Fakultät für Mathematik, Informatik und
Naturwissenschaften der Rheinisch-Westfälischen
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Ore Localization with applications in $\mathcal{D}$-module theory

Niclas Heinrich Kruff

Lehrstuhl D für Mathematik

Betreuer:
Prof. Dr. Eva Zerz und Dr. Viktor Levandovskyy

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§1 Introduction

Let K be a field of characteristic zero and $K[\mathbf{x}, s] := K[x_1, ..., x_n, s]$ the polynomial ring in the variables $x_1, ..., x_n$ and s. The n-th polynomial Weyl algebra is defined as

$$\mathcal{D}_n := K \left\langle \mathbf{x}, \boldsymbol{\partial} \mid \partial_i x_j = x_j \partial_i + \delta(i,j) \right\rangle$$

i.e. the number of differential operators is equal to the numbers of the variables x_i. In $\mathcal{D}$-module theory the Bernstein-Sato polynomial is an object of interest which has plenty of applications in algebraic geometry. For a given polynomial $f \in R$ the global Bernstein-Sato polynomial $b(s)$ is defined as the monic generator of the ideal which consists of all polynomials in the variable s satisfying the functional equation $P(s)f(\mathbf{x})^{s+1} = q(s)f(\mathbf{x})^s$. The polynomial $b(s)$ is also called the global b-function. Hence $P \in \mathcal{D}_n \otimes K[s]$ is a differential operator. It is well known that $b(s) \neq 0$. The statement is still true if one works in the local Weyl algebra defined by $(\mathcal{D}_n)_{\mathfrak{m}_a} := K[\mathbf{x}, s]_{\mathfrak{m}_a} \left\langle \partial_1, ..., \partial_n \mid \partial_i f = f\partial_i + \frac{\partial f}{\partial x_i} \right\rangle$, i.e. $K[\mathbf{x}]$ is localized at the maximal ideal $\mathfrak{m}_a := \langle x_1 - a_1, ..., x_n - a_n \rangle$ which corresponds to the point $a \in K^n$, if K is a algebraic closed field like $\mathbb{C}$. Now fix a monomial ordering on $\mathrm{Mon}(\mathbf{x}, \boldsymbol{\partial}, s) := \mathrm{Mon}(x_1, ..., x_n, \partial_1, ..., \partial_n, s)$, such that $\partial_i > 1$, $x_i < 1$. In this work tools of Ore localization will be used to develop algorithms to compute the local Bernstein-Sato polynomial in a rational point. Actually, the variable s is a global variable, i.e. $s > 1$. Therefore, it is impossible to use elimination orderings for computing the local Bernstein-Sato polynomial, because the variables $x_1, ..., x_n$ are local variables. A new approach in this work is to treat s as a local variable, see chapter 8.1. Especially, Theorem 8.2 and Lemma 8.3 provide a new approach to check whether a number $\beta \in \mathbb{Q}$ is a root of the local b-function. The advantage is that there is the possibility to use a quasi-elimination ordering. That allows to assess the variable s with a smaller weight. At first glance, there is a disadvantage, because one has to work in the product localization $T^{-1}K[\mathbf{x}, s]$, with

$$T := \{f \in K[\mathbf{x}, s] \mid f(\mathbf{x}, s) = g(\mathbf{x}) \cdot h(s),\ f(0) \neq 0\}.$$

During my research, I found out that there is no necessity to work in the product localization, however it suffices to calculate in the localization $K[\mathbf{x}, s]_{\langle \mathbf{x}, s \rangle}$. Obviously computations in this localization are easier. In addition to that, the algorithm '*checkRoot*' written in the computer algebra system Singular which determines whether a rational number is a root of the global Bernstein-Sato polynomial has been extended to the local case. I developed and implemented an algorithm in the computer algebra system Singular which decides whether a complex number is a root of the local

Bernstein-Sato polynomial. Furthermore, this algorithm computes the multiplicity of the root. I proved a corollary which is helpful to compute the multiplicity and which also reduces the computational costs, see Corollary 8.4. Furthermore, there is an approach to compute the local Bernstein-Sato polynomial in an algebraic but non-rational point, see chapter 8.2. One idea is to add a zero-dimensional prime ideal which contains information about all algberaic numbers to the set of generators of a certain ideal, see Lemma 8.15 and Corollary 8.17.
Moreover this work provides some ideas to work with the multivariate case, i.e. there is more than one variable s. Concerning that, one may be interested in the Bernstein-Sato ideal. Unfortunately, it is impossible to transfer the idea from one variable to more than one variable, because the polynomial ring $K[\mathbf{x}, s]$ is not a principal ideal domain.
Additionally, this work will summarize some important results and proofs about homological algebra which are important to understand the theory of Bernstein-Sato polynomials and Bernstein-Sato ideals respectively. One important result is that the embedding $K[\mathbf{x}]_{\langle \mathbf{x} \rangle} \subseteq K[[\mathbf{x}]]$ is faithfully flat.
All algorithms are implemented in the computer algebra system Singular.

§2 Definitions and Theorems

(2.1) Definition (Stable *I*-filtration)
Let R be a Noetherian commutative ring, $I \subseteq R$ an ideal and M a R-module. Let $\{M_n\}_{n\in\mathbb{N}_0}$ be a set of submodules of M with the following properties:

1. $M =: M_0 \supseteq M_1 \supseteq M_2 \supseteq ...$
2. $IM_n \subseteq M_{n+1}$, for all $n \in \mathbb{N}_0$
3. there exists $n_0 \in \mathbb{N}_0$ such that $IM_n = M_{n+1}$, for all $n \geq n_0$.

The set $\{M_n\}_{n\in\mathbb{N}_0}$ is called a stable I-filtration of the R-module M.

(2.2) Theorem (Artin-Rees ([13]))
Let R be a commutative Noetherian ring and $I \subseteq R$ an ideal. Furthermore, let M be a finitely generated R-module and $\{M_n\}_{n\in\mathbb{N}_0}$ be a stable I-filtration of M. If $N \subseteq M$ is a submodule then the set $\{M_n \cap N\}_{n\in\mathbb{N}_0}$ is a stable I-filtration of N.

(2.3) Definition
Let K be a field. The non-commutative ring

$$\mathcal{D}_n := K\left\langle x_1, ..., x_n, \partial_1, ..., \partial_n \mid \partial_j x_i = x_i \partial_j + \delta(i,j) \right\rangle$$

is called the n-th Weyl algebra.

(2.4) Definition
The non commutative ring $\mathcal{D}_n[s]$ is defined as

$$\mathcal{D}_n[s] := \mathcal{D}_n \otimes_K K[s].$$

(2.5) Definition
Let R be a domain and $S \subseteq R$ a multiplicatively closed set such that $1 \in S$ and $0 \notin S$. The set S has the left Ore property if

$$\forall s \in S\ \forall r \in R\ \exists t \in S\ \exists p \in R : tr = ps.$$

(2.6) Lemma
Let K be a field. The set

$$S := \{f \in K[x_1, ..., x_n] \mid f(0) \neq 0\} \subseteq \mathcal{D}_n$$

has the left Ore property with respect to $\mathcal{D}_n$.

(2.7) Definition
Let $K[\mathbf{x}] := K[x_1, ..., x_n]$ the polynomial ring in the variables $x_1, ..., x_n$ and K be a field. A total ordering $\geq$ on $\mathrm{Mon}(\mathbf{x}) := \mathrm{Mon}(x_1, ..., x_n) := \left\{x^\alpha \mid \alpha \in \mathbb{N}_0^n\right\}$ is called a monomial ordering if the following condition is fulfilled:

$$\text{if } p \leq q \text{ then } r \cdot p \leq r \cdot q \text{ for all } p, q, r \in \mathrm{Mon}(\mathbf{x}).$$

(2.8) Definition
A monomial ordering is called global if $1 \leq r$ for all $r \in \mathrm{Mon}(\mathbf{x})$. Respectively a monomial ordering is called local if $1 \geq r$ for all $r \in \mathrm{Mon}(\mathbf{x})$. Otherwise the monomial ordering is called a mixed ordering.

(2.9) Definition
Let $\alpha := (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$. Then define

$$|\alpha| := \sum_{i=1}^{n} \alpha_i.$$

(2.10) Definition
Let $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \mathbf{x}^\alpha \in K[\mathbf{x}]$ and $U \subseteq \{x_1, ..., x_n\}$ a subset. Then one can define the degree of f with respect to U.

$$\mathrm{tdeg}_U(f) := \max\left\{\sum_{i, x_i \in U} \alpha_i \mid a_\alpha \neq 0\right\}.$$

If $U = \{x_1, ..., x_n\}$ one just writes $\mathrm{tdeg}(f) := \mathrm{tdeg}_U(f)$.

(2.11) Definition
Let $<$ be a monomial ordering on $\mathrm{Mon}(\mathbf{x})$. The inverse monomial ordering is defined by

$$m_1 <^{-1} m_2 \Longleftrightarrow m_1 > m_2$$

where $m_1, m_2 \in \mathrm{Mon}(\mathbf{x})$.

(2.12) Lemma
Let R be a commutative ring and $S \subseteq R$ a multiplicatively closed Ore set such that $1 \in S$. The functor $S^{-1}\bullet$ is exact, i.e. if $M_1 \longrightarrow M_2 \longrightarrow M_3$ is an exact sequence of left R-modules then the sequence $S^{-1}M_1 \longrightarrow S^{-1}M_2 \longrightarrow S^{-1}M_3$ is also exact.

(2.13) Lemma
Let M_1, M_2 be two left R-modules such that M_1/M_2 is well defined. Furthermore, let $S \subseteq R$ a multiplicatively closed Ore set such that $1 \in S$. By using Lemma 2.12 one gets the following isomorphism:

$$S^{-1}(M_1/M_2) \cong S^{-1}M_1/S^{-1}M_2.$$

(2.14) Lemma
Let R be a commutative ring and M a left R-module. Then the following properties are equivalent.

1. $M = 0$
2. $M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p} \subseteq R$
3. $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \subseteq R$.

(2.15) Definition
Let $f \in K[\mathbf{x}]$ be a polynomial. The singular locus of f is the variety

$$\mathrm{Sing}(f) := \mathcal{V}\left(\left\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right\rangle\right).$$

(2.16) Definition (direct system of sets)
Let $(I, \leq)$ be a partial ordered set of indices such that every finite subset of I has an upper bound. Moreover, let $(X_i)_{i\in I}$ be a family of sets and let $f_{j,i} : X_i \to X_j$, $i \leq j$, be mappings having the property

- $f_{i,i} = \mathrm{id}_{X_i}$
- $f_{j,i} = f_{j,k} \circ f_{k,i}$, $i \leq k \leq j$.

Then one calls $\langle (X_i)_{i\in I}, (f_{j,i})_{i\leq j}\rangle$ a direct system.

(2.17) Definition (direct limit of vector spaces)
Let $\langle (V_i)_{i\in I}, (f_{j,i})_{i\leq j}\rangle$ be a direct system of vector spaces V_i and linear mappings $f_{j,i}$. Furthermore, let V be a vector space and $f_i : V_i \to V$, $i \in I$, linear mappings with the following properties:

- $f_i = f_j \circ f_{j,i}$ with $i \leq j$
- if W is another vector space with linear mappings $g_i : V_i \to W$ with $g_i = g_j \circ g_{j,i}$ then there exists a unique linear mapping $g : V \to W$ such that $g_i = g \circ f_i$ for all $i \in I$.

Then one calls $\langle V, (f_i)_{i\in I}\rangle$ a direct limit of $\langle (V_i)_{i\in I}, (f_{j,i})_{i\leq j}\rangle$. Another common notation is $V := \lim_{i\in I} V_i$.

(2.18) Definition (contravariant functor Hom)
Let R be a commutative ring and M, N two R-modules. Define

$$\mathrm{Hom}(M, N) := \{f : M \to N \mid f \textit{ is } R\text{-linear}\}$$

the set of all R-linear mappings from M to N. Now fix N and see M as a variable. This gives rise to the contravariant functor $\mathrm{Hom}(\cdot, N)$. If $\phi : M_1 \to M_2$ is an R-module homomorphism between two R-modules one defines $\phi^* := F(\phi) := \phi \circ f$.

(2.19) Definition (Ext-module)
Let R be a commutative ring and M be an R-module. Furthermore, let

$$\cdots \mathfrak{F}_2 \xrightarrow{f_2} \mathfrak{F}_1 \xrightarrow{f_1} \mathfrak{F}_0 \xrightarrow{f_0} M \to 0$$

be a free resolution of M, i.e. $\mathfrak{F}_i$ is a free R-module for all $i \geq 1$. Moreover, let $\widetilde{M}$ be another R-module. By applying the contravariant functor $\mathrm{Hom}(\cdot, \widetilde{M})$ to the complex

$$\cdots \mathfrak{F}_2 \xrightarrow{f_2} \mathfrak{F}_1 \xrightarrow{f_1} \mathfrak{F}_0 \to 0$$

one gets the complex

$$0 \longrightarrow \mathrm{Hom}(\mathfrak{F}_0, \widetilde{M}) \xrightarrow{f_1^*} \mathrm{Hom}(\mathfrak{F}_1, \widetilde{M}) \xrightarrow{f_2^*} \mathrm{Hom}(\mathfrak{F}_2, \widetilde{M}) \cdots$$

where $f_i^*(g) := g \circ f_i$. Now define the i-th Ext-module as

$$\mathrm{Ext}_R^i(M, \widetilde{M}) := \ker(f_{i+1}^*) / \mathrm{im}(f_i^*).$$

§3 Standard bases

This chapter will deal with the concept of standard bases which is a generalization of Gröbner bases introduced in the book *A Singular introduction to commutative algebra* written by Gert-Martin Greuel and Gerhard Pfister, chapter 1, Normal Forms and Standard bases (see [13]). It is necessary to study standard bases to realize some important computations in localizations. This chapter provides two algorithms to compute standard bases. On the one hand it introduces the algorithm of Mora which includes a generalization of a division algorithm. On the other hand, the reader will see the method of Lazard which reduces the problem of standard bases to Gröbner bases.

Let $<$ be a monomial ordering on $\mathrm{Mon}(\mathbf{x})$ and one can define the following terms.

(3.1) Definition
Let $f \in K[\mathbf{x}] \setminus \{0\}$ a polynomial.

- $\mathrm{lm}_<(f) := \max\{\mathbf{x}^\alpha \mid a_\alpha \neq 0\} =: \mathbf{x}^\gamma$, the leading monomial
- $\mathrm{lc}_<(f) := a_\gamma$, the leading coefficient
- $\mathrm{lt}_<(f) := a_\gamma \mathbf{x}^\gamma$, the leading term
- $\mathrm{lexp}_<(f) := \gamma$, the leading exponent

Of course, there is a bijection between the exponents $\alpha \in \mathbb{N}^n$ and the monomials $\mathbf{x}^\alpha$. Therefore, one can use he following equivalent notations. Let $\mathbf{x}^\alpha, \mathbf{x}^\beta \in \mathrm{Mon}(\mathbf{x})$.

$$\mathbf{x}^\alpha \leq \mathbf{x}^\beta \Longleftrightarrow \alpha \leq \beta.$$

The following theorem presents a relation between monomial orderings and matrices.

(3.2) Theorem (L. Robbiano, Ostrowski)
Let $<$ be a monomial ordering on $\mathrm{Mon}(x_1, ..., x_n)$. There exists a matrix $A \in \mathbb{R}^{m \times n}$, $m \in \mathbb{N}$, such that

$$\alpha \leq \beta \Longleftrightarrow A\alpha \leq_{lp} A\beta$$

where $\leq_{lp}$ denotes the lexicographical ordering and $\alpha, \beta \in \mathbb{N}^n$.

On the other hand, each invertible matrix $A \in \mathbb{R}^{n \times n}$ induces a monomial ordering $<_A$ by

$$\alpha \leq \beta :\Longleftrightarrow A\alpha \leq_{lp} A\beta.$$

If $I \subseteq K[\mathbf{x}]$ is an ideal one can define the leading ideal:

$$\text{Lead}_{<}(I) := \langle \text{lm}_{<}(f) | \, f \in I \setminus \{0\} \rangle_{K[\mathbf{x}]} \,.$$

In order to proceed to define standard bases one needs the localization of the polynomial ring $K[\mathbf{x}]$ with respect to a monomial ordering.

(3.3) Definition
Let $<$ be a monomial ordering on $\text{Mon}(\mathbf{x})$ and define

$$S := \left\{ f \in K[\mathbf{x}] \setminus \{0\} \;\mid\; \underset{<}{\text{lm}}(f) = 1 \right\}.$$

This set is multiplicatively closed and $1 \in S$. The localization at the monomial ordering $<$ is defined as:

$$K[\mathbf{x}]_< := S^{-1}K[\mathbf{x}].$$

The following lemma presents some properties of $K[\mathbf{x}]_<$.

