong S^{-1}(M/t_S(M)$ as before).

Lemma 5.34. *Let $\mathcal{D}$ be a domain and $\mathcal{A}$ a left $\mathcal{D}$-module.*

(a) *Let $\varphi : M_1 \to M_2$ be a homomorphism of left $\mathcal{D}$-modules. Then*

$$\operatorname{Hom}_{\mathcal{D}}(M_2/\varphi(M_1), \mathcal{A}) \to \operatorname{Hom}_{\mathcal{D}}(M_2, \mathcal{A})$$

is injective. If φ is a surjection, then $\operatorname{Hom}_{\mathcal{D}}(M_2, \mathcal{A}) \to \operatorname{Hom}_{\mathcal{D}}(M_1, \mathcal{A})$ is also injective.

(b) *Let I and J be proper ideals in $\mathcal{D}$ and $I \subsetneq J$. Then $\operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/J, \mathcal{A}) \to \operatorname{Hom}_{\mathcal{D}}(\mathcal{D}/I, \mathcal{A})$ is injective.*

Proof: *(a)* Follows by applying the left-exact functor $\operatorname{Hom}_{\mathcal{D}}(\cdot, \mathcal{A})$ to the exact sequences

$$M_1 \xrightarrow{\varphi} M_2 \longrightarrow M_2/\varphi(M_1) \to 0 \quad \text{and} \quad \ker(\varphi) \longrightarrow M_1 \xrightarrow{\varphi} M_2 \to 0.$$

(b) The canonical mapping $\varphi : \mathcal{D}/I \to \mathcal{D}/J, p + I \mapsto p + J$ is a well-defined surjection. The statement now follows from applying $\operatorname{Hom}_{\mathcal{D}}(\cdot, \mathcal{A})$ to the exact sequence

$$\ker(\varphi) \longrightarrow \mathcal{D}/I \xrightarrow{\varphi} \mathcal{D}/J \to 0.$$

□

Corollary 5.35. *Let S be a left Ore set in a domain $\mathcal{D}$ and M a left $\mathcal{D}$-module. Then*

$$\operatorname{Hom}_{\mathcal{D}}(M/t_S(M), \mathcal{A}) \to \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{A})$$

is injective.

Proof: Follows from Lemma 5.34 with the canonical surjection $M \to M/t_S(M)$. □

Main example, part 4

Remark 5.36. Consider the matrix $A := \begin{bmatrix} x & 0 & 0 \\ 0 & \partial & 0 \end{bmatrix} \in \mathcal{D}^{2\times 3}$ and let $S = K[x] \setminus \{0\}$. The system module associated to A is

$$\mathcal{M} = \mathcal{D}^{1\times 3}/\mathcal{D}^{1\times 2}A \cong (\mathcal{D}/\mathcal{D}x)e_1 \oplus (\mathcal{D}/\mathcal{D}\partial)e_2 \oplus \mathcal{D}e_3,$$

where e_1, e_2, e_3 is the canonical standard basis of $\mathcal{D}^{1\times 3}$. We have

$$\begin{aligned} t_S(\mathcal{M}) &= \{[m] \in \mathcal{M} \mid \exists s \in S : s[m] = 0\} = \left\{[m] \in \mathcal{M} \mid \exists s \in S : sm \in \mathcal{D}^{1\times 2}A\right\} \\ &= \{[(m_1, m_2, m_3)] \in \mathcal{M} \mid \exists s \in S, a_1, a_2 \in \mathcal{D} : sm_1 = a_1x \wedge sm_2 = a_2\partial \wedge sm_3 = 0\}. \end{aligned}$$

Since $\mathcal{D}$ is a domain and $s \neq 0$, we have $sm_3 = 0$ if and only if $m_3 = 0$. Furthermore, $sm_2 = a_2\partial$ with $s \in K[x] \setminus \{0\}$ can only hold if $m_2 \in \mathcal{D}\partial$. Lastly, given x and $m_1 \in \mathcal{D}$, the left Ore condition on S provides us with $s \in S$ and $a_1 \in \mathcal{D}$ such that $sm_1 = a_1x$ holds. With this information, we get

$$t_S(\mathcal{M}) = \{[(m_1, m_2\partial, 0)] \in \mathcal{M} \mid m_1, m_2 \in \mathcal{D}\} \cong \mathcal{D}^{1\times 3}e_1/\mathcal{D}^{1\times 2}A \cong (\mathcal{D}/\mathcal{D}x)e_1$$

and thus $\mathcal{M}/t_S(\mathcal{M}) \cong (\mathcal{D}/\mathcal{D}\partial)e_2 \oplus \mathcal{D}e_3$. Rewriting the exact sequence

$$0 \to t_S(M) \to \mathcal{M} \to \mathcal{M}/t_S(\mathcal{M}) \to 0$$

with concrete data, we obtain

$$0 \to (\mathcal{D}/\mathcal{D}x)e_1 \to (\mathcal{D}/\mathcal{D}x)e_1 \oplus (\mathcal{D}/\mathcal{D}\partial)e_2 \oplus \mathcal{D}e_3 \to (\mathcal{D}/\mathcal{D}\partial)e_2 \oplus \mathcal{D}e_3 \to 0.$$