(3.4) Lemma

- $K[\mathbf{x}]_<$ is Noetherian
- $K[\mathbf{x}]_<$ is a domain
- if $<$ is a global monomial ordering then
$$K[\mathbf{x}]_< = K[\mathbf{x}].$$
- if $<$ is a local monomial ordering then
$$K[\mathbf{x}]_< = K[\mathbf{x}]_{\langle \mathbf{x} \rangle}.$$

In order to talk about standard bases in the localization one needs the following definition.

(3.5) Definition
Let $f := \frac{p}{q} \in K[\mathbf{x}]_<$ and $u \in S$ such that $\text{lt}_<(u) = 1$ and $uf \in K[\mathbf{x}]$.

- $\text{lm}_<(f) := \text{lm}_<(uf)$, the leading monomial

- $\mathrm{lc}_<(f) := \mathrm{lc}_<(uf)$, the leading coefficient
- $\mathrm{lt}_<(f) := \mathrm{lt}_<(uf)$, the leading term
- $\mathrm{lexp}_<(f) := \mathrm{lexp}_<(uf)$, the leading exponent

It is easy to see that this definition is independent of the choice of u.

(3.6) Definition
Let $0 \neq I \subseteq K[\mathbf{x}]_<$ be an ideal. A finite set G is called a standard basis of I with respect to $<$ if G has the following two properties:

1. $G \subseteq I$
2. $\mathrm{Lead}_<(G) = \mathrm{Lead}_<(I)$

In conclusion, a set G is a standard basis of I if the following equality holds

$$\langle \mathrm{lm}_<(f) |\, f \in I \setminus \{0\} \rangle_{K[\mathbf{x}]} = \langle \mathrm{lm}_<(g) |\, g \in G \setminus \{0\} \rangle_{K[\mathbf{x}]}.$$

One immediately observes similarities with the definition of a Gröbner basis. However, it is important to understand that this definition is a generalization. The choice of the monomial ordering is no longer restricted to global ones. Plenty of examples will deal with local and mixed orderings respectively which are not well-orderings on the set $\mathbb{N}_0^n$.

(3.7) Lemma
Let $0 \neq I \subseteq K[\mathbf{x}]_<$ be an ideal and $<$ an arbitrary monomial ordering on $\mathrm{Mon}(\mathbf{x})$. Then I has a standard basis.

Proof
Let $G_0 \subseteq I$ a nonempty set. If $\mathrm{Lead}_<(G_0) = \mathrm{Lead}_<(I)$ holds G_0 is already a standard basis. Therefore, let $\mathrm{Lead}_<(G_0) \subsetneq \mathrm{Lead}_<(I)$. Then there exists an element $g_1 \in I$ such that $\mathrm{lm}(_<g_1) \notin \mathrm{Lead}_<(G_0)$. Now one defines $G_1 = G_0 \cup \{g_1\}$, and one has the inclusion $\mathrm{Lead}(G_0) \subsetneq \mathrm{Lead}(G_1)$. By iteration one gets a strictly increasing chain of ideals in $K[\mathbf{x}]_<$. However, $K[\mathbf{x}]_<$ is Noetherian, and consequently the chain must be finite. Let l be the chain's last index. In conclusion one has $\mathrm{Lead}_<(G_l) = \mathrm{Lead}_<(I)$ and G_l is a standard basis of I. □

The existence of standard basis is now verified. Having a closer look this proof is exactly the same as in the case of Gröbner bases because it is only important that $K[\mathbf{x}]_<$ is Noetherian.

The following three definitions will be very useful:

(3.8) Definition

1. Let $\varnothing \neq G \subseteq I$. G is called interreduced if $0 \notin G$ and for all $f, g \in G$ with $f \neq g$ $\mathrm{lm}_<(g) \nmid \mathrm{lm}_<(f)$.
2. $f \in R$ is called reduced with respect to G if no monomial in the representation of f is an element of $\mathrm{Lead}_<(G)$.
3. G is called reduced if G is interreduced and if the leading coefficient is equal to 1 for all $g \in G$ and if $\mathrm{tail}_<(g)$ is reduced with respect to G.

The next step is to find algorithms which compute standard bases. In the theory of Gröbner bases one computes S-polynomials and reduces them by applying a division algorithm with respect to a given set G. Now the strategy will be very similar. It is convenient to recall the concepts of S-polynomials and normal forms.

(3.9) Definition

Let $f, g \in K[\mathbf{x}]_< \setminus \{0\}$ and $\mathrm{lm}_<(f) = \mathbf{x}^\alpha$, $\mathrm{lm}_<(g) = \mathbf{x}^\beta$. One defines

$$\gamma := \mathrm{lcm}(\alpha, \beta) := [\max\{\alpha_1, \beta_1\}, ..., \max\{\alpha_n, \beta_n\}],$$

and $\mathrm{lcm}(\mathbf{x}^\alpha, \mathbf{x}^\beta) := \mathbf{x}^\gamma$. The S-polynomial of f and g is defined as

$$\mathrm{spoly}(f, g) := \mathbf{x}^{\gamma-\alpha} f - \frac{\mathrm{lc}_<(f)}{\mathrm{lc}_<(g)} \mathbf{x}^{\gamma-\beta} g.$$

(3.10) Definition

Let $\mathcal{G} := \{G \subseteq K[\mathbf{x}]_< \mid |G| < \infty\}$. Consider the mapping $\mathrm{NF} : K[\mathbf{x}]_< \times \mathcal{G} \to K[\mathbf{x}]_<, (f, G) \mapsto \mathrm{NF}(f|G)$. NF is called a normal form if the following three conditions are true for all $G \in \mathcal{G}$:

1. $\mathrm{NF}(0|G) = 0$
2. $\mathrm{NF}(f|G) \neq 0 \Rightarrow \mathrm{lm}_<(\mathrm{NF}(f|G)) \notin \mathrm{Lead}(G)\ \forall f \in K[\mathbf{x}]_< \setminus \{0\}$
3. If $G = \{g_1, ..., g_s\}$ the element $f - \mathrm{NF}(f|G)$ has a representation of the form

$$f - \mathrm{NF}(f|G) = \sum_{i=1}^{n} a_i g_i,\ a_i \in K[\mathbf{x}]_<$$

and for all $1 \leq i \leq n$ one has either $a_i g_i = 0$ or

$$\mathrm{lm}_<(\sum_{i=1}^{n} a_i g_i) \geq \mathrm{lm}_<(a_i g_i).$$

These definitions are also quite similar to the global case. Condition three says that there exists a $1 \leq j \leq n$, such that

$$\operatorname{lm}_<(a_j g_j) = \operatorname{lm}_<(\sum_{i=1}^{n} a_i g_i) = \max\{\operatorname{lm}_<(a_k g_k | \ 1 \leq k \leq n)\}.$$

In case of standard bases it is necessary to have another definition of a normal form, a so called *weak normal form*. A weak normal form also satisfies the conditions one and two of the Definition 3.10. Only condition three will be modified. Concerning this condition, one requires that for all $f \in K[\mathbf{x}]_<$ and for all $G \in \mathcal{G}$ there exists a unit $u \in (K[\mathbf{x}]_<)^*$, such that uf has a representation

$$uf = \sum_{i=1}^{n} a_i g_i + h$$

with $\operatorname{lm}_<(h) \notin \operatorname{lm}_<(G)$. One calls the element h a weak normal form of f with respect to G. As a remark, if the monomial ordering $>$ is global one chooses $u = 1$ because one has $(K[\mathbf{x}]_<)^* = K \setminus \{0\}$, i.e. it is necessary to have local variables to get a unit not equal to one. Some properties of standard basis will be dicussed in the following. Again, some properties are well-known from the Gröbner bases theory.

(3.11) Lemma
Let $0 \neq I \subseteq K[\mathbf{x}]_<$ be an ideal, $G \subseteq I$ a standard basis of I, and $\operatorname{NF}(\cdot|G)$ a weak normal form of R with respect to G. This implies the following four results:

1. $\forall f \in R$: $f \in I \ \Leftrightarrow \ \operatorname{NF}(f|G) = 0$.
2. Let $J \subseteq K[\mathbf{x}]_<$ be an ideal such that $I \subseteq J$. If $\operatorname{Lead}_<(I) = \operatorname{Lead}_<(J)$ then $I = J$.
3. $I = \langle G \rangle_{K[\mathbf{x}]_<}$.
4. If $\operatorname{NF}(\cdot|G)$ is a reduced normal form the normal form is unique.

Proof
ad 1. : Let $\operatorname{NF}(f|G) = 0$. By assumption $\operatorname{NF}(\cdot|G)$ is a weak normal form of R with respect to G. Therefore, there exists a unit $u \in K[\mathbf{x}]_<$ such that $uf \in I$. The element u is a unit in $K[\mathbf{x}]_<$, and this implies $f \in I$.
On the other hand, let $\operatorname{NF}(f|G) \neq 0$. By definition of a weak normal form one gets $\operatorname{lm}_<(\operatorname{NF}(f|G)) \notin \operatorname{Lead}_<(G)$. However, G is a standard basis of I and consequently $\operatorname{Lead}_<(G) = \operatorname{Lead}_<(I)$. In conclusion, $\operatorname{NF}(f|G) \notin I$ and that means that $f \notin I$.

ad 2. : The inclusion $I \subseteq J$ evidently holds. Therefore, one has to check the other inclusion. Let $f \in J$ and assume that $\mathrm{NF}(f|G) \neq 0$. Consequently,

$$\mathrm{lm}_<(\mathrm{NF}(f|G)) \notin \mathrm{Lead}_<(G) = \mathrm{Lead}_<(I) = \mathrm{Lead}_<(J).$$

But this contradicts the fact that $\mathrm{NF}(f|G) \in J$. In conclusion, $\mathrm{NF}(f|G) = 0$ and by using 1 the claim follows.

ad 3. : One has $\langle G \rangle_{K[\mathbf{x}]_<} \subseteq I$ and $\mathrm{Lead}_<(\langle G \rangle_{K[\mathbf{x}]_<}) = \mathrm{Lead}(I)$ because

$$\mathrm{Lead}_<(G) = \mathrm{Lead}_<(I).$$

By statement 2 the claim follows.

ad 4. : Let $f \in K[\mathbf{x}]_<$ and assume that h_1, h_2 are two reduced normal forms of f with respect to G. That means that no monomial appearing in the representation of h_1 and h_2 respectively is an element of $\mathrm{Lead}_<(G)$. Moreover, the fact

$$h_1 - h_2 = (f - h_2) - (f - h_1) \in \langle G \rangle_{K[\mathbf{x}]_<} = I,$$

implies $h_1 - h_2 = 0$. Otherwise assume that $h_1 - h_2 \neq 0$ is true. That implies

$$\mathrm{lm}_<(h_1 - h_2) \in \mathrm{Lead}_<(I) = \mathrm{Lead}_<(G).$$

But that is a contradiction because $\mathrm{lm}_<(h_1 - h_2)$ is either a monomial in the representation of h_1 or h_2. $\square$

Especially statement 4 shows that a reduced normal form is unique. The following lemma will show, that similarly to the case of Gröbner bases, a reduced standard basis is unique if it exists.

(3.12) Lemma
Let $<$ a monomial ordering on $\mathrm{Mon}(\mathbf{x})$ and $I \subseteq K[\mathbf{x}]_<$ an ideal.
Then: If I has a reduced standard basis it is unique.

The following example will show one difficulty in case of local monomial orderings.

(3.13) Example
Consider the polynomial ring $R := K[x]$ in one variable. Let $f := x$ and $G := \{g\}$, $g := x - x^2$. Furthermore, fix a local monomial ordering $<$ on $\mathrm{Mon}(x)$, i.e. $x < 1$. Now try to apply Buchberger's Algorithm. At first, one has $\mathrm{lm}_<(f) = x$

and $\mathrm{lm}_<(g) = x$. Therefore, divide f by G. The result is $f - g = x^2 =: f_1$. Now $\mathrm{lm}_<(f_1) = x^2$. Next reduce f_1 and consequently $f_1 - xg = x^3 =: f_2$. Proceeding in this manner one has the strictly decreasing chain of monomials $x > x^2 > x^3 > x^4 > ....$ Obviously, this chain is infinite and in conclusion Buchberger's Algorithm is inapplicable. However, one has the equation $f - g - x \cdot f = 0$. This implies $(1 - x) \cdot f = g$. Moreover, $\mathrm{lm}_<(1 - x) = 1$, i.e. $1 - x$ is a unit in the localization $R_<$. This observation leads to weak normal forms and the Mora algorithm which will be introduced in this chapter.

Another approach to compute standard bases by avoiding weak normal forms is the method of Lazard which uses homogenization.

(3.14) Definition
Let $f \in K[\mathbf{x}]$ such that $\mathrm{tdeg}(f) = d$. The new element
$f^h := t^d \cdot f(\frac{x_1}{t}, ..., \frac{x_n}{t}) \in K[t, \mathbf{x}]$ is called the homogenization of f with respect to t.

(3.15) Example
Let $g = x_1 x_2^2 + x_1 x_3 + 3x_1^3 x_3 + 4x_2^2 \in \mathbb{Q}[x_1, x_2, x_3]$. The total degree is $\mathrm{tdeg}(g) = 4$ thus $g^h = t x_1 x_2^2 + t^2 x_1 x_3 + 3x_1^3 x_3 + 4t^2 x_2^2 \in \mathbb{Q}[t, x_1, x_2, x_3]$.

(3.16) Remark
Consider the linear map $x_i \mapsto \frac{x_i}{t}$ which is multiplicative. Let $f = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \mathbf{x}^\alpha,\ a_\alpha \in K$.

Now compute the homogenization:

$$f^h = t^{\mathrm{tdeg}(f)} \cdot f(\frac{x_1}{t}, ..., \frac{x_n}{t}) = t^{\mathrm{tdeg}(f)} \cdot \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha (\frac{x_1}{t})^{\alpha_1} \cdots (\frac{x_n}{t})^{\alpha_n}.$$

By multiplicativity one has

$$f = \sum_{i=1}^{n} a_i \cdot f_i,\ a_i, f_i \in K[\mathbf{x}],\ n \in \mathbb{N}$$

and this implies

$$\begin{aligned} f^h =& t^{\mathrm{tdeg}(f)} \cdot f(\frac{x_1}{t}, ..., \frac{x_n}{t}) \\ =& t^{\mathrm{tdeg}(f)} \cdot \sum_{i=1}^{n} (a_i \cdot f_i)(\frac{x_i}{t}, ..., \frac{x_n}{t}) \\ =& t^{\mathrm{tdeg}(f)} \cdot \sum_{i=1}^{n} a_i(\frac{x_i}{t}, ..., \frac{x_n}{t}) \cdot f_i(\frac{x_i}{t}, ..., \frac{x_n}{t}). \end{aligned}$$

.

Obviously, homogenization is not additive for $f, g \in K[\mathbf{x}]$, i.e.

$$(f+g)^h = f^h + g^h$$

is wrong in general. As an example let $f := x+1$, $g := -x+1 \in K[x]$. This implies $(f+g)^h = 2 \neq 2t = f^h + g^h$. Obviously, one has $(f^h)_{|t=1} = f$. This process of setting $t=1$ is called dehomogenization. On the other hand, the equation $(f(\mathbf{x},t)_{|t=1})^h = f$ is false in general considering the polynomial $g = tx - t^2x$. The following subsections will present two methods for computing standard bases.

§3.1 Lazard

This chapter concentrates on the method of Lazard. The following lemma shows an important compatibility relation. As a preparation for computing standard bases define the following monomial ordering on $\mathrm{Mon}(t, \mathbf{x})$.

(3.17) Definition

The matrix $A_<$ induces a monomial ordering $<$ on $\mathrm{Mon}(\mathbf{x})$ and this leads to a global monomial ordering $<_h$ on $\mathrm{Mon}(t, \mathbf{x})$ by defining the matrix

$$A_{<_h} := \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix} \in \mathbb{Q}^{(n+1)\times(n+1)}.$$

(3.18) Lemma

Consider $A_{<_h}$ from definition 3.17 and let $m_1, m_2 \in \mathrm{Mon}(t, \mathbf{x})$ be two monomials such that $\mathrm{tdeg}(m_1) = \mathrm{tdeg}(m_2)$. This implies:

$$m_1 >_h m_2 \Leftrightarrow (m_1)_{|t=1} > (m_2)_{|t=1}.$$

Let $g \in K[t, \mathbf{x}]$ be a homogeneous polynomial of degree $d \geq 0$. There exists a representation

$$G = \sum_{k+|\beta|=d} c_{k,\beta} t^k \mathbf{x}^\beta$$

Let $c_{k_1,\beta_1}, c_{k_2,\beta_2} \neq 0$ be two coefficients in this representation. By Lemma 3.18 one concludes:

$$t^{k_1}\mathbf{x}^{\beta_1} >_h t^{k_2}\mathbf{x}^{\beta_2} \Leftrightarrow \mathbf{x}^{\beta_1} > \mathbf{x}^{\beta_2}.$$

This yields:

$$(\mathrm{lm}_{<_h}(G))_{|t=1} = \mathrm{lm}_<((G)_{|t=1})$$

(3.19) Theorem (method of Lazard)
Let $<$ be a monomial ordering on $\mathrm{Mon}(\mathbf{x})$ induced by a matrix $A \in \mathrm{GL}(n, \mathbb{Q})$ and $I := \langle f_1, ..., f_m \rangle \subseteq K[\mathbf{x}]$ an ideal. The matrix

$$A_{<_h} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix} \in \mathbb{Q}^{(n+1)\times(n+1)}$$

induces the monomial ordering $<_h$ on $\mathrm{Mon}(t, \mathbf{x})$. Additionally, let $G := \{g_1, ..., g_s\}$ be a homogeneous Gröbner basis of $\langle f_1^h, ..., f_m^h \rangle \subseteq K[t, x_1, ..., x_n]$.
This implies that $(G)_{|t=1} := \left\{ (g_1)_{|t=1}, ..., (g_s)_{|t=1} \right\}$ is a standard basis of I.

Algorithm 1: Lazard

Input: $<$ a monomial ordering on $\mathrm{Mon}(\mathbf{x})$, $\varnothing \neq G \subseteq K[\mathbf{x}]$ finite set of polynomials, a normal form algorithm NF.
Output: a standard basis of $\langle G \rangle$.
$S := G$;
$T := \left\{ g^h | g \in S \right\}$;
$T := \mathrm{Buchberger}(T | \mathrm{NF})$;
$S := \left\{ (f)_{|t=1} | f \in T \right\}$;
return S;

Have a look at the following example.

(3.20) Example
Let $S := \left\{ x + y^2, xy + y^4 \right\}$ and $<$ be the negative lexicographical ordering.
$T := \{ \underbrace{\underline{tx} + y^2}_{=:f_1}, \underbrace{\underline{t^2xy} + y^4}_{=:f_2} \}$.
$f_3 := \mathrm{spoly}(f_1, f_2) = ty^3 - y^4$.
Applying Buchberger's reduction yields $\mathrm{NF}_{<_h}(f_3 | T) = 0$.
By Buchberger's criterion the set T is a standard basis of $I := K[t, x, y] \langle T \rangle$ and by the method of Lazard the set S is a standard basis of $J := K[x, y] \langle S \rangle$.

§ 3.2 Mora

Consider the following definition.

(3.21) Definition
Let $f \in K[\mathbf{x}] \setminus \{0\}$ and $<$ be a monomial ordering on $\mathrm{Mon}(\mathbf{x})$. The ecart of f is defined as

$$\mathrm{ecart}_<(f) := \mathrm{tdeg}(f) - \mathrm{tdeg}(\mathrm{lm}_<(f)).$$

The following algorithm by Mora computes a weak normal form of polynomial with respect to finite set G.

Algorithm 2: Mora

Input: Let $<$ be a monomial ordering on $\mathrm{Mon}(\mathbf{x})$ and $0 \neq f \in K[\mathbf{x}]$, $\varnothing \neq G \subseteq K[\mathbf{x}]$ a finite set of polynomials.
Output: a weak normal form of f with respect to G.

$q := f$;
$T := G$;
while $q \neq 0$ & $T_q := \{g \in T \mid \mathrm{lm}_<(g) \mid \mathrm{lm}_<(q)\} \neq \varnothing)$ **do**
  choose $g \in T_q$ such that $\mathrm{ecart}(g)$ minimal;
  if $\mathrm{ecart}(g) > \mathrm{ecart}(q)$ **then**
    $T := T \cup \{q\}$;
  end
  $q := \mathrm{spoly}(q, g)$;
end
return q;

By using the previous algorithm one can compute standard basis similar to the case of Gröbner bases.

Algorithm 3: computing standard bases with Mora

Input: $<$ a monomial ordering on $\mathrm{Mon}(\mathbf{x})$, $\varnothing \neq G \subseteq K[\mathbf{x}]$ finite set of polynomials, a normal form algorithm NF (for example Mora).
Output: a standard basis of $\langle G \rangle$.

```
S := G;
P := {(f,g) | f,g ∈ S, f ≠ g};
while P ≠ ∅) do
    choose (f,g) ∈ P;
    P := P \ {(f,g)};
    h := NF(spoly(f,g) | S);
    if (h ≠ 0) then
        P := P ∪ {(h,f) | f ∈ S};
        S := S ∪ {h};
    end
end
return S;
```

Mora's algorithm will be applied in the following example.

(3.22) Example
Let $R := K[x,y]_{\langle x,y \rangle}$, and $<_{Ds}$ denotes the negative degree lexicographical ordering. Furthermore, define $T := \left\langle \underline{y^3} =: g_1, \underline{x} - 2y =: g_2, \underline{y} + 3x^2 =: g_3 \right\rangle_R$ and $f := \underline{x}$. The underlined terms always represent the leading terms of a given polynomial.

$h := f = \underline{x}$
$T := G = \left\{ \underline{y^3}, \underline{x} - 2y, \underline{y} + 3x^2 \right\}$
First step:
Choose $g = g_2$, $\mathrm{ecart}(g) = \mathrm{tdeg}(g) - \mathrm{tdeg}(\mathrm{lm}(g)) = 0$
$h := \mathrm{spoly}(h,g) = \underline{2y}$
Second step:
Choose $g = g_3$, $\mathrm{ecart}(g) = 1 > 0 = \mathrm{ecart}(h)$
$T := T \cup \{h = 2y\}$
$h := \mathrm{spoly}(h,g) = \underline{-6x^2}$
Third step:
Choose $g = g_2$, $\mathrm{ecart}(g) = 0$
$h := \mathrm{spoly}(h,g) = \underline{-12xy}$
Fourth step:
Choose $g = 2y$, $\mathrm{ecart}(g) = 0$
$h := \mathrm{spoly}(h,g) = 0$

$h = 0$
By the above calculations one has $f \in T$. Of course it is possible to compute the unit u in the standard representation. Let $h_1 := 2y$. Then:

$$\begin{aligned} x - g_2 &= h_1 \\ h_1 - 2g_3 &= -6x^2 \\ -6x^2 + 6xg_2 &= -12xy \\ -12xy + 6xh_1 &= 0 \\ h_1 &= f - g_2 \\ (1+6x)f &= g_2 + 2g_3 \end{aligned}$$

§4 Local cohomology

This chapter will give a short introduction to local cohomology bases on the notes of Craig Huneke and Amelia Taylor (see [15]). An application of local cohomology is the computation of standard bases in a special case. Throughout this chapter, let R be a commutative ring with 1.

§4.1 Injective modules

First, it is necessary to repeat some definitions concerning injective modules. Let R be a commutative ring and let M, N be two R-modules.

(4.1) Definition
The set

$$\mathrm{Hom}(M, N) := \{f : M \to N \mid f \textit{ is } R\text{-linear}\}$$

denotes the set which contains all R-linear maps from M to N.

(4.2) Remark
The set $\mathrm{Hom}(M, N)$ is also an R-module.

Of course one can fix N and treat M as a variable. The result is the functor $F := \mathrm{Hom}(\cdot, N)$. Let $M, P1, P2$ be R-modules and $f : P_1 \to P_2$ be an R-module homomorphism.

- $FM := \mathrm{Hom}(M, N)$

- $Ff : FP_1 \to FP_2, g \mapsto g \circ f$.

This functor is called the contravariant Hom-functor and the functor is left exact, i.e. if

$$P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3 \longrightarrow 0$$

is exact then

$$0 \longrightarrow FP_3 \xrightarrow{Ff_2} FP_2 \xrightarrow{Ff_2} FP_1$$

is exact too. The following theorem defines and characterizes the property injective.

(4.3) Theorem
Let M, P_1, P_2, P_3 be R-modules an $F := \operatorname{Hom}(\cdot, M)$. The following statements are equivalent.

- $0 \longrightarrow P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3$ exact implies $FP_3 \xrightarrow{Ff_2} FP_2 \xrightarrow{Ff_2} FP_1 \longrightarrow 0$ is exact.
- $P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3$ exact implies $FP_3 \xrightarrow{Ff_2} FP_2 \xrightarrow{Ff_2} FP_1$ is exact.

If one of these equivalent conditions is true the module M is called injective.

Another very useful characterization of injectivity will be presented in the next theorem.

(4.4) Theorem (Baer‘s Criterion)
Let M be an R-module. This implies that M is injective if and only if every R-module homomorphism $\psi : I \to M$, $I \subseteq R$ ideal can be extended to an R-module homomorphism $\tilde{\psi} : R \to M$ and $\tilde{\psi}|_I = \psi$.

For certain modules one can characterize injectivity by the property divisible.

(4.5) Definition (divisible)
Let M be an R-module. The module M is called divisible if

$$\forall r \in R\ \forall m \in M \text{ such that } \operatorname{ann}(r) \subseteq \operatorname{ann}(m)\ \exists \tilde{m} \in M :\ r\tilde{m} = m.$$

(4.6) Definition (essential)
Let M, N be two R-modules such that $M \subseteq N$. The module N is called essential over the module M if for all submodules $\{0\} \neq T$ of N one has $T \cap M \neq \{0\}$.

If one of the three equivalent conditions of the next lemma is fulfilled the module N is called an injective hull of M.

(4.7) Lemma
Let M, N be two R-modules such that $M \subseteq N$. The following statements are equivalent:

- The module N is injective and essential over M.
- If $M \subseteq P \subseteq N$ and P is injective then $N = P$.
- If $N \subseteq Q$ and Q is essential over M then $N = Q$.

(4.8) Lemma
Let M, N be two R-modules, M injective and essential over N. Furthermore, let P be an injective R-module such that $N \subseteq P \subseteq M$. This implies that $P = M$.

Proof
Let $N \subseteq P \subseteq M$. Let Q be a submodule of M. The module M is essential over N and therefore $Q \cap N \neq 0$. However, one has $N \subset P$ and consequently $0 \neq Q \cap N \subseteq Q \cap P$ and this implies that M is essential over P. Moreover, by assumption the module P is injective. Consider the following exact sequence:

$$0 \longrightarrow P \longrightarrow M \longrightarrow M/P.$$

By the injectivity of N this exact sequence splits, i.e. there exists an R-module T such that $M = P \oplus T$ and $P \cap T = \{0\}$. However, the module T is a submodule of M and M is essential over P. Consequently, $T = \{0\}$ and therefore $M = P$. $\square$

The injective hull of an R-module M is an interesting object.

(4.9) Lemma
An injective hull of M is unique up to isomorphism.

Proof
Let M_1, M_2 be two injective hulls of M. Consider the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\mathrm{id}_M"] \arrow[d, "\mathrm{id}_M"] & M_1 \\ & M_2 & \end{tikzcd}$$

By the injectivity of M_2 there exists a map $\chi : M_1 \to M_2$ be such that the diagram commutes. First let $n \in M$ such that $\chi(n) = 0$, i.e. $m \in M \cap \ker(\chi)$. The diagram commutes and consequently one has $m = \chi(m) = 0$ and therefore $M \cap \ker(\chi) = \{0\}$.

The module $\ker(\chi)$ is a submodule of M_1 and M_1 is essential over M, i.e. $\ker(\chi) = 0$. This implies that χ is injective. By the homomorphism theorem one gets

$$M_1 \cong \operatorname{im}(\chi)$$

i.e. $\operatorname{im}(\chi)$ is injective, and one has the chain of inclusions $M \subseteq \operatorname{im}(\chi) \subseteq M_2$. Moreover, the module M_2 is also essential over M and by previous considerations one gets $\operatorname{im}(\chi) = M_2$ and therefore χ is also surjective. □

Consider the following lemma which will play an important role in this chapter.

(4.10) Lemma
Let R be a domain. The injective hull of R is its quotient field $\operatorname{Quot}(R)$.

Proof
If one treats $\operatorname{Quot}(R)$ as an R-module it is torsion-free. Therefore, $\operatorname{Quot}(R)$ is injective if and only if it is divisible. Let $r \in R \setminus \{0\}$ and $s := \frac{s_1}{s_2} \in \operatorname{Quot}(R)$. Choose the element $t := \frac{s_1}{s_2 \cdot r}$ and this implies that $r \cdot t = s$. By these calculations the module $\operatorname{Quot}(R)$ is divisible and hence injective. Next one has to prove that $\operatorname{Quot}(R)$ is essential over R. Let $\frac{s}{t} \in \operatorname{Quot}(R) \setminus \{0\}$ and choose $0 \neq t \in R$. This implies that $t \cdot \frac{s}{t} = s \in R \setminus \{0\}$ and therefore $\operatorname{Quot}(R)$ is essential over R. The claim follows by Lemma 4.9 and Lemma 4.7. □

(4.11) Definition (injective resolution)
Let M be an R-module. An exact sequence

$$0 \longrightarrow M \longrightarrow P_0 \xrightarrow{f_0} P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} \cdots$$

is called an injective resolution of M if P_i is an injective R-module for all i. Furthermore, the resolution is called minimal if P_0 is an injective hull of M and P_i is an injective hull of $\ker(f_i)$ for all $i \geq 1$.

(4.12) Lemma
A minimal injective resolution is unique up to isomorphism.

Now let M be an R-module, fix an ideal $I \subseteq R$ and consider the following set

$$\mathcal{F}(M) := \mathcal{F}_I(M) := \{x \in M \mid \exists n \in \mathbb{N} : I^n x = 0\}.$$

Obviously, $\mathcal{F}_I(M)$ is a submodule of M and moreover it gives rise to a functor between R-modules. Let $\mathcal{M}$ be the set of all R-modules. Denote

$$\mathcal{F}_I : \mathcal{M} \longrightarrow \mathcal{M}, M \mapsto \mathcal{F}(M).$$

(4.13) Remark
Let M_1, M_2 be two R-modules and $\phi : M_1 \to M_2$ be an R-module homomorphism. The functor $\mathcal{F}$ maps ϕ to

$$\mathcal{F}\phi : \mathcal{F}_I(M_1) \to \mathcal{F}_I(M_2),\ x \mapsto \phi(x).$$

Let $x \in \mathcal{F}_I(M_1)$, that is $x \in M_1$ and there exists $n \in \mathbb{N}$ such that $I^n x = 0$. By definition of ϕ the element $y := \phi(x) \in M_2$ and one gets

$$0 = \phi(I^n x) \overset{R\text{-linear}}{=} I^n \phi(x) = I^n y$$

i.e. $y \in \mathcal{F}_I(M_2)$. Therefore, the map $\mathcal{F}\phi$ is well-defined. Consequently, the functor $\mathcal{F}_I$ is covariant.

(4.14) Lemma
The covariant functor $\mathcal{F}$ is left-exact, i.e. if

$$0 \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$$

is exact, the sequence

$$0 \longrightarrow \mathcal{F}_I M_1 \xrightarrow{\mathcal{F}_I f_1} \mathcal{F}_I M_2 \xrightarrow{\mathcal{F}_I f_2} \mathcal{F}_I M_3$$

is also exact.

Proof
Let $x \in \mathcal{F}_I M_1$ such that $\mathcal{F}_I f_1(x) = 0$, i.e. $f_1(x) = 0$. By the injectivity of f_1 this yields $x = 0$ and therefore $\mathcal{F}_I f_1$ is injective. By the same calculations $\operatorname{im}(\mathcal{F}_I f_1) \subseteq \ker(\mathcal{F}_I f_2)$. Now let $a \in \ker(\mathcal{F}_I f_2)$, i.e. $f_2(a) = 0$. By the inclusion $\ker(f_2) \subseteq \operatorname{im}(f_1)$ there exists $b \in M_1$ such that $a = f_1(b)$. By the choice of a there exists $m \in \mathbb{N}$ such that $I^m a = 0$ and consequently $0 = I^m a = f_1(I^m b)$. By the injectivity of f_1 one has $I^m b = 0$ and thus $b \in \mathcal{F}_I M_1$. Thus the claim follows. □

Now let

$$0 \longrightarrow M \longrightarrow P_0 \xrightarrow{f_0} P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} \cdots$$

be a minimal injective resolution of M, apply the functor $\mathcal{F}$, and ignore the term $\mathcal{F}M$. This yields the complex

$$0 \longrightarrow \mathcal{F}P_0 \xrightarrow{\mathcal{F}f_0} \mathcal{F}P_1 \xrightarrow{\mathcal{F}f_1} \mathcal{F}P_2 \xrightarrow{\mathcal{F}f_2} \cdots .$$

The defect of exactness is given by the modules

$$H^j_I(M) := \ker(\mathcal{F} f_{j+1}) / \operatorname{im}(\mathcal{F} f_j)$$

if $j > 0$ and

$$H^0_I(M) := \ker(\mathcal{F} f_1)$$

if $j = 0$. By left exactness of functor $\mathcal{F}\bullet$ one gets the isomorphism $H^0_I(M) \cong \mathcal{F}_I M$. Consider the following example.