Analogously, we get $t(\mathcal{M}) \cong (\mathcal{D}/\mathcal{D}x)e_1 \oplus (\mathcal{D}/\mathcal{D}\partial)e_2$ and $\mathcal{M}/t(\mathcal{M}) \cong \mathcal{D}e_3$, which gives us

$$0 \to (\mathcal{D}/\mathcal{D}x)e_1 \oplus (\mathcal{D}/\mathcal{D}\partial)e_2 \to (\mathcal{D}/\mathcal{D}x)e_1 \oplus (\mathcal{D}/\mathcal{D}\partial)e_2 \oplus \mathcal{D}e_3 \to \mathcal{D}e_3 \to 0.$$

As we can see, this gives us a finer description of the torsion submodule of $\mathcal{M}$ via S-torsion.

6. Algorithms

Convention 6.1. In this chapter, let $n, m \in \mathbb{N}$.

6.1. Orderings and monoideals in $\mathbb{N}_0^n$

Definition 6.2. The (partial) ordering $\leq_{cw}$ on $\mathbb{N}_0^n$, defined by

$$\alpha \leq_{cw} \beta \quad :\Leftrightarrow \quad \alpha_i \leq \beta_i \text{ for all } i \in \{1, \ldots, n\}$$

for $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$, is called the *component-wise ordering*. Equivalently, we have $\alpha \leq_{cw} \beta$ if and only if $\beta \in \alpha + \mathbb{N}_0^n$.

Definition 6.3. A total order $\leq$ on $\mathbb{N}_0^n$ with least element 0 is called *admissible*, if $\alpha \leq \beta$ implies $\alpha + \gamma \leq \beta + \gamma$ for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$.

Lemma 6.4. *Any admissible order $\leq$ on $\mathbb{N}_0^n$ is a refinement of the component-wise ordering, that is, $\alpha \leq_{cw} \beta$ implies $\alpha \leq \beta$ for all $\alpha, \beta \in \mathbb{N}_0^n$.*

Proof: Let $\alpha \leq_{cw} \beta$, then $\beta - \alpha \in \mathbb{N}_0^n$. We have $\alpha = \alpha + 0 \leq \alpha + \beta - \alpha = \beta$, since $\leq$ is admissible. $\square$

Definition 6.5. Let $\leq$ be an admissible ordering on $\mathbb{N}_0^{n+m} \cong \mathbb{N}_0^n \times \mathbb{N}_0^m$. We call $\leq$ an *elimination ordering* for the last m components, if for all $\alpha, \beta \in \mathbb{N}_0^{n+m}$, $\beta \in \mathbb{N}_0^n \times \{0\}$ and $\alpha \leq \beta$ imply $\alpha \in \mathbb{N}_0^n \times \{0\}$.

Definition 6.6. Let $\leq_n$ resp. $\preceq_m$ be an admissible order on $\mathbb{N}_0^n$ resp. $\mathbb{N}_0^m$. The ordering $\leq\ = (\leq_n, \preceq_m)$ on $\mathbb{N}_0^{n+m} \cong \mathbb{N}_0^n \times \mathbb{N}_0^m$, defined by

$$(\alpha, \beta) \leq (\gamma, \delta) \quad :\Leftrightarrow \quad \beta \prec_m \delta \vee (\beta = \delta \wedge \alpha \leq_n \gamma)$$

for $\alpha, \gamma \in \mathbb{N}_0^n$ and $\beta, \delta \in \mathbb{N}_0^m$, is called (n, m)*-antiblock ordering*.

Lemma 6.7. *Let $\leq\ = (\leq_n, \preceq_m)$ be a (n, m)-antiblock ordering. Then $\leq$ is an elimination ordering for the last m components.*

Proof: Let $\beta = (\beta_1, 0) \in \mathbb{N}_0^n \times \{0\}$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^{n+m}$ such that $\alpha \leq \beta$. Then $\alpha_2 \preceq_m 0$, which implies $\alpha_2 = 0$ and thus $\alpha \in \mathbb{N}_0^n \times \{0\}$. $\square$

Definition 6.8 ([BGTV03])**.** A non-empty subset $E \subseteq \mathbb{N}_0^n$ is called a $\mathbb{N}_0^n$*-monoideal* if $E + \mathbb{N}_0^n = E$. The $\mathbb{N}_0^n$-monoideal *generated* by E is $E + \mathbb{N}_0^n$.

6.2. Gröbner bases in G-algebras

Let us recall the basics of the theory of Gröbner bases in G-algebras. For the proofs omitted here as well as a more exhaustive treatment of the subject we refer to [Lev05].

Convention 6.9. In this section, let A be a G-algebra generated by $\underline{x} = \{x_1, \ldots, x_n\}$ over a field K and $\leq$ and admissible ordering on $\mathbb{N}_0^n$ satisfying the order condition for G-algebras from Definition 1.34.