(4.15) Example
Let $R := K[\mathbf{x}]$ be the polynomial ring in the variables $x_1, ..., x_n$ and $I := \langle \mathbf{x} \rangle := \langle x_1, ..., x_n \rangle$. The ring R is a domain and by applying Lemma 4.10 the injective hull is given by $\operatorname{Quot}(R)$. Of course $\operatorname{Quot}(R)$ is an R-module and R is a submodule of $\operatorname{Quot}(R)$ and this implies that $\operatorname{Quot}(R)/R$ is divisible. If R is a principal ideal domain it is clear that $\operatorname{Quot}(R)/R$ is injective. In case $n = 1$ a minimal injective resolution is given by

$$0 \longrightarrow R \longrightarrow \operatorname{Quot}(R) \longrightarrow \operatorname{Quot}(R)/R \longrightarrow 0.$$

Furthermore

$$\mathcal{F}_I(R) = \{x \in R \mid \exists n \in \mathbb{N} : I^n x = 0\} = \{0\}$$

because R is a domain and by the same reason one gets $\mathcal{F}_I(K) = \{0\}$. Therefore, one has $H^j_I(R) = 0$ for all $j \neq 1$. If $j = 1$ the result is $H^1_I(R) = \mathcal{F}(\operatorname{Quot}(R)/R)$. Now concentrate on this module:

$$\mathcal{F}(\operatorname{Quot}(R)/R) = \left\{ \left[\frac{r}{s}\right] \in \operatorname{Quot}(R)/R \mid \exists n \in \mathbb{N} : I^n \left[\frac{r}{s}\right] = 0 \right\}.$$

However, $I = \langle x \rangle$ and therefore

$$\mathcal{F}(\operatorname{Quot}(R)/R) = \left\{ \left[\frac{r}{s}\right] \in \operatorname{Quot}(R)/R \mid \exists n \in \mathbb{N} : \left[\frac{x^n r}{s}\right] = 0 \right\}.$$

Now let $[\frac{r}{s}] \neq 0$ and $\gcd(r, s) = 1$. Obviously, the equality $\left[\frac{x^n r}{s}\right] = 0$ means that $x^n r \in \langle s \rangle$ and consequently $s \in I$, i.e. there exist $m \in \mathbb{N}$ and $p \in R$ with $p(0) \neq 0$ such that $s = p \cdot x^m$. Considering the prime factor decomposition p must be a divisor of r. By these considerations one gets $\left[\frac{r}{s}\right] = \left[\frac{t}{x^l}\right]$, $l \in \mathbb{N}_0$ and $t \in R$ appropriate. Of course each element $\frac{a}{b} \in R[x^{-1}]/R$ is an element of $\mathcal{F}(\operatorname{Quot}(R)/R)$ and therefore $\mathcal{F}(\operatorname{Quot}(R)/R) \cong R[x^{-1}]/R$.

(4.16) Remark
If $n > 1$ the polynomial ring R is no principal ideal domain and consequently the calculations presented in 4.15 can not be transferred. However, the module $H_I^n(R)$ is given by
$$H_I^n(R) = R[x_1^{-1}, ..., x_n^{-1}]/R,$$
see [15].

§4.2 Local cohomology and standard bases

In this chapter the objective is to compute standard bases for a special class of ideals by using local cohomology. All results and ideas presented in this chapter derive from the paper *Change of ordering for zeroÂ■dimensional standard bases via algebraic local cohomology classes* written by Katsusuke Nabeshima and Shinichi Tajima and published in 2015 (see [24]).

(4.17) Lemma
Let $R := K[\mathbf{x}]$ and $I := \langle x_1, ..., x_n \rangle$. This implies
$$H_I^n(R) = \lim_{k\to\infty} \operatorname{Ext}_R^k(R/I^k, R) \tag{1}$$
where the limit is the direct limit of the K-vector spaces $\operatorname{Ext}_R^k(R/I^k, R)$. Furthermore, each element can be represented as a finite sum of terms
$$\sum_{\alpha\in\mathbb{N}_0^n} a_\alpha \left[\frac{1}{\mathbf{x}^{\alpha+\mathbf{1}}}\right],$$
$a_\alpha \in K$.

The case $n = 1$ was already addressed in Example 4.15. To compute the modules
$$\operatorname{Ext}_R^k(R/I^k, R)$$
for $n = 1$ consider the following example.

(4.18) Example
Let $R := K[x]$, $I := \langle x \rangle$ and $M_k := R/I^k$. First of all compute $\operatorname{Ext}_R^k(R/I^k, R)$. If $k = 1$ one gets a free resolution of $M_1 = R/I$ by
$$0 \longrightarrow R \xrightarrow{i} R \xrightarrow{\pi} M_1 \longrightarrow 0$$

where i is given by

$$i : R \longrightarrow R, r \mapsto r \cdot x.$$

This yields the complex

$$0 \longrightarrow \mathrm{Hom}(R, R) \xrightarrow{i^*} \mathrm{Hom}(\langle x \rangle, R) \longrightarrow 0.$$

By using the isomorphisms

$$\mathrm{Hom}_R(R, R) \cong R$$

one obtaines the complex

$$0 \longrightarrow R \xrightarrow{\psi} R \longrightarrow 0.$$

where $\psi = i^*$ is given by

$$\psi : K[x] \longrightarrow K[x], p \mapsto x \cdot p.$$

Therefore one has $\mathrm{Ext}^1_R(R/I, R) = R/I$. If $k \neq 1$ one has gets by similar considerations the result $\mathrm{Ext}^k_R(R/I, R) = 0$.

(4.19) Remark
In the following substitute all elements of the form $\left[\frac{1}{\mathbf{x}^\alpha + \mathbf{1}}\right]$ by the polynomial $\mathbf{y}^\alpha$, i.e. the sum $\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \left[\frac{1}{\mathbf{x}^{\alpha+\mathbf{1}}}\right]$ will be transformed to $\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \mathbf{y}^\alpha$. The transforming map will be denoted by Ψ. Moreover, by the structure of the local cohomology which was discussed in Example 4.15 for $n = 1$ one uses the following multiplication:

$$\mathbf{x}^\alpha * \mathbf{y}^\beta = \mathbf{y}^{\beta - \alpha}$$

if $\beta \geq_{\mathrm{cw}} \alpha$ and zero otherwise.

In order to describe the following algorithms the next remark will be useful.

(4.20) Remark
Let $<$ be a monomial ordering on $\mathrm{Mon}(\mathbf{y})$ and let $f = a_{\tilde{\alpha}} \mathbf{y}^{\tilde{\alpha}} + \sum_{\alpha < \tilde{\alpha}} a_\alpha \mathbf{y}^\alpha$ be an element of $H^n_{\langle \mathbf{x} \rangle}(K[\mathbf{x}])$, i.e. $\mathrm{lm}(f) = \mathbf{y}^{\tilde{\alpha}}$, $\mathrm{lc}(f) = a_{\tilde{\alpha}}$. The set of all monomials in f with nonzero coefficients will be denoted as $\mathrm{Term}(f)$ and all lower terms in the representation of f as $\mathrm{lowerterm}(f) \subseteq \mathrm{Term}(f)$. If $F \subseteq K[\mathbf{y}]$ is a non empty finite set of polynomials the set of terms is the union of all terms in the representation of each $f \in F$ and the same holds for the lower terms. Furthermore, let $m := \mathbf{y}^\alpha \in K[\mathbf{y}]$ be a monomial. The set of neighbours of m is given by

$$\mathrm{Neighbour}(m) := \{m \cdot y_i \mid 1 \leq i \leq n\}.$$

Now let $F := \{f_1, ..., f_l\} \subseteq \mathbb{C}[\mathbf{x}]$ be a set of polynomials such that there exists a neighbourhood, with respect to the euclidean norm, U of the origin such that

$$\{u \in U \mid f_i(u) = 0 \ \forall\, 1 \leq i \leq l\} = \{\underline{0}\} \tag{2}$$

Now define the set

$$H_F := \left\{ g \in H^n_{\langle \mathbf{x} \rangle}(K[\mathbf{x}]) \mid g * f_i = 0 \ \forall\, 1 \leq i \leq l \right\}$$

which is a $\mathbb{C}$ vector space. The idea is to compute a generating set of H_F with respect to a local monomial ordering $<$ on $\mathrm{Mon}(\mathbf{y})$ and finally one can use this generating system to compute a standard basis of $\langle F \rangle$ with respect to the inverse monomial ordering $<^{-1}$.

(4.21) Definition
Let $\varnothing \neq F := \{f_1, ..., f_s\} \subseteq K[\mathbf{x}]$. Denote by T a interreduced standard basis of $\mathrm{Term}(F)$ and let M be the set of standard monomials of $\langle T \rangle$. Define the set $\mathrm{MB}(H_F)$ as

$$\mathrm{MB}(H_F) := \Psi(M).$$

The next two results are essential for the algorithm '*LocalCohomology*'.

(4.22) Theorem
Let G be an interreduced standard basis of $\langle F \rangle$, $\varnothing \neq F = \{f_1, ..., f_s\} \subseteq K[\mathbf{x}]$ and F finite, with respect to a local ordering $<$. Moreover let T be the set of standard monomials of $\langle \mathrm{Lead}(G) \rangle$, $\{\mathbf{y}^{\alpha_1}, ..., \mathbf{y}^{\alpha_k}\} = \Psi(T) \setminus \mathrm{MB}(H_F)$ and let $c_{\lambda,\beta}$ denote the coefficient of the monomial $\mathbf{x}^\beta$ in the representation of

$$\mathbf{x}^\gamma + \sum_{\mathbf{x}^\lambda < \mathbf{x}^\gamma} c_{\lambda,\gamma} \mathbf{x}^\lambda.$$

This implies that for each $i \in \{1, ..., k\}$ there exists $\nu_i \in K[\mathbf{y}]$ such that

- $f_j * \psi_i = 0$ for all $j \in \{1, ..., s\}$, where $\psi_i := \mathbf{y}^{\alpha_i} - \sum\limits_{\mathbf{x}^{\alpha_i} \in \mathrm{Term}(G) \setminus \mathrm{Lead}(G)} c_{\lambda,\alpha_i} \mathbf{y}^\lambda + \nu_i$
- $\mathrm{lm}_<(\nu_i) > \alpha_i$.

(4.23) Lemma
Let $\phi \in M_1 := \{\psi_1, ..., \psi_l\} \subseteq K[\mathbf{y}]$ and define $M_2 := \{\tilde{\psi} \in M_1 \mid \tilde{\psi} > \phi\}$. The relation $\mathbf{y}^\alpha \in \mathrm{lowerterm}(\phi)$ implies

$$\text{if } \alpha_i \geq 1 \text{ then } \mathbf{y}^{\alpha - e_i} \in \mathrm{MB}(H_F) \cup \mathrm{Term}(M_2). \tag{3}$$

Algorithm 4: LocalCohomology(F,G,<)

Input: Given $F := \{f_1, ..., f_l\} \subseteq \mathbb{C}[\mathbf{x}]$ fulfilling property 2, a local monomial ordering $<$ and an interreduced standard basis of F with respect to $<$, called G.
Output: A basis of the vector space H_F.
$M :=$ Compute $\mathrm{MB}(H_F)$;
$T :=$ Compute a standard monomial of $\langle \mathrm{Lead}(G) \rangle$;
$S := \varnothing$;
$\Phi := \Psi(T) \setminus M$;
$U :=$ Neighbour$(\Psi(T)) \setminus \Psi(T)$;
LList $:= \varnothing$;
NC $:= \{\mathbf{y}^\alpha \in U \mid \mathbf{y}^\alpha$ fulfill condition 3$\}$;
while $\Phi \neq \varnothing$ **do**
  $\mathbf{y}^\alpha :=$ the minimum of Φ w.r.t. $<^{-1}$;
  $\Phi := \Phi \setminus \{\mathbf{y}^\alpha\}$; $\psi := \mathbf{y}^\alpha - \sum_{\mathbf{y}^\alpha \in \mathrm{Term}(G) \setminus \mathrm{Lead}(G)} b_{\beta,\alpha} \mathbf{y}^\beta$;
  CL $:= \{\mathbf{y}^\gamma \in \mathrm{NC} \mid \mathbf{y}^\gamma > \mathbf{y}^\alpha\} \cup$ LList;
  $\phi := \psi + \sum_{\mathbf{y}^\gamma \in \mathrm{CL}} c_\gamma \mathbf{y}^\gamma$, $c_\gamma \in \mathbb{C}$;
  $f_i * \psi = 0$ for all $1 \leq i \leq l \Rightarrow$ solve the system of linear equations;
  LList $:= LL(\phi) \cup$ LList;
  NC $:=$ NC $\setminus LL(\phi)$;
  NC $:=$ NC $\cup \{\mathbf{y}^\gamma \in$ Neighbour$(LL(\phi)) \mid \mathbf{y}^\gamma$ fulfill condition 3$\}$;
  $S := S \cup \{\phi\}$;
end
return $S \cup M$;

The algorithm will become clear in the next example.

(4.24) Example
Let $f = xy^2 + y^3 + x^2y$ and

$$F := \left\{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\} = \left\{ y^2 + 2xy, 2xy + 3y^2 + x^2 \right\}.$$

Moreover, let $<$ be the negative degree reverse lexicographical ordering. First compute a standard basis G of F. Using Singular a standard basis of F with respect to $<_{ds}$ is $G = \left\{ \underline{2xy} + y^2, \underline{x^2} + 2xy + 3y^2, \underline{y^3} \right\}$. Therefore, one gets the sets:

- $\mathrm{Term}(F) = \{y^2, xy, x^2\}$
- $\mathrm{Lead}(G) = \{xy, x^2, y^3\}$

- Initialization:
  - $M := \{1, y_1, y_2\}$
  - $T := \{1, x, y, y^2\}$
  - $S := \varnothing$
  - $\Phi := \{y_2^2\}$
  - $U := \{y_2^3, y_1 y_2^2\}$
  - LList $:= \varnothing$
  - NC $:= \varnothing$
- **while** $\Phi \neq \varnothing$ **do**
  - $\mathbf{y}^\alpha := y_2^2$
  - $\Phi := \varnothing$
  - CL $= \varnothing$
  - $\phi := y_2^2 + \frac{1}{2} y_1 y_2 + \frac{2}{3} y_1^2$
  - ϕ already satisfies $f_i * \phi = 0$.
- Reinitialization
  - $S := \left\{ y_2^2 + \frac{1}{2} y_1 y_2 + \frac{2}{3} y_1^2 \right\}$

Finally, the result is $\left\{1, y_1, y_2, y_2^2 - \frac{1}{2} y_1 y_2 - 2 y_1^2 \right\}$.

Using a basis of the vector space H_F Katsusuke Nabeshima and Shinichi Tajima present a possibility to compute a reduced standard basis of F. In their paper they provide the following definition.

(4.25) Definition (the transfer)
Let $<$ be a global monomial ordering on $\mathrm{Mon}(\mathbf{y})$ and $A \subseteq H^n_{\langle \mathbf{x} \rangle}(K[\mathbf{x}])$ be a finite subset. Moreover, let $f := \mathbf{y}^{\tilde{\alpha}} + \sum_{\tilde{\alpha} > \alpha} a_{\tilde{\alpha},\alpha} \mathbf{y}^\alpha \in A$, f not a monomial, and $B \subseteq \mathrm{Mon}(\mathbf{y})$. The transfer of $\mathbf{y}^\beta$ is defined as

$$\mathrm{SB}_A(\mathbf{y}^\beta) := \mathbf{x}^\beta - \sum_{\mathbf{y}^\alpha \in \mathrm{Lead}(\mathrm{SL}(A))} a_{\tilde{\alpha},\alpha} \mathbf{x}^\alpha$$

if $\mathbf{y}^\alpha \in \text{lowerterm}(A)$ and

$$\text{SB}_A(\mathbf{y}^\beta) := \mathbf{x}^\beta$$

otherwise.

Now one can compute a standard basis of a given set $F \subseteq K[\mathbf{x}]$ satisfying 2 by using the following algorithm.