Definition 6.10. Let $f \in A \setminus \{0\}$, then $f = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \underline{x}^\alpha$ for some $c_\alpha \in K$, where $c_\alpha = 0$ for almost all $\alpha \in \mathbb{N}_0^m$, but $c_\alpha \neq 0$ for at least one $\alpha \in \mathbb{N}_0^n$. Now we define

- $\mathcal{N}_\leq(f) := \{\alpha \in \mathbb{N}_0^n \mid c_\alpha \neq 0\} \subseteq \mathbb{N}_0^n$, the *Newton diagram* of f,
- $\mathrm{le}_\leq(f) := \max_\leq(\mathcal{N}(f)) \in \mathbb{N}_0^n$, the *leading exponent* of f,
- $\mathrm{lc}_\leq(f) := c_{\mathrm{le}_\leq(f)} \in K$, the *leading coefficient* of f,
- $\mathrm{lm}_\leq(f) := \underline{x}^{\mathrm{le}_\leq(f)} \in \mathrm{Mon}(A)$, the *leading monomial* of f.

Let further $S \subseteq A \setminus \{0\}$ and define

- $\mathcal{L}_\leq(S) := \mathrm{Exp}_\leq(S) := \{\alpha \in \mathbb{N}_0^n \mid \exists s \in S : \mathrm{le}_\leq(s) = \alpha\} + \mathbb{N}_0^n \subseteq \mathbb{N}_0^n$, the *monoideal of leading exponents* of S,
- $L_\leq(S) := {}_K\langle \underline{x}^\alpha \mid \alpha \in \mathrm{Exp}_\leq(S)\rangle \subseteq A$, the *span of leading monomials* of S.

If the ordering is clear from the context, we sometimes omit the index $\leq$.

Remark 6.11. In general, the product of two monomials of A is a polynomial. Nevertheless, for all $\alpha, \beta \in \mathbb{N}_0^n$ we have $\mathrm{lm}(\underline{x}^\alpha \cdot \underline{x}^\beta) = \underline{x}^{\alpha+\beta}$ due to the restrictions imposed on the relations between the variables. Thus, for all $f, g \in A \setminus \{0\}$ we get the property $\mathrm{le}(f \cdot g) = \mathrm{le}(f) + \mathrm{le}(g)$. Furthermore, if A is of Lie type we have $\mathrm{lc}(f \cdot g) = \mathrm{lc}(f) \cdot \mathrm{lc}(g)$.

Definition 6.12. Given $\underline{x}^\alpha, \underline{x}^\beta \in \mathrm{Mon}(A)$, we say that $\underline{x}^\alpha$ *divides* $\underline{x}^\beta$ (written $\underline{x}^\alpha \mid \underline{x}^\beta$) if $\alpha \leq_{cw} \beta$.

Lemma 6.13. *Let A be a G-algebra generated by two blocks of variables $\underline{x} = \{x_1, \ldots, x_n\}$ and $\underline{y} = \{y_1, \ldots, y_m\}$ and $\leq$ be an admissible ordering on $\mathbb{N}_0^{n+m}$. Then the following are equivalent:*

(1) The ordering $\leq$ is an elimination ordering for the last m components.

(2) For any $f \in A \setminus \{0\}$, $\mathrm{le}(f) \in \mathbb{N}_0^n \times \{0\} \subseteq \mathbb{N}_0^{n+m}$ implies that no monomial of f contains any variable from $\underline{y}$.

***Proof:* *(1)⇒(2)*:** Let $f \in A \setminus \{0\}$ such that $\beta := \mathrm{le}(f) \in \mathbb{N}_0^n \times \{0\}$. Take a term $t := c_\alpha \underline{x}^{\alpha_1} \underline{y}^{\alpha_2}$ appearing in f with $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^{n+m}$, then $\alpha \leq \beta$. Now *(1)* implies $\alpha \in \mathbb{N}_0^n \times \{0\}$, therefore no variable from $\underline{y}$ occurs in t, and by iteration in f.

***(2)⇒(1)*:** Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^{n+m}$ and $\beta = (\beta_1, 0) \in \mathbb{N}_0^n \times \{0\}$ such that $\alpha \leq \beta$. Then $f := \underline{x}^{\beta_1} \underline{y}^0 + \underline{x}^{\alpha_1} \underline{y}^{\alpha_2} \in A \setminus \{0\}$ and $\mathrm{le}(f) = \beta \in \mathbb{N}_0^n \times \{0\}$. By *(2)* f does not contain any variable from $\underline{y}$, thus $\alpha_2 = 0$, which implies $\alpha \in \mathbb{N}_0^n \times \{0\}$.

□

Definition 6.14. Let $L \subseteq A$ be a left ideal and $G \subseteq L \setminus \{0\}$ a finite subset. We call G a *left Gröbner basis* of L with respect to $\leq$ if for all $f \in L \setminus \{0\}$ there exists a $g \in G$ such that $\mathrm{lm}(g) \mid \mathrm{lm}(f)$.

Theorem 6.15. *Let $L \subseteq A$ be a left ideal and $G \subseteq L \setminus \{0\}$ a finite subset. Then the following are equivalent:*

(1) G is a left Gröbner basis of L with respect to $\leq$.

(2) $L_{\leq}(G) = L_{\leq}(I)$ *as vector spaces.*

(3) $\mathrm{Exp}_{\leq}(G) = \mathrm{Exp}_{\leq}(L)$ *as* $\mathbb{N}_0^n$*-monoideals.*

Definition 6.16. Denote by $\mathcal{G}$ the set of all finite ordered subsets of A.