Input: Given $F := \{f_1, ..., f_l\} \subseteq \mathbb{C}[\mathbf{x}]$ fulfilling property 2, local monomial orderings $<_1, <_2$ and an interreduced standard basis of F, called G, with respect to $<_1$.
Output: A reduced standard basis of F w.r.t. $<_2$
$\psi :=$ LocalCohomology$(F, G, <_1)$;
compute a vector v which is built by the elements of $\text{Term}(\psi) \setminus \text{MB}(HF)$ ordered w.r.t. $<_2^{-1}$;
$A :=$coefficient matrix of $\text{Term}(\psi) \setminus \text{MB}(HF)$ w.r.t. v;
$B :=$row reduced echelon matrix of A;
$\phi := B\psi$;
$\Psi := \phi \cup \text{MB}(H_F)$;
$\Phi :=$ minimal basis of $\text{Neighbour}(\text{Lead}(\Psi)) \setminus \text{Lead}(\Psi)$ w.r.t. $<_2^{-1}$;
$S := \text{SB}_\Psi(\Phi)$;
return S;

(4.26) Example
Again consider $f = xy^2 + y^3 + x^2y$, $<_1 = <_{ds}$, $<_2 = <_{ls}$,

$$F := \left\{ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\} = \left\{ y^2 + 2xy, 2xy + 3y^2 + x^2 \right\}$$

and

$$G := \left\{ 2xy + y^2, x^2 + 2xy3y^2, y^3 \right\}$$

By using Example 4.24 one has

$$\psi = \left\{ 1, y_1, y_2, y_2^2 - \frac{1}{2}y_1y_2 - 2y_1^2 \right\}.$$

Moreover, $\text{Term}(\psi) \setminus \text{MB}(H_F) = \{y_2^2, y_1y_2, y_1^2\}$ and therefore $v = (y_1^2, y_1y_2, y_2^2)$. Next compute the coefficient matrix.

$$A := \begin{array}{c} \begin{array}{ccc} y_1^2 & y_1y_2 & y_2^2 \end{array} \\ \begin{pmatrix} -2 & -\frac{1}{2} & 1 \end{pmatrix} \end{array}$$

$$B := \begin{array}{c} \begin{array}{ccc} y_1^2 & y_1y_2 & y_2^2 \end{array} \\ \begin{pmatrix} 1 & \frac{1}{4} & -\frac{1}{2} \end{pmatrix} \end{array}$$

$\phi = Bv = (y_1^2 + \frac{1}{4}y_1y_2 - \frac{1}{2}y_2^2)$.
$\Psi = \left\{1, y_1, y_2, y_1^2 + \frac{1}{4}y_1y_2 - \frac{1}{2}y_2^2)\right\}$
$\Phi = \text{Neighbour}(\text{Lead}(\Psi)) \setminus \text{Lead}(\Psi) = \{y_1y_2, y_2^2, y_1^3\}$
$S = \left\{xy - \frac{1}{4}x^2, y^2 + \frac{1}{2}x^2, x^3\right\}$.
By the previous algorithm S is a reduced standard basis of F w.r.t. $<_{ls}$.

(4.27) Remark
The Algorithm 4 and the algorithm 5 has been implemented in the computer algebra system Singular by the author of this thesis, see A.9.

§ 5 Flat embeddings

Most of the ideas in this chapter come from the book *A Singular introduction to commutative algebra* written by Gert-Martin Greuel and Gerhard Pfister, chapter 6 about complete local rings and chapter 7 about flatness (see [13]). Let $K[[\mathbf{x}]]$ be the ring of formal power series. The goal of this chapter is to prove that the embedding $K[\mathbf{x}]_{\langle \mathbf{x} \rangle} \subseteq K[[\mathbf{x}]]$ is faithfully flat, i.e. $K[[\mathbf{x}]]$ is a faithfully flat $K[\mathbf{x}]_{\langle \mathbf{x} \rangle}$-module. First of all, it is helpful to recall some facts concerning the tensor product and $\mathfrak{m}$-adic completions.

§ 5.1 The tensor product

(5.1) Lemma
Let R be a commutative ring and M an R-module. The following statements are equivalent.

- M is flat.
- For all ideals $I \subseteq R$: The map $I \otimes_R M \longrightarrow M$, $r \otimes m \mapsto mr$ is injective.

(5.2) Lemma
Let R be a commutative ring, N an R-module and M a flat R-module, i.e. the functor $F := \bullet \otimes_R M$ is exact. Then the following statements are equivalent:

- $FN = 0 \implies N = 0$ for all N.

- $FN = 0 \implies N = 0$ for all finitely generated N.
- $FN = 0 \implies N = 0$ for all cyclic N.
- M is faithfully flat.

(5.3) Lemma
Let R be a commutative ring, M an R-module and $I \subseteq R$ an ideal. This implies:

$$R/I \otimes_R M \cong M/IM.$$

Proof
Consider the mapping $b : R/I \times M \longrightarrow M/IM,\ ([r]_I, m) \mapsto [rm]_{IM}$.
b is well-defined: Let $r \in I$ and $m \in M$. This implies $rm \in IM$, i.e. $[rm]_{IM} = 0$.
By the universal property of the tensor product there exists a unique R-linear map $\phi_b : R/I \otimes_R M \longrightarrow M/IM,\ [r]_I \otimes m \mapsto [rm]_{IM}$.
ϕ_b is surjective: Let $[m]_{IM} \in M/IM$, $m \in M$. Obviously, $\phi_b([1]_I \otimes m) = [m]_{IM}$.
ϕ_b is injective: $\alpha := \sum\limits_j [r_j]_I \otimes m_j \in R/I \otimes_R M$ such that $\phi_b(\alpha) = \sum\limits_j [r_j m_j]_{IM} = 0$.
Equivalently, this means $\sum\limits_j r_j m_j \in IM$ and therefore there exist $s_k \in I$ and $\widetilde{m}_k \in M$ such that $\sum\limits_j r_j m_j = \sum\limits_k s_k \widetilde{m}_k$. One gets the following calculations:

$$\begin{aligned} \sum_j [r_j]_I \otimes m_j &= \sum_j r_j [1]_I \otimes m_j = \sum_j [1]_I \otimes r_j m_j \\ &= [1]_I \otimes \sum_j r_j m_j = [1]_I \otimes \sum_k s_k \widetilde{m}_k \\ &= \sum_k \underbrace{[s_k]_I}_{=0} \otimes \widetilde{m}_k = 0. \end{aligned}$$

Consequently, the mapping ϕ_b is an isomorphism. □

The next lemma will give a criterion to verify faithfulness.

(5.4) Lemma
Let M be a flat R-module.
M is faithfully flat if and only if $\mathfrak{m}M \subsetneq M$ for all maximal ideals $\mathfrak{m} \subseteq R$.

Proof
First assume that M is faithfully flat and let $\mathfrak{m}$ be a maximal ideal in R. This implies that the module $R/\mathfrak{m} \neq 0$ because $R/\mathfrak{m}$ is a field, by the maximality of $\mathfrak{m}$. The

fact that M is faithfully flat implies that the tensor product $M \otimes R/\mathfrak{m}$ is not zero. By using the previous lemma one concludes that $M \otimes R/\mathfrak{m} \cong M/\mathfrak{m}M$ is not zero. Therefore $\mathfrak{m}M \subsetneq M$.
To prove the other implication one may assume by using Lemma 5.2 that $\widetilde{M}$ is a cyclic module, i.e. $\widetilde{M} = Rq$, $q \in \widetilde{M}$. Only the case $\widetilde{M} \neq 0$ is interesting. Consider the epimorphism

$$\phi : R \longrightarrow \widetilde{M},\ r \mapsto rq.$$

The kernel of ϕ is $\operatorname{ann}_R^{\widetilde{M}}(q) := \operatorname{ann}_R(q)$ which is an ideal in R. The isomorphism

$$R/\operatorname{ann}_R(q) \cong \widetilde{M} \neq 0$$

implies that $\operatorname{ann}_R(q) \subsetneq R$ and, as a result, there exists a maximal ideal $\mathfrak{m} \subsetneq R$ such that $\operatorname{ann}_R(q) \subseteq \mathfrak{m}$. This implies $M \operatorname{ann}_R(q) \subseteq M\mathfrak{m} \subsetneq M$ and one gets

$$M \otimes_R \widetilde{M} \cong M \otimes_R R/\operatorname{ann}_R(q) \cong M/M\operatorname{ann}_R(q) \neq 0.$$

This proves the lemma. □

(5.5) Example
Consider $R = \mathbb{Z}$ and $M = \mathbb{Q}$. Let $I \subseteq \mathbb{Q}$ be an ideal, i.e. $I = \mathbb{Z}b$, $b \in \mathbb{Z}$ appropriate. The mapping $\mathbb{Q} \otimes_{\mathbb{Z}} I \longrightarrow \mathbb{Q}$, $q \otimes r \mapsto q \cdot r$ is injective. To see that let $q_i := \frac{\alpha_i}{\beta_i} \in \mathbb{Q}$ and $bs_i \in \mathbb{Z}$, $s_i \in \mathbb{Z}$ and let

$$b \cdot \sum_i q_i s_i = 0.$$

In the case $b = 0$ the map is of course injective. If $b \neq 0$ one has $b \cdot \sum_i q_i s_i = 0$. Now let $0 \neq \beta := \prod_i \beta_i$ and consequently $\sum_i \gamma_i s_i = 0$, $\gamma_i := \beta q_i \in \mathbb{Z}$. This yields

$$\begin{aligned} \sum_i q_i \otimes bs_i &= \sum_i bq_i \otimes s_i \\ &= \sum_i b\frac{\gamma_i}{\beta} \otimes s_i = \sum_i b\frac{1}{\beta} \otimes \gamma_i s_i \\ &= b\frac{1}{\beta} \otimes \underbrace{\sum_i \gamma_i s_i}_{=0} = 0. \end{aligned}$$

Consequently, the mapping is injective and using Lemma 5.1, this implies that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. However, let $\mathfrak{m} := \mathbb{Z}2$ the maximal ideal generated by the element 2 and let $\frac{r}{s} \in \mathbb{Q}$. Of course, one gets $2 \cdot \frac{r}{2s} = \frac{r}{s}$ and therefore $\mathfrak{m}\mathbb{Q} = \mathbb{Q}$. By the previous Lemma $\mathbb{Q}$ is not a faithfully flat $\mathbb{Z}$-module.

§ 5.2 The $\mathfrak{m}$-adic completion

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M an R-module. To define the completion of M one needs the definition of a stable I-filtration, see Definition 2.1. The following Lemma is of particular importance.

(5.6) Lemma
Let $\{M_n\}_{n\in\mathbb{N}_0}$ and $\{N_n\}_{n\in\mathbb{N}_0}$ be two stable I-filtrations of a S-module M. This implies the existence of an index $k_0 \in \mathbb{N}_0$, such that

$$\begin{aligned} M_{n+k_0} &\subseteq N_n \\ N_{n+k_0} &\subseteq M_n \end{aligned}$$

for all $n \in \mathbb{N}_0$.

Proof
First of all, consider the special stable I-filtration $P_n := I^n M$. By assumption there exists $n_0 \in \mathbb{N}_0$ such that $M_{n+1} = IM_n$ for all $n \geq n_0$. Therefore, $IM_{n_0} = M_{n_0+1}$ and by induction this implies $I^n M_{n_0} = M_{n_0+n}$ for all $n \in \mathbb{N}_0$. This implies further that $M_{n_0+n} = I^n M_{n_0} \subseteq I^n M = P_n$. On the other, hand from the inclusion $IM_n \subseteq M_{n+1}$ one can derive by induction $P_{n+n_0} \subseteq P_n = I^n M \subseteq M_n$ for all $n \in \mathbb{N}_0$. In particular, the lemma is true if one takes a stable I-filtration $\{M_n\}_{n\in\mathbb{N}_0}$ and the special filtration $\{P_n\}_{n\in\mathbb{N}_0}$. That means there exists $l_0 \in \mathbb{N}_0$ such that $N_{n+l_0} \subseteq P_n$ and $P_{n+l_0} \subseteq N_n$. By defining the maximum $m := \max\{n_0, l_0\}$ one may assume that $n_0 = l_0$ without loss of generality. Then one gets the following steps:

$$\begin{aligned} M_{n+2n_0} &= I^{n+n_0} M_{n_0} = I^{n_0} M_{n_0+n} \subseteq I^{n_0} P_n = I^{n+n_0} M = P_{n+n_0} \subseteq N_n \\ N_{n+2n_0} &= I^{n+n_0} N_{n_0} = I^{n_0} N_{n_0+n} \subseteq I^{n_0} P_n = I^{n+n_0} M = P_{n+n_0} \subseteq M_n \end{aligned}$$

for all $n \geq 0$. A possible choice for the index k_0 is $2n_0$ and the claim follows. □

Now consider the following definition.

(5.7) Definition
Let M be an R-module and $\{M_n\}_{n\in\mathbb{N}_0}$ a stable $\mathfrak{m}$-filtration of M. The set

$$\widehat{M}_{M_n} := \left\{ (m_1, m_2, \ldots) \in \prod_{j=1}^{\infty} M/M_j \mid m_k - m_l \in M_k,\ \forall l \geq k \right\}$$

is called the completion of M with respect to $\{M_n\}_{n\in\mathbb{N}_0}$.

Of course, $\widehat{M}$ is an R-module. Due to Lemma 5.6 the definition of $\widehat{M}$ is independent of the stable $\mathfrak{m}$-filtration.

(5.8) Lemma
The R-module $\widehat{M}$ does not depend on the stable $\mathfrak{m}$-filtration.

Proof
Take two $\mathfrak{m}$-filtrations $\{M_n\}_{n\in\mathbb{N}_0}$ and $\{N_n\}_{n\in\mathbb{N}_0}$ of M. By Lemma 5.6 there exists an index $n_0 \in \mathbb{N}_0$ such that $M_{n_0+n} \subseteq N_n$ and $N_{n_0+n} \subseteq M_n$ for all $n \geq n_0$. By using this inclusions the following sequence of canonical mappings is well-defined:

$$M/M_{n+2n_0} \xrightarrow{\pi_{1,n}} M/N_{n+n_0} \xrightarrow{\pi_{2,n}} M/M_n,$$

where $\pi_{1,n}$ and $\pi_{2,n}$ are projections. It is easy to extend this sequence to the sequence

$$\widehat{M}_{M_n} \xrightarrow{\widehat{\pi_1}} \widehat{M}_{N_n} \xrightarrow{\widehat{\pi_2}} \widehat{M}_{M_n},$$

where the mappings $\widehat{\pi_1}$ and $\widehat{\pi_2}$ are given by

$$\left(\widehat{\pi_1}\left((m_1, m_2, ...)\right)\right)_j = \begin{cases} \pi_{1,j-n_0}(m_{j+n_0}), & j \geq n_0+1 \\ \pi_{1,1}(m_{2n_0+1}) (\text{mod } N_j), & j \leq n_0 \end{cases}.$$

and

$$\left(\widehat{\pi_2}\left((n_1, n_2, ...)\right)\right)_j = \pi_{2,j}(n_{n_0+j})$$

respectively. Now compose these two mappings:

$$\left(\widehat{\pi_2}\left(\widehat{\pi_1}\left(m_1, m_2, ...\right)\right)\right)_j = \left[m_{2n_0+j}\right]_{M_j} = \left[m_j\right]_{M_j},$$

since $2n_0 + j \geq j$. On the other hand, one gets:

$$\begin{aligned} \left(\widehat{\pi_1}\left(\widehat{\pi_2}\left(n_1, n_2, ...\right)\right)\right)_j &= \widehat{\pi_1}\left(\left(\pi_{2,1}(n_{n_0+1}), \pi_{2,2}(n_{n_0+2}), ...\right)\right)_j \\ &= \begin{cases} \left[n_{3n_0+1}\right]_{N_j} & , j \leq n_0 \\ \left[n_{2n_0+j}\right]_{N_j} & , j \geq n_0+1 \end{cases}. \end{aligned}$$

Of course, one has $n_j - n_{3n_0+1} \in N_j$ for all $j \leq n_0$ and $n_{2n_0+j} - n_{n_0+1} \in N_j$ for all $j \geq n_0+1$. Therefore, the equalities $\widehat{\pi_1} \circ \widehat{\pi_2} = id_{\widehat{N}}$ and $\widehat{\pi_2} \circ \widehat{\pi_1} = id_{\widehat{M}}$ are verified and the claim follows. □

Because of the previous lemma it is convenient to write $\widehat{M}$ instead of $\widehat{M}_{M_n}$. The following lemma will show that exact sequences remain exact sequences if one considers the sequence of completions.