(1) A map $\mathrm{NF} : A \times \mathcal{G} \to A,\ (f, G) \mapsto \mathrm{NF}(f|G)$, is called a *left normal form* on A if for all $f \in A$ and $G \in \mathcal{G}$

  (i) $\mathrm{NF}(0|G) = 0$,

  (ii) $\mathrm{NF}(f|G) \neq 0$ implies $\mathrm{lm}(\mathrm{NF}(f|G)) \notin L(G)$,

  (iii) $f - \mathrm{NF}(f|G) \in {}_A\langle G\rangle$.

(2) Let $G = \{g_1, \dots, g_s\} \in \mathcal{G}$. A representation of $f \in {}_A\langle G\rangle$, $f = \sum_{i=1}^s a_i g_i$ where $a_i \in A$, satisfying $\mathrm{le}_{\leq}(a_i g_i) \leq \mathrm{le}_{\leq}(f)$ if $a_i g_i \neq 0$ for all $i \in \{1, \dots, s\}$, is called a *standard left representation* of f with respect to G.

Lemma 6.17. *Let $I \subseteq A$ be a left ideal, $G \subseteq I$ a left Gröbner basis of I with respect to $\leq$ and* $\mathrm{NF}(\cdot|G)$ *a left normal form on A with respect to G.*

(a) For any $f \in A$ we have $f \in I$ if and only if $\mathrm{NF}(f|G) = 0$.

(b) Let $J \subseteq A$ be a left ideal. Then $L(I) = L(J)$ implies $I = J$. In particular, $I = {}_A\langle G\rangle$.

Algorithm 6.18 (LeftNF)**.**

Input: $f \in A$, $G \in \mathcal{G}$.
Output: $h \in A$, a left normal form of f with respect to G and $\leq$.

begin
  $h := f$;
  while $h \neq 0$ **and** $G_h := \{g \in G : \mathrm{lm}(g) \mid \mathrm{lm}(h)\} \neq \emptyset$ **do**
    choose any $g \in G_h$;
    $\alpha := \mathrm{le}(h)$;
    $\beta := \mathrm{le}(g)$;
    $h := \mathrm{LeftSpoly}(h, g) := h - \frac{\mathrm{lc}(h)}{\mathrm{lc}(x^{\alpha-\beta} g)} x^{\alpha-\beta} g$;
  end
  return h;
end

Theorem 6.19. *Let $I \subseteq A$ be a left ideal, $G = \{g_1, \dots, g_s\} \subseteq I$. Let* $\mathrm{LeftNF}(\cdot|G)$ *be a left normal form on A with respect to G. Equivalent are:*

(1) G is a left Gröbner basis of I.

(2) $\mathrm{LeftNF}(f|G) = 0$ *for all $f \in I$.*

(3) Each $f \in I$ has a standard left representation with respect to G.

Definition 6.20. Let $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$. Define $\mu(\alpha, \beta) \in \mathbb{N}_0^n$ via $\mu(\alpha, \beta)_i := \max\{\alpha_i, \beta_i\}$.

Algorithm 6.21 (LeftGröbnerBasis).

Input: $F \in \mathcal{G}$.

Output: A left Gröbner basis $G \in \mathcal{G}$ of $I := {}_A\langle F \rangle$ with respect to $\leq$.

begin
  $G := F$;
  $P := \{(f, g) \in G \times G \mid f \neq g\}$;
  while $P \neq \emptyset$ **do**
    choose any $(f, g) \in P$;
    $P := P \setminus \{(f, g)\}$;
    $\alpha := \mathrm{le}(f)$;
    $\beta := \mathrm{le}(g)$;
    $\gamma := \mu(\alpha, \beta)$;
    $t := \underline{x}^{\gamma-\alpha} f - \frac{\mathrm{lc}(\underline{x}^{\gamma-\alpha} f)}{\mathrm{lc}(\underline{x}^{\gamma-\beta} g)} \underline{x}^{\gamma-\beta} g$;
    $h := \mathrm{LeftNF}(t|G)$;
    if $h \neq 0$ **then**
      $P := P \cup \{(h, f) \mid f \in G\}$;
      $G := G \cup \{h\}$;
    end
  end
  return G;
end

6.3. Gröbner bases in rational OLGAs

Convention 6.22. In this section, let A be a G-algebra generated by two blocks of variables $\underline{x} = \{x_1, \ldots, x_n\}$ and $\underline{y} = \{y_1, \ldots, y_m\}$ such that $\underline{x}$ generates a sub-G-algebra B of A. Assume further that $S := B \setminus \{0\}$ is a left Ore set in A and $L \subseteq A$ is left ideal.

Definition 6.23. The set of *monomials* in $S^{-1}A$ is $\mathrm{Mon}(S^{-1}A) := \left\{\underline{y}^\alpha \mid \alpha \in \mathbb{N}_0^m\right\}$. Let $\alpha, \beta \in \mathbb{N}_0^m$. Then $\underline{y}^\alpha$ *divides* $\underline{y}^\beta$ (written $\underline{y}^\alpha \mid \underline{y}^\beta$), if $\alpha \leq_{cw} \beta$.