(5.9) Lemma
Let M, N, P are finitely generated R-modules such that the sequence

$$0 \longrightarrow N \xrightarrow{\alpha} M \xrightarrow{\beta} P \longrightarrow 0$$

is exact. Then the induced sequence

$$0 \longrightarrow \widehat{N} \xrightarrow{\widehat{\alpha}} \widehat{M} \xrightarrow{\widehat{\beta}} \widehat{P} \longrightarrow 0$$

is also exact where

$$\widehat{\alpha} : \widehat{N} \longrightarrow \widehat{M},\ (n_1, n_2, \dots) \mapsto (\alpha(n_1), \alpha(n_2), \dots)$$

and

$$\widehat{\beta} : \widehat{M} \longrightarrow \widehat{P},\ (m_1, m_2, \dots) \mapsto (\beta(m_1), \beta(m_2), \dots).$$

Proof
Without loss of generality let N be a submodule of M and $\alpha(n) = n$, i.e. $\alpha = \mathrm{id}_N$. The set $\left\{\mathfrak{m}^i M\right\}_{i \in \mathbb{N}_0}$ is a stable $\mathfrak{m}$-filtration with respect to the R-module M. By the Lemma of 'Artin Rees' the set $\left\{\mathfrak{m}^i M \cap N\right\}_{i \in \mathbb{N}_0}$ is a stable $\mathfrak{m}$-filtration of N and the inclusion $\mathfrak{m}^i M \cap N \subseteq \mathfrak{m}^i M$ holds anyway. That is why the mappings

$$\phi_j : N/\left(\mathfrak{m}^j M \cap N\right) \longrightarrow M/\mathfrak{m}^j M,\ [n]_{\mathfrak{m}^j M \cap N} \mapsto [n]_{\mathfrak{m}^j M}$$

are well-defined and injective. Moreover, the induced map

$$\psi_j : M/\mathfrak{m}^j M \longrightarrow P/\mathfrak{m}^j P,\ [m]_{\mathfrak{m}^j M} \mapsto [\beta(m)]_{\mathfrak{m}^j P}$$

is well-defined and surjective for all j. To see that ψ_j is well-defined let $m \in \mathfrak{m}^{\mathrm{i}} M$. Then $\beta(m) \in \beta(\mathfrak{m}^{\mathrm{i}} M) = \mathfrak{m}^{\mathrm{i}} \underbrace{\beta(M)}_{=P}$. Furthermore, let $\beta(m) \in \mathfrak{m}^{\mathrm{i}} P$, i.e. there exists $c \in \mathfrak{m}^{\mathrm{i}} M$ such that $\beta(m) = \beta(c)$. This implies:

$$m - c \in \ker(\beta) = \mathrm{im}(\alpha).$$

Therefore, there exists $n \in N$ such that $m - c = n$ and in conclusion

$$[m]_{\mathfrak{m}^{\mathrm{i}} M} \in \mathrm{im}(N/\left(\mathfrak{m}^i M \cap N\right)).$$

On the other hand,

$$\psi_j(\phi_j([n])) = \left[\underbrace{\beta(\alpha(n))}_{=0}\right] = 0.$$

This proves that the sequence

$$0 \longrightarrow N/\left(\mathfrak{m}^j M \cap N\right) \xrightarrow{\phi_j} M/\mathfrak{m}^j M \xrightarrow{\psi_j} P/\mathfrak{m}^j P \longrightarrow 0$$

is exact and, moreover, the induced sequence

$$0 \longrightarrow \widehat{N} \xrightarrow{\widehat{\alpha}} \widehat{M} \xrightarrow{\widehat{\beta}} \widehat{P}$$

is exact because by using the Lemma 5.8 the $\mathfrak{m}$-adic completion does not depend on the stable filtration. The last step is to prove that the mapping $\widehat{\beta}$ is surjective. Let $p = (p_1, p_2, ...) \in \widehat{P}$ and let $m_1, m_2 \in M$ such that $\beta(m_i) - p_i \in \mathfrak{m}^i P$ which exist by the surjectivity of β. By definition $p_1 - p_2 \in \mathfrak{m}P$ and this yields

$$\beta(m_1) - \beta(m_2) = \beta(\tau)$$

for some $\tau \in \mathfrak{m}M$. This implies $m_1 - m_2 - \tau \in \ker(\beta) = \operatorname{im}(\alpha)$ and there exists $n \in N$ such that $m_1 - m_2 = \tau + n$. Now define $m_2' := m_2 + n$. By definition

$$m_2' - m_1 = m_2 - m_1 + n = -\tau \in \mathfrak{m}M$$

and

$$\beta(m_2 + n) - p_2 = \beta(m_2) - p_2 + \underbrace{\beta(\alpha(n))}_{=0 \text{ (exactness)}} \in \mathfrak{m}^2 M.$$

Iterating this procedure one gets an element $m := (m_1, m_2', m_3', ...) \in \widehat{M}$ such that $\widehat{\beta}(m) = p$ holds. □

(5.10) Lemma
The map

$$\begin{aligned} &\Phi : \left(\widehat{R}\right)^n \longrightarrow \widehat{R^n}, \\ &((m_{1,1}, m_{1,2}, ...), ..., (m_{n,1}, m_{n,2}, ...)) \mapsto ((m_{1,1}, ..., m_{n,1}), (m_{1,2}, ..., m_{n,2}), ...) \end{aligned}$$

is an isomorphism.

From this lemma one can deduce a very useful corollary.

(5.11) Corollary
Let M be a finitely generated R-module. This implies the isomorphism

$$M \otimes_R \widehat{R} \cong \widehat{M}.$$

Proof
The R-module M is finitely generated and R is a Noetherian ring, i.e. M is finitely presented. Let $M \cong R^n/AR^m$ and $A \in R^{n\times m}$. That gives the exact sequence

$$R^m \xrightarrow{A\cdot} R^n \xrightarrow{\pi} M \longrightarrow 0,$$

where π is the projection mapping. Then by Lemma 5.9 and keeping in mind that the functor $\bullet \otimes_R \widehat{R}$ is right exact one gets the commutative diagram

$$\begin{tikzcd}
\widehat{R^m} \arrow[r] & \widehat{R^n} \arrow[r] & \widehat{M} \arrow[r] & 0 \\
R^m \otimes_R \widehat{R} \arrow[r] \arrow[u, "\phi_1"] & R^n \otimes_R \widehat{R} \arrow[r] \arrow[u, "\phi_2"] & M \otimes_R \widehat{R} \arrow[r] \arrow[u, "\psi"] & 0
\end{tikzcd}$$

with exact rows. The map ψ is given by

$$\psi : M \otimes_R \widehat{R} \longrightarrow \widehat{M}, m \otimes (r_1, r_2, ...) \mapsto (mr_1, mr_2, ...).$$

The maps ϕ_1 and ϕ_2 are isomorphisms. By diagram chasing this implies that ψ is also an isomorphism. $\square$

(5.12) Theorem
Let M_1, M_2 be two finitely generated R-modules and $f : M_1 \longrightarrow M_2$ be an injective map. The module $\widehat{R}$ is flat as an R-module with respect to finitely generated modules, i.e. the induced map

$$f \otimes id_{\widehat{R}} : M_1 \otimes_R \widehat{R} \longrightarrow M_2 \otimes_R \widehat{R}$$

is injective.

Proof
The sequence

$$0 \longrightarrow M_1 \longrightarrow M_2$$

is exact and by conclusion the sequence

$$0 \longrightarrow \widehat{M_1} \longrightarrow \widehat{M_2}$$

is also exact. Using Corollary 5.11 one gets $\widehat{M_i} \cong M_i \otimes_R \widehat{R}$, $i \in \{1, 2\}$. In other words, the map $f \otimes id_{\widehat{R}}$ is injective. $\square$

The next question which arises is whether the functor $\bullet \otimes_R \widehat{R}$ is faithful. Actually, it is convenient to use Lemma 5.4 and that is why the maximal ideals of the ring $\widehat{R}$ are of interest.

(5.13) Lemma
The ring $\widehat{R}$ is a local ring with maximal ideal

$$\widehat{\mathfrak{m}} := \left\{ (m_1, m_2, ...) \in \widehat{R} \mid m_1 = 0 \right\}.$$

Proof
Let $n := (n_1, n_2, ...) \in \widehat{R}$ such that $n_1 \neq 0$, i.e. $n_1 \notin \mathfrak{m}$. The ring R is local, i.e. n_1 is a unit. Moreover, $n_j - n_1 \in \mathfrak{m}$ for all $j > 1$. The fact $n_j \in \mathfrak{m}$ would imply $n_1 \in \mathfrak{m}$ a contradiction. In conclusion, $n_j \in R^*$ for all $j \geq 1$. Therefore, $n \in \widehat{R}^*$. This implies that $\widehat{R}$ is a local ring with the maximal ideal $\widehat{\mathfrak{m}}$. □

Furthermore, by using Lemma 5.4 the following lemma proves that $\bullet \otimes_R \widehat{R}$ is faithful.

(5.14) Lemma
The extension of $\mathfrak{m}$ with respect to $\widehat{R}$ is not equal to $\widehat{R}$, i.e. $\mathfrak{m}\widehat{R} \subsetneq \widehat{R}$.

Proof
By Lemma 5.13 the maximal ideal of $\widehat{R}$ is equal to

$$\widehat{\mathfrak{m}} := \left\{ (m_1, m_2, ...) \in \widehat{R} \mid m_1 = 0 \right\}.$$

Let $r := (r_1, r_2, ...,) \in \widehat{R}$ and $m \in \mathfrak{m}$. Then:

$$m \cdot r = (\underbrace{mr_1}_{\in \mathfrak{m}}, mr_2, ...) \in \widehat{\mathfrak{m}} \subsetneq \widehat{R} \tag{4}$$

Now let $\delta := \sum_j m_j r_j \in \mathfrak{m}\widehat{R}$, $m_j \in \mathfrak{m}$ and $r_j \in \widehat{R}$. By inclusion (4) it is clear that $\delta \in \widehat{\mathfrak{m}}$ and this implies $\mathfrak{m}\widehat{R} \subseteq \widehat{\mathfrak{m}} \subsetneq \widehat{R}$ and the claim follows. □

As a special case consider $R = K[\mathbf{x}]_{\langle \mathbf{x} \rangle}$, where K is a field. The maximal ideal is $\mathfrak{m} := \langle \mathbf{x} \rangle$.

(5.15) Lemma
The completion of R is given by $K[[\mathbf{x}]]$.

Proof
Consider $f \in R$, i.e. $f = \frac{p}{q}$ with $p, q \in R$ and $q(0) \neq 0$. Without loss of generality the polynomial q has the structure

$$q(x_1, ..., x_n) = 1 - \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} a_\alpha x^\alpha$$

and the set $\{\alpha \mid a_\alpha \neq 0\}$ is finite. By using the geometric series it is clear that the equality

$$\frac{p}{q} = p \cdot \sum_{j=0}^{\infty} \left(\sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} a_\alpha x^\alpha \right)^j$$

is true in $K[[\mathbf{x}]]$. That means that one has the inclusion $K[\mathbf{x}]_{\langle \mathbf{x} \rangle} \subseteq K[[\mathbf{x}]] =: S$. The next step is to construct an isomorphism. Consider the following R-linear map:

$$\Psi : S \longrightarrow \widehat{R}, f(x_1, ..., x_n) = \sum_{\beta \in \mathbb{N}^n} b_\beta x^\beta \mapsto \left(f_0 + \langle \mathbf{x} \rangle, f_0 + f_1 + \langle \mathbf{x} \rangle^2, ... \right),$$

where $b_\beta \in K$, $\left| \{\beta \mid b_\beta \neq 0\} \right| \in \mathbb{N}_0$ and $f_j = \sum_{\beta \in \mathbb{N}^n,\ |\beta| = j} b_\beta x^\beta$. The map Ψ is well-defined, because choosing $j > i$ one observes that

$$\sum_{k=0}^{j} f_k - \sum_{l=0}^{i} f_l = \sum_{\beta \in \mathbb{N}^n,\ i < |\beta| \leq j} b_\beta x^\beta \in \langle \mathbf{x} \rangle^i$$

holds.
Ψ is injective: Let $\Psi(f) = 0$, i.e. one has $\sum_{k=0}^{j} f_k \in \mathfrak{m}^{j+1}$ for all $j \geq 0$. However, $\operatorname{tdeg}(f_k) < j+1$ for all $k \leq j$ and this implies $f_k = 0$ for all $k \leq j$ because

$$\mathfrak{m}^{j+1} = \langle \{x^\gamma \mid |\gamma| = j+1\} \rangle .$$

In conclusion f has to be zero.
Ψ is surjective: Let $p := (p_1 + \mathfrak{m}, p_2 + \mathfrak{m}^2, ...) \in \widehat{R}$. Without loss of generality it is possible to choose $p_i \in K[\mathbf{x}]$ by using the geometric series. Considering equivalence classes it is convenient to represent p as

$$p = (p_{0,0} + \mathfrak{m}, p_{1,0} + p_{1,1} + \mathfrak{m}^2, p_{2,0} + p_{2,1} + p_{2,2} + \mathfrak{m}^3, ...)$$

with tdeg $= j$ for all monomials in the representation of $p_{k,j}$ or $p_{k,j}$ is equal to zero. However, the fact $p \in \widehat{R}$ implies $\underbrace{p_{1,1}}_{\text{tdeg}=1} + \underbrace{p_{1,0}}_{\text{tdeg}=0} - \underbrace{p_{0,0}}_{\text{tdeg}=0} \in \mathfrak{m}$. This implies $p_{0,0} = p_{1,0}$ because $p_{1,1} \in \mathfrak{m}$. By induction it is evident that $p_{k,j} = p_{k+1,j}$ for all k and for all j. Therefore, one can represent p as

$$p = (p_{0,0} + \mathfrak{m}, p_{0,0} + p_{1,1} + \mathfrak{m}^2, p_{0,0} + p_{1,1} + p_{2,2} + \mathfrak{m}^3, ...) = \Psi(\sum_{j=0}^{\infty} p_{j,j}).$$

This proves the claim. □

In conclusion, using the previous lemma, the embedding

$$K[\mathbf{x}]_{\langle \mathbf{x} \rangle} \subseteq K[[\mathbf{x}]]$$

is faithfully flat for all finitely generated $K[\mathbf{x}]_{\langle \mathbf{x} \rangle}$-modules.

(5.16) Lemma
The embedding

$$K[\mathbf{x}]_{\langle \mathbf{x} \rangle} \subseteq K[[\mathbf{x}]]$$

is faithfully flat for all $K[\mathbf{x}]_{\langle \mathbf{x} \rangle}$-modules.

Proof
Let $I \subseteq K[\mathbf{x}]_{\langle \mathbf{x} \rangle} =: R$ be an ideal. The ring R is Noetherian and consequently I is a finitely generated R-module. Of course, R itself is a finitely generated R-module and by Lemma 5.12 one has the injective mapping

$$f \otimes id_{\widehat{R}} : I \otimes_R \widehat{R} \longrightarrow \underbrace{R \otimes_R \widehat{R}}_{\cong \widehat{R}}$$

where f is the natural embedding

$$f : I \longrightarrow R, \; q \mapsto q$$

which is injective. The ideal $I \subseteq R$ was arbitrary and by applying Lemma 5.1 this yields that $\widehat{R}$ is a flat R-module. By using Lemma 5.14 the embedding is faithful. □

This fact will be very important for studying Bernstein-Sato ideals and the Bernstein-Sato polynomial respectively.

§6 The Weyl algebra

This chapter will summarize some important facts about the Weyl algebra, especially the polynomial Weyl algebra (for more see [18]).

(6.1) Definition
Let K be a field. The Weyl algebra is the non-commutative ring

$$\mathcal{D}_n = (K[\mathbf{x}])[\partial_1; id, \frac{\partial}{\partial x_1}] \cdots [\partial_n; id, \frac{\partial}{\partial x_n}].$$

Another common notation is

$$\mathcal{D}_n = K \left\langle \mathbf{x}, \boldsymbol{\partial} \mid \partial_i \partial_j = \partial_j \partial_i, x_i x_j = x_j x_i, \partial_j x_i = x_i \partial_j + \delta(i,j) \right\rangle,$$

where $\delta(i,j)$ denotes the Kronecker-Delta.

The following lemma gives some properties of $\mathcal{D}_n$

(6.2) Lemma

- The set $\mathcal{B} := \left\{ x_1^{\alpha_1} \cdots x_n^{\alpha_n} \cdot \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} \mid \alpha_k, \beta_l \in \mathbb{N}_0 \right\}$ is a K-basis of $\mathcal{D}_n$.
- $\mathcal{D}_n$ is a Noetherian domain.
- If $\mathrm{char}(K) = 0$: $\mathcal{D}_n$ is simple and $\mathrm{Z}(\mathcal{D}_n) = K$.
- $\mathcal{D}_n \cong \underbrace{\mathcal{D}_1 \otimes_K \cdots \otimes_K \mathcal{D}_1}_{n \text{ times}}$

§6.1 Standard bases in $\mathcal{D}_n$

This chapter will always deal with left ideals, i.e. one talks about left division, left standard bases and left S-polynomials. Moreover, the definition of standard basis will be generalized to the non-commutative case. The set of monomials in $\mathcal{D}_n$ is $\mathrm{Mon}(\mathbf{x}, \boldsymbol{\partial})$. Let $<$ be a total ordering on $\mathrm{Mon}(\mathbf{x}, \boldsymbol{\partial})$. Given a polynomial $f \in \mathcal{D}_n$ one can define the leading monomial, the leading coefficient, the leading exponent and the leading term of f as usual.

(6.3) Definition

Let $h_1, h_2, h_3 \in \mathrm{Mon}(\mathbf{x}, \boldsymbol{\partial})$. A monomial ordering $<$ on $\mathrm{Mon}(\mathbf{x}, \boldsymbol{\partial})$ has the following properties:

- $<$ is a total ordering
- $h_1 < h_2 \Longrightarrow \mathrm{lm}_<(h_3 h_1) < \mathrm{lm}_<(h_3 h_2)$
- $h_1 < h_2 \Longrightarrow \mathrm{lm}_<(h_1 h_3) < \mathrm{lm}_<(h_2 h_3)$.

The following lemma presents a monomial ordering with a special property.