Definition 6.24. Let $\leq$ be an admissible order on $\mathbb{N}_0^m$ and $f \in S^{-1}A \setminus \{0\}$, then $f = \sum_{\alpha \in \mathbb{N}_0^m} c_\alpha \underline{y}^\alpha$ for some $c_\alpha \in K(\underline{x})$, where $c_\alpha = 0$ for almost all $\alpha \in \mathbb{N}_0^m$. Now we define

- $\mathcal{N}_\leq(f) := \{\alpha \in \mathbb{N}_0^m \mid c_\alpha \neq 0\} \subseteq \mathbb{N}_0^m$, the *Newton diagram* of f,
- $\mathrm{le}_\leq(f) := \max_\leq(\mathcal{N}(f)) \in \mathbb{N}_0^m$, the *leading exponent* of f,
- $\mathrm{lc}_\leq(f) := c_{\mathrm{le}(f)} \in K(\underline{x})$, the *leading coefficient* of f,
- $\mathrm{lm}_\leq(f) := \underline{y}^{\mathrm{le}(f)} \in \mathrm{Mon}(S^{-1}A)$, the *leading monomial* of f.

Definition 6.25. Let $G \subseteq L \setminus \{0\}$ be a finite subset and $\leq$ an admissible order on $\mathbb{N}_0^m$. We call G a *left Gröbner basis* of L with respect to $\leq$ if for every $f \in L \setminus \{0\}$ there exists a $g \in G$ such that $\mathrm{lm}_\leq(g) \mid \mathrm{lm}_\leq(f)$.

The main result of this section is the fact that with respect to an antiblock ordering, Gröbner bases of ideal in A induce Gröbner bases of the extension of the ideal in the rational Ore localization of A:

Proposition 6.26 (cf. Lemma 1.5.9 in [Lev15])**.** *Let* $\leq\ = (\leq_n, \preceq_m)$ *be a* (n,m)*-antiblock ordering and* G *a left Gröbner basis of* L *with respect to* $\leq$*. Then* $\rho_{S,A}(G)$ *is a left Gröbner basis of* $J := S^{-1}L$ *with respect to* $\preceq_m$*.*

Proof: Let $f \in J \setminus \{0\}$, then $f = (s,l)$ for some $s \in S$ and $l \in L$. Then there exists $g \in G$ such that $\mathrm{lm}_{\leq}(g) \mid \mathrm{lm}_{\leq}(l)$. Let $\alpha := (\alpha_1, \alpha_2) := \mathrm{le}_{\leq}(g)$ and $\beta := (\beta_1, \beta_2) := \mathrm{le}_{\leq}(l)$. Then $\alpha \leq_{cw} \beta$, which implies $\alpha_2 \leq_{cw} \beta_2$. But then $\mathrm{le}_{\preceq_m}(\rho_{S,A}(g)) = \alpha_2 \leq_{cw} \beta_2 = \mathrm{le}_{\preceq_m}(f)$ and therefore $\mathrm{lm}_{\preceq_m}(\rho_{S,A}(g)) \mid \mathrm{lm}_{\preceq_m}(f)$. Thus, $\rho_{S,A}(G)$ is a left Gröbner basis of J with respect to $\preceq_m$. □

6.4. Central saturation

Definition 6.27. Let R be a left Noetherian ring, $I \subseteq R$ a left ideal and $q \in Z(R)$.

- The *quotient* of I by q is the left ideal $I : q := \{r \in R \mid qr \in I\} = \{r \in R \mid rq \in I\}$.
- Since R is left Noetherian, the left ideal chain $I \subseteq I : q \subseteq I : q^2 \subseteq \ldots$ becomes stationary. Thus, there exists a $k \in \mathbb{N}$ minimal with the property that $I : q^k = \bigcup_{i \in \mathbb{N}} (I : q^i)$. Then $I : q^\infty := I : q^k$ is called the *central saturation* of I by q and k is called the *(central) saturation index* of I by q, denoted by $\mathrm{Satindex}(I,q)$. Note that even in an arbitrary ring the saturation index of I by q may exist.

Computational Remark 6.28. In the situation of Definition 6.27, consider the left R-module homomorphism $\phi : R \to R/I,\ r \mapsto rq$. We have

$$\ker(\phi) = \{r \in R \mid \phi(r) = 0 \text{ in } R/I\} = \{r \in R \mid rq = \phi(r) \in I\} = I : q.$$

Thus, if we can compute kernels of left R-module homomorphisms, we can also compute central quotients.
Furthermore, if we can decide equality of left ideals in R, then we can also compute the central saturation index by iteratively computing $I : q^{k+1}$ and comparing it to $I : q^k$.
In G-algebras, which are left Noetherian by Theorem 1.37, both can be done using Gröbner-driven algorithms (cf. [Lev05]).

As an application of central saturation we give a generalization of a classical ideal decomposition that is well-known in the commutative setting (cf. Lemma 8.95 of [BW93]).