(6.4) Lemma

Let $f, g \in \mathcal{D}_n \setminus \{0\}$ and let $<$ be a monomial ordering on $\mathrm{Mon}(\mathbf{x}, \boldsymbol{\partial})$ with the additional property

$$\mathrm{lm}_<(\partial_i x_i) > 1,$$

i.e. $\mathrm{lm}_<(\partial_i x_i) = x_i \partial_i$ for all i. This implies:

$$\mathrm{lm}_<(f \cdot g) = \mathrm{lm}_<(\mathrm{lm}_<(f) \cdot \mathrm{lm}_<(g)).$$

(6.5) Remark
Let $f, g \in \mathcal{D}_n$ such that $\mathrm{lm}_<(f) = \mathbf{x}^{\alpha_1} \cdot \boldsymbol{\partial}^{\beta_1}$ and $\mathrm{lm}_<(g) = \mathbf{x}^{\alpha_2} \cdot \boldsymbol{\partial}^{\beta_2}$. Lemma 6.4 yields

$$\mathrm{lm}_<(f \cdot g) = \mathbf{x}^{\alpha_1+\alpha_2} \cdot \boldsymbol{\partial}^{\beta_1+\beta_2}.$$

Since there exists a one to one correspondence between monomials in $\mathcal{D}_n$ and tuples in $\mathbb{N}_0^{2n}$ a monomial ordering on $\mathcal{D}_n$ can be encoded by a matrix, see Theorem 3.2 for the commutative case. Moreover, the concept of monomial orderings in local Weyl algebras like

$$(\mathcal{D}_n)_{\mathfrak{p}} := K[\mathbf{x}]_{\mathfrak{p}} \left\langle \boldsymbol{\partial} \mid \partial_i f = f\partial_i + \frac{\partial f}{\partial x_i} \right\rangle,$$

where $\mathfrak{p} \subseteq K[\mathbf{x}]$ is a prime ideal, is similar to the commutative case as well.

(6.6) Example
Let $n = 1$ and consider the matrix

$$A_< := \begin{array}{c} \begin{array}{cc} x & \partial_x \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}.$$

This matrix induces a monomial ordering $<$ as in the commutative case. The exponent vector of the monomial $x\partial_x$ is $v_1 := (1,1)$ and $v_2 := (0,0)$ of the monomial 1. This implies:

$$A_< v_1 = (1,1) >_{lp} (0,0) = A_< v_2$$

and therefore $x\partial_x > 1$.

(6.7) Definition
Let

$$0 \neq f = \mathbf{x}^{\alpha_1}\boldsymbol{\partial}^{\beta_1}, g = \mathbf{x}^{\alpha_2}\boldsymbol{\partial}^{\beta_2} \in \mathrm{Mon}(\mathbf{x}, \boldsymbol{\partial})$$

be two monomials. One says that f is left divisible by g if

$$\alpha_2 \leq_{\mathrm{cw}} \alpha_1 \text{ and } \beta_2 \leq_{\mathrm{cw}} \beta_1.$$

A common notation is $g \mid f$.

By using this definition one can define standard bases.

(6.8) Definition
Let $0 \neq I \subseteq \mathcal{D}_n$ be a left ideal and $<$ be a monomial ordering:
A finite, non-empty set G is called a standard basis for I with respect to $<$ if:

- $G \subseteq I$
- $\forall 0 \neq f \in I \; \exists g \in G$ with $\mathrm{lm}_<(g) | \mathrm{lm}_<(f)$.

One possibility to compute standard bases will be presented in the following lemma.

(6.9) Lemma
Let $G \subseteq \mathcal{D}_n$ be a finite subset, $I := \langle G \rangle$ and $<$ be a monomial ordering. Furthermore, assume that one has a normal form algorithm $\mathrm{NF}_<$ to divide a polynomial by a given set of polynomials.

$$G \text{ standard basis of } I \iff \mathrm{NF}_<(\mathrm{spoly}(g,h)|G) = 0 \text{ for all } g, h \in G,\; g \neq h.$$

(6.10) Remark
The method of Lazard introduced in chapter 3 can be modified to compute standard bases in the non-commutative case (see [9]). The technique of homogenization is similar to the commutative case, one, however, has to use a modified relation, i.e.

$$\partial_i x_j = x_j \partial_i + t^2 \cdot \delta(i,j).$$

To finish this section have a closer look at the following example.

(6.11) Example
Let $F := \left\{ x\partial_x^2 + x^2\partial_x, \partial_x^2 + \partial_x \right\} \subseteq \mathcal{D}_1$ and $I := \langle F \rangle \subseteq (\mathcal{D}_1)_{\langle x \rangle}$. Furthermore, choose

$$A_< := \begin{array}{cc} x & \partial_x \end{array} \atop \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and let $<$ be the monomial ordering on $\mathrm{Mon}(x, \partial_x)$ induced by the matrix $A_<$, i.e. the variable x is local and the differential operator ∂_x is global. One can compute a standard basis by using homogenization and the method of Lazard. The homogenization of F looks as follows:

$$F^h = \left\{ x\partial_x^2 + x^2\partial_x, \partial_x^2 + \partial_x \cdot t \right\}.$$

The following code calculates a standard basis of F in Singular.

```
LIB "nctools.lib";
intmat m[3][3]=1,1,1,
               0,1,0,
              -1,0,0;
```

```
ring r=0,(t,x,dx),M(m);
matrix c[3][3];
c[2,3]=t*t;
def S=nc_algebra(1,c);
setring S;
poly f=x*dx*dx+x*x*dx;
poly g=dx*dx+t*dx;
ideal I=f,g;
ideal G=slimgb(I);
//G[1]=dx^2+t*dx
//G[2]=x^2*dx-t*x*dx
//G[3]=2*t^2*x*dx-t^3*dx
//G[4]=t^4*dx
```

By Lazard a standard basis of I is given by

$$\left\{\partial_x^2 + \partial_x, x^2\partial_x - x\partial_x, 2x\partial_x - \partial_x, \partial_x\right\}.$$

Another, obviously more simple, standard basis is $\{\partial_x\}$.

§7 Bernstein-Sato ideals

§7.1 General observations

This chapter will summarize some ideas of the paper "Remarques sur l'idéal de Bernstein associé a des polynômes" written by Joël Briançon and Philippe Maisonobe (see [7]). Let K be a field of characteristic zero and $\mathcal{D}_n$ the n-th polynomial Weyl algebra. This chapter deals with Bernstein-Sato ideals and Bernstein-Sato polynomials respectively.

(7.1) Definition

Let $\mathcal{D}_n[\mathbf{s}] := \mathcal{D}_n[s_1,..,s_m] := \mathcal{D}_n \otimes_K K[s_1,...,s_m]$ and $f_1,...,f_m \in K[x_1,...,x_n]$. Define the set

$$\begin{aligned} I_{\mathbf{f}} &:= I_{f_1,...,f_m} \\ &:= \left\{ b(s_1,...,s_m) \in K[s_1,...,s_m] \mid \exists P \in \mathcal{D}_n[s_1,..,s_m] :\ P \bullet \prod_{i=1}^m f_i^{s_i+1} = b \prod_{i=1}^m f_i^{s_i} \right\}. \end{aligned}$$

This set is an ideal in $K[s_1,...,s_m]$ and is called the global Bernstein-Sato ideal or the global b-function.

Consider a short example.

(7.2) Example
Let $n = 1$ and $f = x^2$. Consider the following calculations:

$$\partial_x \bullet f^{s+1} = (s+1) \cdot f^s \cdot 2x$$
$$\partial_x^2 \bullet f^{s+1} = s(s+1) \cdot f^{s-1} \cdot 4x^2 + 2(s+1) \cdot f^s = (4s(s+1) + 2(s+1)) \cdot f^s.$$

In conclusion, choosing $P := \frac{1}{4}\partial_x^2$ one has:

$$P \bullet f^{s+1} = (s^2 + \frac{3}{2}s + \frac{1}{2})f^s = (s+1) \cdot (s + \frac{1}{2})f^s.$$

This polynomial is monic and the one with smallest total degree having the desired property. Therefore, the global Bernstein-Sato polynomial is $b_f = (s+1) \cdot (s + \frac{1}{2})$.

(7.3) Lemma ([8],[19])
$I_{f_1,\ldots,f_p} \neq 0$.

First, consider the case $m = 1$, i.e. there is one polynomial $f \in K[x_1, ..., x_n]$ and there is one variable s. The symbol f^s is the generator of the module

$$K[x_1, ..., x_n, s_1, ..., s_m, \frac{1}{f}]f^s$$

which is a free

$$K[x_1, ..., x_n, s_1, ..., s_m, \frac{1}{f}]\text{-module.}$$

The module

$$K[x_1, ..., x_n, s_1, ..., s_m, \frac{1}{f}]f^s$$

can also be interpreted as a module over the Weyl algebra by using the following relations:

1. $x_i \bullet g(s,x)f^{s+j} = x_i \cdot g(s,x)f^{s+j}$
2. $s \bullet g(s,x)f^{s+j} = s \cdot g(s,x)f^{s+j}$
3. $\partial_i \bullet g(s,x)f^{s+j} = \frac{\partial g(s,x)}{\partial x_i} f^{s+j} + (s+j)g(s,x)\frac{\partial f}{\partial x_i} f^{s+j-1}$.

The element g is a polynomial in the variables $\mathbf{x}, s$ and the number j is an integer. Especially the case $m = 1$ will be studied in more detail. In this case the polynomial ring $K[s]$ is a principal ideal domain, i.e there exists a unique monic generator of the ideal I_f. This element is called the global Bernstein-Sato polynomial and will be denoted by b_f. The following lemma presents an approach to compute this polynomial.

(7.4) Lemma
With the notations above one has the equation

$$\left(\operatorname{ann}_{\mathcal{D}_n[s]}(f^s) +_{\mathcal{D}_n[s]} \langle f\rangle\right) \cap K[s] = \langle b_f\rangle .$$

Proof
By definition of the global Bernstein-Sato ideal there exists an operator $P \in \mathcal{D}_n[s]$ such that

$$P \bullet f^{s+1} = b_f(s) \cdot f^s.$$

In particular, this implies $(P \cdot f - b_f) \bullet f^s = 0$ and consequently

$$b_f \in \left(\operatorname{ann}_{\mathcal{D}_n[s]}(f^s) +_{\mathcal{D}_n[s]} \langle f\rangle\right) \cap K[s].$$

On the other hand, assume there exist $P, Q \in \mathcal{D}_n[s]$ having the property

$$Q + P \cdot f \in \left(\operatorname{ann}_{\mathcal{D}_n[s]}(f^s) +_{\mathcal{D}_n[s]} \langle f\rangle\right) \cap K[s],$$

i.e. there exists a polynomial $h \in K[s]$ such that $Q + P \cdot f = h$. Now interpret both sides as operators and let them act on the symbol f^s.

$$\begin{aligned} &(Q + P \cdot f) \bullet f^s = h(s) \cdot f^s \\ \Longrightarrow &\underbrace{Q \bullet f^s}_{=0} + P \bullet f^{s+1} = h(s) \cdot f^s \\ \Longrightarrow &P \bullet f^{s+1} = h(s) \cdot f^s. \end{aligned}$$

Concluding $h(s) \in I_f = K[s]\,\langle b(s)\rangle$. This proves the claim. □

This lemma provides a possibility to compute the polynomial b_f in three steps.

- Compute $\operatorname{ann}_{\mathcal{D}_n[s]}(f^s)$.
- Add the polynomial f to $\operatorname{ann}_{\mathcal{D}_n[s]}(f^s)$.
- Eliminate the variables $\mathbf{x}, \boldsymbol{\partial}$ from the ideal

$$\operatorname{ann}_{\mathcal{D}_n[s]}(f^s) + \mathcal{D}_n[s]f.$$

(7.5) Remark
Consider the product $\partial_i \bullet f^{s+1}$. One gets the following computations:

$$\begin{aligned} &\partial_i \bullet f^{s+1} = (s+1)f^s \cdot \frac{\partial f}{\partial x_i} \\ \Longleftrightarrow &(\partial_i f - (s+1)\frac{\partial f}{\partial x_i}) \bullet f^s = 0. \end{aligned}$$

Consequently, the ideal

$$\mathcal{D}_n[s]\left\langle\left\{\partial_i f-(s+1)\frac{\partial f}{\partial x_i}\mid 1\leq i\leq n\right\}\right\rangle$$

is a subset of $\operatorname{ann}_{\mathcal{D}_n[s]}(f^s)$.

(7.6) Remark
Assume that $s=-1$. By the existence of the global Bernstein-Sato polynomial there exists an operator $P\in\mathcal{D}_n[s]$ such that

$$(P\bullet f^{s+1})_{|s=-1}=b_f(-1)\cdot\frac{1}{f}.$$

If $f\in K[\mathbf{x}]\setminus K$ the right hand side has to be zero because it has a non constant denominator and the left hand side has not. Consequently, one gets $b_f(-1)=0$.

So far the the discussion dealt with the global Bernstein-Sato ideal and the global Bernstein-Sato polynomial respectively. Actually, there is a local analogon. Let $\mathfrak{p}\subseteq K[\mathbf{x}]$ be a prime ideal and consider the algebra

$$(\mathcal{D}_n)_{\mathfrak{p}}[\mathbf{s}]=(\mathcal{D}_n)_{\mathfrak{p}}\otimes_K K[\mathbf{s}],$$

with $(\mathcal{D}_n)_{\mathfrak{p}}:=K[\mathbf{x}]_{\mathfrak{p}}\left\langle\boldsymbol{\partial}\mid[\partial_i,f]=\frac{\partial f}{\partial x_i}\right\rangle$. This set is called the geometric localization of the Weyl algebra or the local polynomial Weyl algebra. As described in the global case one gets the following definition.

(7.7) Definition
The set of polynomials in $K[s_1,...,s_m]$

$$\begin{aligned}I_{\mathfrak{p},f_1,\dots,f_p}&:=I_{\mathfrak{p},\mathbf{f}}\\&:=\left\{b(s_1,...,s_m)\in K[s_1,...,s_m]\mid\exists P\in(\mathcal{D}_n)_{\mathfrak{p}}[s_1,..,s_m]:\ P\bullet\prod_{i=1}^m f_i^{s_i+1}=b\prod_{i=1}^m f_i^{s_i}\right\}.\end{aligned}$$

is called the local Bernstein-Sato ideal with respect to $\mathfrak{p}$.

The next lemma gives a relation between the global and the local case.

(7.8) Lemma (Briançon-Maisonobe([7]))

$$I_{f_1,\dots,f_p}=\bigcap_{\mathfrak{p}\in\operatorname{Spec}(K[\mathbf{x}])}I_{\mathfrak{p},f_1,\dots,f_p}=\bigcap_{\mathfrak{m}\text{ maximal}}I_{\mathfrak{m},f_1,\dots,f_p}.$$

Proof
Consider the sets $\mathcal{D}_n[\mathbf{s}]\mathbf{f}^{\mathbf{s}}$ and $\mathcal{D}_n[\mathbf{s}]\mathbf{f}^{\mathbf{s}+1}$ as $K[\mathbf{x}]$-modules. Furthermore, let $\mathfrak{p} \in K[\mathbf{x}]$ be a prime ideal and $b(\mathbf{s}) \in K[\mathbf{s}]$ a polynomial. By using Lemma 2.13 one gets the following isomorphism:

$$\left(b(\mathbf{s})\frac{\mathcal{D}_n[\mathbf{s}]\mathbf{f}^{\mathbf{s}}}{\mathcal{D}_n[\mathbf{s}]\mathbf{f}^{\mathbf{s}+1}}\right)_{\mathfrak{p}} \cong b(\mathbf{s})\frac{(\mathcal{D}_n)_{\mathfrak{p}}[\mathbf{s}]\mathbf{f}^{\mathbf{s}}}{(\mathcal{D}_n)_{\mathfrak{p}}[\mathbf{s}]\mathbf{f}^{\mathbf{s}+1}}.$$

If $b(\mathbf{s}) \in I_{\mathbf{f}}$ then $b(\mathbf{s})\frac{\mathcal{D}_n[\mathbf{s}]\mathbf{f}^{\mathbf{s}}}{\mathcal{D}_n[\mathbf{s}]\mathbf{f}^{\mathbf{s}+1}} = 0$. This implies, by using the above isomorphism, that $b(\mathbf{s})\frac{(\mathcal{D}_n)_{\mathfrak{p}}[\mathbf{s}]\mathbf{F}^{\mathbf{s}}}{(\mathcal{D}_n)_{\mathfrak{p}}[\mathbf{s}]\mathbf{f}^{\mathbf{s}+1}} = 0$ for all prime ideals. In conclusion, $b(\mathbf{s}) \in I_{\mathfrak{p},\mathbf{f}}$ is true for all prime ideals and therefore $b(\mathbf{s}) \in \bigcap_{\mathfrak{p}\in\mathrm{Spec}(K[\mathbf{x}])} I_{\mathfrak{p},\mathbf{f}}$.
On the other hand, let $b(\mathbf{s}) \in \bigcap_{\mathfrak{p}\in\mathrm{Spec}(K[\mathbf{x}])} I_{\mathfrak{p},\mathbf{f}}$, i.e. by using the isomorphism in the other direction one gets $\left(b(\mathbf{s})\frac{\mathcal{D}_n[\mathbf{s}]\mathbf{f}^{\mathbf{s}}}{\mathcal{D}_n[\mathbf{s}]\mathbf{f}^{\mathbf{s}+1}}\right)_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p} \subseteq K[\mathbf{x}]$. Now the Lemma 2.14 says that $b(\mathbf{s})\frac{\mathcal{D}_n[\mathbf{s}]\mathbf{f}^{\mathbf{s}}}{\mathcal{D}_n[\mathbf{s}]\mathbf{f}^{\mathbf{s}+1}} = 0$ and therefore one has $b(\mathbf{s}) \in I_{\mathbf{f}}$. Again by using 2.14 one can replace the interection over all prime ideals by the intersection over all maximal ideals which completes the proof. □

In the case $m = 1$ it is again clear that $K[s]$ is a principal ideal domain. Now assume that $K = \mathbb{C}$ and let $\mathfrak{m}$ be a maximal ideal in $K[\mathbf{x}]$. Obviously, the maximal ideal has the form $\mathfrak{m} =: \mathfrak{m}_a := \langle x_1 - a_1, ..., x_n - a_n \rangle$, where $a := (a_1, ..., a_n) \in \mathbb{C}^n$ is a point. The unique monic generator of the ideal $I_{\mathfrak{m},f}$ will be denoted by $b_{f,a}$. The following lemma is the local analogon to Lemma 7.4.