Lemma 6.29. *Let* R *be a ring,* $I \subseteq R$ *a left ideal and* $q \in Z(R)$*. Further, let* $k \in \mathbb{N}$ *be the saturation index of* I *by* q*. Then* $I = {}_R\langle I, q^k\rangle \cap (I : q^k)$*.*

Proof: Let $J := {}_R\langle I, q^k\rangle \cap (I : q^k)$. Since $I \subseteq {}_R\langle I, q^k\rangle$ and $I \subseteq (I : q^k)$, we clearly have $I \subseteq J$. Now let $a \in J$, then $q^k a \in I$ (since $a \in (I : q^k)$), and $a = b + rq^k$ for some $b \in I$ and $r \in R$ (since $a \in {}_R\langle I, q^k\rangle$). Then

$$q^{2k} r = q^k r q^k = q^k (b - a) = q^k b - q^k a \in I,$$

thus $r \in (I : q^{2k}) = (I : q^k)$, which implies $rq^k = q^k r \in I$. But then $a = b + rq^k \in I$. □

As a direct consequence we get another generalization of a well-known statement from commutative algebra:

Corollary 6.30. *Let* I *be a left ideal in a domain* R*,* $q \in Z(R)$*,* $k := \mathrm{Satindex}(I,q)$ *and* $S := [q] = \{q^n \mid n \in \mathbb{N}_0\}$*. Then* $I^S = I : q^k$*.*

6.5. S-closure algorithm

In this section we give a Gröbner-based algorithm to compute the S-closure of an ideal in a situation that is of interest in the theory of D-modules.

Convention 6.31. In this section, let A be a G-algebra generated by two blocks of variables $\underline{x} = \{x_1, \dots, x_n\}$ and $\underline{y} = \{y_1, \dots, y_m\}$ over a field K such that $\underline{x}$ generates a sub-G-algebra $B \subseteq Z(A)$ of A. Let $\leq = (\leq_n, \preceq_m)$ be a (n, m)-antiblock ordering and $S := B \setminus \{0\}$.

Remark 6.32.

- Since $B \subseteq Z(A)$, we can identify B with the commutative polynomial ring $K[\underline{x}]$.
- By Lemma 2.25, S is a left Ore set in B as well as in A. Furthermore, $S^{-1}B \cong K(\underline{x})$.
- We can view $S^{-1}A$ as a G-algebra over the field $K(\underline{x})$, since $S^{-1}A \cong K(\underline{x})\langle \underline{y} \mid Q\rangle$, where Q is the set of relations inherited from A.

Algorithm 6.33 (S-Closure)**.**
Input: A left ideal $I \subseteq A$.
Output: A Gröbner basis $G \subseteq A$ of I^S with respect to $\leq$.

begin
 $H := \text{LeftGröbnerBasis}(I, \leq)$;
 $h := \operatorname{lcm}(\{\operatorname{lc}_{\preceq_m}(\rho_{S,A}(g)) \mid g \in H\}) \in K[\underline{x}]$;
 $m := \operatorname{Satindex}(I, h)$;
 $G := \text{LeftGröbnerBasis}(I : h^m, \leq)$;
 return G;
end

Proposition 6.34. *Algorithm 6.33 terminates and is correct.*

Proof: Termination follows directly from the fact that both the computation of left Gröbner bases and the computation of saturation indices are algorithmic in G-algebras (see also Algorithm 6.21 and Computational Remark 6.28). To prove correctness, we have to show that $I^S = I : h^m = I : h^\infty$:

(i) Let $\rho := \rho_{S,A}$, then we have

$$\rho^{-1}(S^{-1}I) = \left\{r \in A \mid \rho(r) \in S^{-1}I\right\} = \{r \in A \mid \exists s \in S : sr \in I\} = I^S.$$

Furthermore, we have $h \in K[\underline{x}] \setminus \{0\}$ and $\operatorname{lc}_{\preceq_m}(\rho(g)) \mid h$ for all $g \in H$.

(ii) Let $f \in I : h^m$, then $h^m f \in I$ and $\rho(f) = (1, f) = (h^m, h^m f) \in S^{-1}I$. Thus we have $f \in \rho^{-1}(S^{-1}I) = I^S$, which implies $I : h^m \subseteq I^S$.

(iii) Let $f \in I^S$, then $\rho(f) \in S^{-1}I$. By Proposition 6.26, $\rho(H)$ is a left Gröbner basis of $S^{-1}I$. By Theorem 6.19 we have $\operatorname{LeftNF}(\rho(f)|\rho(H)) = 0$. We now prove $f \in I : h^m$ by an induction on the minimal number n of steps necessary in Algorithm 6.18 to reduce $\rho(f)$ to zero with respect to $\rho(H)$:

(IB) If $n = 0$, then $\rho(f) = 0$ and thus $f = 0$, which trivially implies $f \in I : h^m$.

(IH) Assume that for any $f \in I^S$, such that $\rho(f)$ can be reduced to zero with respect to $\rho(H)$ in $n-1$ steps, we have $f \in I : h^m$.

(IS) Let $f \in I^S$ such that Algorithm 6.18 needs at least n steps to reduce $\rho(f)$ to zero with respect to $\rho(H)$. Then there exists $g \in H$ such that $\mathrm{lm}_{\preceq_m}(\rho(g)) \mid \mathrm{lm}_{\preceq_m}(\rho(f))$ and

$$f_1 := \rho(f) - \frac{\mathrm{lc}_{\preceq_m}(\rho(f))}{\mathrm{lc}_{\preceq_m}(\underline{y}^{\alpha-\beta} g)} \underline{y}^{\alpha-\beta} \rho(g) \in S^{-1}A,$$

where $\alpha := \mathrm{le}_{\preceq_m}(\rho(f))$ and $\beta := \mathrm{le}_{\preceq_m}(\rho(g))$, can be reduced to zero in $n-1$ steps with respect to $\rho(H)$. Since the relations between the variables in $S^{-1}A$ have the form $y_j y_i = c_{i,j} y_i y_j + d_{i,j}$ for some $c_{i,j} \in K \setminus \{0\} = U(K)$ and a $d_{i,j} \in A$ which is of lower order than $y_i y_j$, we have

$$\mathrm{lc}_{\preceq_m}(\underline{y}^{\alpha-\beta} g) = u \cdot \mathrm{lc}_{\preceq_m}(\underline{y}^{\alpha-\beta}) \cdot \mathrm{lc}_{\preceq_m}(g) = u \cdot 1 \cdot \mathrm{lc}_{\preceq_m}(g) = u \cdot \mathrm{lc}_{\preceq_m}(g)$$

for some $u \in K \setminus \{0\}$, which is just the product of all $c_{i,j}$ that occur while bringing $\underline{y}^{\alpha-\beta} g$ in standard monomial form, and thus

$$f_1 = \rho(f) - \frac{\mathrm{lc}_{\preceq_m}(\rho(f))}{u\,\mathrm{lc}_{\preceq_m}(g)} \underline{y}^{\alpha-\beta} \rho(g). \tag{1}$$

Since $\mathrm{lc}_{\preceq_m}(g)$ divides h we have $c := \frac{h}{u\,\mathrm{lc}_{\preceq_m}(g)} \in K[\underline{x}] \setminus \{0\}$ and therefore

$$t := hf - c\,\mathrm{lc}_{\preceq_m}(\rho(f))\underline{y}^{\alpha-\beta} g \in I^S,$$

as $f, g \in I^S$. Multiplying both sides of equation (1) with $h \in S$ yields

$$hf_1 = h\rho(f) - \frac{h}{u\,\mathrm{lc}_{\preceq_m}(g)} \mathrm{lc}_{\preceq_m}(\rho(f))\underline{y}^{\alpha-\beta}\rho(g) = \rho(hf - c\,\mathrm{lc}_{\preceq_m}(\rho(f))\underline{y}^{\alpha-\beta} g) = \rho(t).$$

Since $\rho(I^S) = \rho(\rho^{-1}(S^{-1}I)) \subseteq S^{-1}I$, we get $hf_1 = \rho(t) \in S^{-1}I$ and thus $t \in I^S$. Now we can apply the induction hypothesis: we have $t \in I^S$ such that $\rho(t) = hf_1$ can be reduced to zero in $n-1$ steps with respect to $\rho(H)$, since $h \in S$ is invertible in $S^{-1}A$ and thus does not change the reducibility of f_1. This gives us $t \in I : h^m$ or $h^m t \in I$. Now

$$h^{m+1} f = h^m t + h^m c\,\mathrm{lc}_{\preceq_m}(\rho(f))\underline{y}^{\alpha-\beta} g \in I$$

implies $f \in I : h^{m+1} = I : h^m$, which shows $I^S \subseteq I : h^m$.

$\square$

Remark 6.35. The theory of Gröbner bases in G-algebras can be extended to submodules of the free module A^k via monomial module orderings like position over term. Similarly, central saturation of a submodule I of A^k by $q \in Z(A)$ can be defined via

$$I : q := \left\{ r \in A^k \mid qr \in I \right\}.$$

Therefore Algorithm 6.33 should be extendable to this setting as well; a question we will further investigate in future works.

6.6. Application: D-module theory

Let K be a field and $R := K[x_1, \ldots, x_n]$ a commutative polynomial ring.

Given a set of non-zero polynomials $f_1, \ldots, f_m \in R$, define $f := f_1 \cdot \ldots \cdot f_m$ and consider the free $R[s, \frac{1}{f}] = R[s_1, \ldots, s_m, \frac{1}{f_1 \cdot \ldots \cdot f_m}]$-module of rank one generated by the formal symbol $f^s := f_1^{s_1} \cdot \ldots \cdot f_m^{s_m}$, that is $M = R[s, \frac{1}{f}] \cdot f^s$. Let D be the n-th Weyl algebra containing R as a subring, then M naturally becomes a left $D[s]$-module via

$$g(s,x) \bullet f^s = g(s,x) \cdot f^s \quad \text{and} \quad \partial_i \bullet f^s = \left(\sum_{j=1}^{m} s_j \frac{\partial f_j}{\partial x_i} \frac{1}{f_j} \right) \cdot f^s \in M.$$

Let $\operatorname{Ann}_{D[s]}(f^s)$ be the left ideal of elements from $D[s]$ that annihilate f^s, then $M \cong D[s]/\operatorname{Ann}_{D[s]}(f^s)$ as $D[s]$-module. Since $D[s]$ is Noetherian, there exists a finite generating set for $\operatorname{Ann}_{D[s]}(f^s)$. Moreover, $\operatorname{Ann}_{D[s]}(f^s) \cap K[x,s] = \{0\}$ and for $f = f_1 \cdot \ldots \cdot f_m$ we define

$$B_f(s) = (\operatorname{Ann}_{D[s]}(f^s) + D[s]f) \cap K[s]$$

to be the *Bernstein-Sato ideal* of $(f_1, \ldots, f_m)$, which is known to be non-zero (e.g. [Lev15]).

From the action of $D[s]$ on M above, we conclude that $D[s]/\operatorname{Ann}_{D[s]}(f^s)$ has no $R[s]$-torsion.

Now, the *order* of an operator $P \in D[s] \setminus \{0\}$ is defined to be the total degree of P with respect to variables $\partial_1, \ldots, \partial_n$ (equivalently, one sets weighted degrees $\deg(x_i) = \deg(s_j) = 0$ and $\deg(\partial_i) = 1$). The set A^1 of operators of order 1 that annihilate f^s is non-empty, since from the action above we can see that

$$\left(f \cdot \partial_i - \left(\sum_{j=1}^{m} s_j \frac{\partial f_j}{\partial x_i} \prod_{k \neq j} f_k \right) \right) \bullet f^s = 0$$

in M.

Let us define the *logarithmic annihilator* of f^s, $\operatorname{Ann}^{(1)}_{D[s]}(f^s)$ to be the left ideal of $D[s]$, generated by A^1. Then $\operatorname{Ann}^{(1)}_{D[s]}(f^s) \subseteq \operatorname{Ann}_{D[s]}(f^s)$ and one of the questions is how to detect the equality without determining $\operatorname{Ann}_{D[s]}(f^s)$.

Though $\operatorname{Ann}_{D[s]}(f^s)$ is $K[x,s] \setminus \{0\}$-saturated ([Lev15]), $\operatorname{Ann}^{(1)}_{D[s]}(f^s)$ is, in general, neither $K[s] \setminus \{0\}$- nor $K[x] \setminus \{0\}$-saturated. On the other hand, the $K[x,s] \setminus \{0\}$-closure of $\operatorname{Ann}^{(1)}_{D[s]}(f^s)$ is precisely $\operatorname{Ann}_{D[s]}(f^s)$ ([Lev15]).

By applying Algorithm 6.33 we can compute the $K[s] \setminus \{0\}$-closure of $\operatorname{Ann}^{(1)}_{D[s]}(f^s)$. If it strictly contains $\operatorname{Ann}^{(1)}_{D[s]}(f^s)$, we conclude that $\operatorname{Ann}^{(1)}_{D[s]}(f^s)$ is strictly contained in $\operatorname{Ann}_{D[s]}(f^s)$. Otherwise the $K[x,s] \setminus \{0\}$-closure of $\operatorname{Ann}^{(1)}_{D[s]}(f^s)$ contains the $K[x] \setminus \{0\}$-closure of the $K[s] \setminus \{0\}$-closure of $\operatorname{Ann}^{(1)}_{D[s]}(f^s)$. If $\operatorname{Ann}^{(1)}_{D[s]}(f^s)$ is $K[s] \setminus \{0\}$-closed, we can treat both modules faithfully over the localization at $K[s] \setminus \{0\}$. Since the latter is central in $D[s]$, we can view $D(s)$ as the Weyl algebra over the field $K(s)$. Now, since $D(s)/\operatorname{Ann}_{D(s)}(f^s)$ is a module of holonomic rank 1 ([Lev15]), we can apply Weyl closure ([Tsa00]) to $\operatorname{Ann}^{(1)}_{D(s)}(f^s)$ to compute its $K[x] \setminus \{0\}$-closure. As above, if the result strictly contains $\operatorname{Ann}^{(1)}_{D[s]}(f^s)$, then $\operatorname{Ann}^{(1)}_{D[s]}(f^s)$ is strictly contained in $\operatorname{Ann}_{D[s]}(f^s)$

Conclusion and future work

Since the definition of LSat is inherently unconstructive there is no obvious way to formulate an algorithm, i.e. a terminating procedure for its computation in general. While in a special case of S-closure of an ideal I we have presented an algorithm that uses central saturation to compute $I^S = \mathrm{LSat}_S(I)$, as of yet there is no viable strategy to automatically compute the left saturation of left Ore sets and represent them in a finitely parametrized form.

Furthermore, we see potential in further analyzing the interplay between local torsion and algebraic systems theory, where the study of chains of left Ore sets and subsequently chains of local torsion modules might give more insight into the structure of autonomous systems.

Lastly, we are working to generalize the S-closure algorithm to submodules of free modules.

To the best knowledge of the author, this is the first work to consider the notion of LSat as described in Chapter 4 not only as a general concept that encompasses many problems, but specifically as a way to describe the units in Ore localized domains and to regard any Ore localization as a localization where the set of denominators is saturated.

The algorithm to compute S-closure in a special case given in Chapter 6 is also a new contribution. There are already efforts to implement the algorithm in the computer algebra SINGULAR in the context of the already extensive collection of libraries concerning themselves with D-module theory.

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