(7.9) Lemma

$$\left(\mathrm{ann}_{(\mathcal{D}_n)_{\mathfrak{m}_a}[s]}(f^s) +_{(\mathcal{D}_n)_{\mathfrak{m}_a}[s]} \langle f \rangle\right) \cap K[s] = \langle b_{f,a} \rangle .$$

The proof is quite similar to the proof of Lemma 7.4. Similar to the global case, one has the following lemma:

(7.10) Lemma
Let $f \in \mathbb{C}[\mathbf{x}]$ and $a \in \mathcal{V}(f)$. Then:

$$(s+1) \mid b_{f,a}.$$

Proof
There exists an operator $P \in (\mathcal{D}_n)_{\mathfrak{m}_a}[s]$ such that

$$P \bullet f^{s+1} = b_{f,a}(s) f^s. \tag{5}$$

Now let $s = -1$. Multiplying both sides of equation (5) by f and replacing s by -1 yields

$$f \cdot (P \bullet f^{s+1})_{|s=-1} = b_{f,a}(-1).$$

By assumption, $f \notin (\mathbb{C}[\mathbf{x}]_{\mathfrak{m}_a})^*$ and consequently no denominator in the representation of P contains f. Therefore, $b_{f,a}(-1)$ has to be zero. $\square$

(7.11) Remark
Let $a \notin \mathcal{V}(f)$, i.e. $(f \in K[\mathbf{x}]_{\mathfrak{m}_a})^*$. Consider the operator $P := f^{-1}$:

$$P \bullet f^{s+1} = f^s.$$

Therefore, the local b-function in a is given by

$$b_{f,a} = 1.$$

The next theorem contains a very useful result because it describes a relation between the global Bernstein-Sato polynomial and the local Bernstein-Sato polynomials.

(7.12) Theorem (Briançon-Maisonobe (unpublished),Mebkhout-Narváez ([23]))

$$b_f = \operatorname*{lcm}_{a \in \mathrm{Sing}(f)} b_{f,a}.$$

It is proven that the polynomials $b_{f,a}$ are not zero and by the previous theorem the global Bernstein-Sato polynomial is not zero either. Now consider that the coefficients of the differential operators are not fractions but holomorphic functions with coefficients in $\mathbb{C}$.

(7.13) Definition
The ring of holomorphic functions from $\mathbb{C}^n$ to $\mathbb{C}$ will be denoted as $\mathcal{O}_{\mathbb{C}^n}$.

Similarly to the case of fractions one has the following definition.

(7.14) Definition
The algebra

$$\mathcal{D}_{\mathbb{C}^n}[\mathbf{s}] := \mathcal{D}_{\mathbb{C}^n} \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{s}]$$

is the Weyl algebra with holomorphic functions in $\mathbb{C}^n$ as coefficients.
Furthermore, let $a \in \mathbb{C}^n$ be a point. Define

$$\mathcal{D}_{\mathbb{C}^n,a}[\mathbf{s}] := \mathcal{D}_{\mathbb{C}^n,a} \otimes_{\mathbb{C}} K[\mathbf{s}].$$

The coefficients of the differential operators are functions which are holomorphic in a neighbourhood of a with respect to the Euclidean topology.

Of course, it is again possible to define the Bernstein-Sato ideal.

(7.15) Definition
Let $a \in \mathbb{C}^n$ be a point and define the analytical Bernstein-Sato ideal with respect to a:

$$I_{\mathcal{O}_{\mathbb{C}^n},a,\mathbf{f}} := \left\{ b(\mathbf{s}) \in K[\mathbf{s}] \mid b(\mathbf{s})\mathbf{f}^{\mathbf{s}} \in \mathcal{D}_{\mathbb{C}^n,a}[\mathbf{s}]\mathbf{f}^{\mathbf{s}+\mathbf{1}} \right\}.$$

There is an amazing relation between the analytic and the local algebraic situation.

(7.16) Lemma (Briançon-Maisonobe([7])
Let $a \in \mathbb{C}^n$. Then one has the following equality:

$$I_{\mathcal{O}_{\mathbb{C}^n},a,\mathbf{f}} = I_{\mathfrak{m}_a,\mathbf{f}}.$$

Proof
Without loss of generality let $a = 0$. Remembering the section about flat embeddings (see chapter 5), the embedding

$$\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle} \subseteq \mathcal{O}_{\mathbb{C}^n,0}$$

is faithfully flat. Now let $M_1 \longrightarrow M_2 \longrightarrow M_3$ be an exact sequence of left $\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}$-modules. By definition of flatness the sequence

$$M_1 \otimes_{\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}} \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow M_2 \otimes_{\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}} \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow M_3 \otimes_{\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}} \mathcal{O}_{\mathbb{C}^n,0}$$

is also exact. Consider two left $\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}$-modules M_1 and M_2 with $M_1 \subseteq M_2$. One gets the exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_2/M_1 \longrightarrow 0.$$

Therefore, the sequence

$$0 \longrightarrow M_1 \otimes_{\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}} \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow M_2 \otimes_{\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}} \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow M_2/M_1 \otimes_{\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}} \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow 0$$

is also exact. Using the exactness and the homomorphism theorem leads to the isomorphism

$$M_2/M_1 \otimes_{\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}} \mathcal{O}_{\mathbb{C}^n,0} \cong \frac{M_2 \otimes_{\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}} \mathcal{O}_{\mathbb{C}^n,0}}{M_1 \otimes_{\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}} \mathcal{O}_{\mathbb{C}^n,0}}.$$

Choose $M_1 := (\mathcal{D}_n)_{\mathfrak{m}_a}[\mathbf{s}]\mathbf{f}^{\mathbf{s}+\mathbf{1}}$ and $M_2 := (\mathcal{D}_n)_{\mathfrak{m}_a}[\mathbf{s}]\mathbf{f}^{\mathbf{s}}$. In conclusion, there is the isomorphism

$$b(\mathbf{s}) \cdot M_2/M_1 \otimes_{\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}} \mathcal{O}_{\mathbb{C}^n,0} \cong b(\mathbf{s}) \frac{M_2 \otimes_{\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}} \mathcal{O}_{\mathbb{C}^n,0}}{M_1 \otimes_{\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}} \mathcal{O}_{\mathbb{C}^n,0}}. \tag{6}$$

Moreover, the tensor product $M_1 \otimes_{\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}} \mathcal{O}_{\mathbb{C}^n,0}$ is isomorphic to the module

$$\mathcal{D}_{\mathbb{C}^n,a}[\mathbf{s}]\mathbf{f}^{\mathbf{s}+\mathbf{1}}.$$

To visualize this, remember that an element in $K[\mathbf{x}]_{\mathfrak{m}_a}$ is a rational function and the denominator does not vanish in $a \in \mathbb{C}^n$. In particular, this element can be interpreted as an holomorphic function in an appropriate neighbourhood of a. If one considers the module M_2 and gets a similar result. Now let $b(\mathbf{s}) \in I_{\mathcal{O}_{\mathbb{C}^n},a,\mathbf{f}}$, i.e. the module

$$b(\mathbf{s}) \frac{\mathcal{D}_{\mathbb{C}^n,a}[\mathbf{s}]\mathbf{f}^{\mathbf{s}}}{\mathcal{D}_{\mathbb{C}^n,a}[\mathbf{s}]\mathbf{f}^{\mathbf{s}+\mathbf{1}}}$$

is equal to zero. By the isomorphism (6) and the fact that the functor $\mathcal{O}_{\mathbb{C}^n,a} \otimes_{\mathbb{C}[\mathbf{x}]_{\langle \mathbf{x} \rangle}} \bullet$ is faithful it follows that

$$b(\mathbf{s}) \frac{(\mathcal{D}_n)_{\mathfrak{m}_a}[\mathbf{s}]\mathbf{f}^{\mathbf{s}}}{(\mathcal{D}_n)_{\mathfrak{m}_a}[\mathbf{s}]\mathbf{f}^{\mathbf{s}+\mathbf{1}}}$$

is equal to zero which is equivalent to the statement that $b(\mathbf{s}) \in I_{\mathfrak{m}_a,\mathbf{f}}$. The other implication includes the same considerations. □

(7.17) Lemma ([5])
Let $f_1, ..., f_m \in K[[\mathbf{x}]]$ and $u_1, ..., u_m \in K[[\mathbf{x}]]^*$. This implies:

$$I_{\mathbf{f},0} = I_{\mathbf{u}\cdot\mathbf{f},0}.$$

Proof
The key element in this proof is the following consideration:

$$\begin{aligned}&\partial_i \bullet (\mathbf{f}^{\mathbf{s}+\mathbf{1}} \cdot \mathbf{u}^{\mathbf{s}})\\ &= \left(\left(\sum_{j=1}^{m} s_j \frac{\partial u_j}{\partial x_i} u_j^{-1} + \partial_i \right) \bullet \mathbf{f}^{\mathbf{s}+\mathbf{1}} \right) \mathbf{u}^{\mathbf{s}}.\end{aligned}$$

Moreover, let $P = \sum_{\alpha \in \mathbb{N}_0^n} f_\alpha(\mathbf{x}) \boldsymbol{\partial}^\alpha \in K[[\mathbf{x}]] \left\langle \boldsymbol{\partial} \mid \partial_i f = f\partial_i + \frac{\partial f}{\partial x_i} \right\rangle$. Then

$$\begin{aligned}&\partial_j \bullet ((P \bullet \mathbf{f}^{\mathbf{s}+\mathbf{1}})\mathbf{u}^{\mathbf{s}}) = (\partial_j \bullet (P \bullet \mathbf{f}^{\mathbf{s}+\mathbf{1}}))\mathbf{u}^{\mathbf{s}} + (P \bullet \mathbf{f}^{\mathbf{s}+\mathbf{1}}) \cdot (\partial_j \bullet \mathbf{u}^{\mathbf{s}})\\ =&((\partial_j P) \bullet \mathbf{f}^{\mathbf{s}+\mathbf{1}})\mathbf{u}^{\mathbf{s}} + (P \bullet \mathbf{f}^{\mathbf{s}+\mathbf{1}}) \cdot \left(\sum_{k=1}^{m} s_k u_k^{-1} \frac{\partial u_k}{\partial x_j} \right) \mathbf{u}^{\mathbf{s}}.\end{aligned}$$

Now define $P_1 := \partial_j P$ and $P_2 := \sum_{k=1}^{m} s_k u_k^{-1} \frac{\partial u_k}{\partial x_j} P$, then one gets the equality

$$\partial_j \bullet ((P \bullet \mathbf{f}^{\mathbf{s}+\mathbf{1}})\mathbf{u}^{\mathbf{s}}) = ((P_1 + P_2) \bullet \mathbf{f}^{\mathbf{s}+\mathbf{1}}) \cdot \mathbf{u}^{\mathbf{s}}.$$

By an easy induction one gets the inclusion

$$\mathcal{M}_1 := \mathcal{D}_n[s] \bullet (\mathbf{f}^{\mathbf{s}+\mathbf{1}}\mathbf{u}^{\mathbf{s}}) \subseteq (\mathcal{D}_n[s] \bullet \mathbf{f}^{\mathbf{s}+\mathbf{1}})\mathbf{u}^{\mathbf{s}} =: \mathcal{M}_2.$$

To prove the other inclusion consider the following calculations:

$$\begin{aligned}\underbrace{\partial_j \bullet (\mathbf{f}^{\mathbf{s}+\mathbf{1}}\mathbf{u}^{\mathbf{s}})}_{\in \mathcal{M}_1} &= (\partial_j \bullet \mathbf{f}^{\mathbf{s}+\mathbf{1}}) \cdot \mathbf{u}^{\mathbf{s}} + \underbrace{(\partial_j \bullet \mathbf{u}^{\mathbf{s}}) \cdot \mathbf{f}^{\mathbf{s}+\mathbf{1}}}_{\in \mathcal{M}_1}\\ \underbrace{(\partial_j\partial_k) \bullet (\mathbf{f}^{\mathbf{s}+\mathbf{1}}\mathbf{u}^{\mathbf{s}})}_{\in \mathcal{M}_1} &= \partial_j \bullet ((\partial_k \bullet \mathbf{f}^{\mathbf{s}+\mathbf{1}}) \cdot \mathbf{u}^{\mathbf{s}}) + \partial j \bullet ((\partial_k \bullet \mathbf{u}^{\mathbf{s}}) \cdot \mathbf{f}^{\mathbf{s}+\mathbf{1}})\\ &= (\partial_j\partial_k \bullet \mathbf{f}^{\mathbf{s}+\mathbf{1}})\mathbf{u}^{\mathbf{s}} + \underbrace{(\partial_k \bullet \mathbf{f}^{\mathbf{s}+\mathbf{1}}) \cdot (\partial_j \bullet \mathbf{u}^{\mathbf{s}})}_{\in \mathcal{M}_1}\\ &\quad + \underbrace{\partial j \bullet ((\partial_k \bullet \mathbf{u}^{\mathbf{s}}) \cdot \mathbf{f}^{\mathbf{s}+\mathbf{1}})}_{\in \mathcal{M}_1}.\end{aligned}$$

Again by an induction it is obvious that $(P \bullet \mathbf{f}^{\mathbf{s}+\mathbf{1}})\mathbf{u}^{\mathbf{s}} \in \mathcal{M}_1$ for all P, i.e. $\mathcal{M}_2 \subseteq \mathcal{M}_1$.

Now consider the functional equation:

$$P \bullet \mathbf{f}^{\mathbf{s}+\mathbf{1}} = b(\mathbf{s})\mathbf{f}^{\mathbf{s}}.$$

Let u be a unit. Using the result $\mathcal{M}_2 = \mathcal{M}_1$, this implies that there exists an appropriate Q:

$$\begin{aligned} b(\mathbf{s})\mathbf{f}^{\mathbf{s}}\mathbf{u}^{\mathbf{s}} = (P \bullet \mathbf{f}^{\mathbf{s+1}}) \cdot \mathbf{u}^{\mathbf{s}} &= Q \bullet (\mathbf{u}^{\mathbf{s}}\mathbf{f}^{\mathbf{s+1}}) \\ &= (Qu^{-1}) \bullet (\mathbf{fu})^{\mathbf{s+1}}. \end{aligned}$$

This implies $I_{\mathbf{f},0} \subseteq I_{\mathbf{u}\cdot\mathbf{f},0}$. Analogous calculations lead to the other inclusion. □

Consider a short example.

(7.18) Example
Let $f = x^2 + y^2z^2 + z^3 \in K[x,y,z]$ and $u = 1 + x + y + z \in (K[x,y,z]_{\langle x,y,z \rangle})^*$.

$$\frac{\partial f}{\partial x} = 2x$$
$$\frac{\partial f}{\partial y} = 2yz^2$$
$$\frac{\partial f}{\partial z} = 2zy^2 + 3z^2$$

The singular locus is given by

$$\operatorname{Sing}(f) = \{(0,c,0) \mid c \in \mathbb{C}\}$$

and hence $(0,0,0)$ is not an isolated singularity. Using Singular one gets the following results:

```
LIB "dmod.lib";
ring R = 0,(x,y,z),dp;
poly f = x2+y2z2+z3;
bfct(f);
Singular calculates the global Bernstein-Sato polynomial:
//[1]:
//   _[1]=-1
//   _[2]=-4/3
//   _[3]=-3/2
//   _[4]=-5/3
//[2]:
//   3,1,1,1
```

The global Bernstein-Sato polynomial of f is

$$b_f = (s+1)^3 \cdot (s+\frac{4}{3}) \cdot (s+\frac{3}{2}) \cdot (s+\frac{5}{3}).$$

Now let Singular calculate the local Bernstein-Sato polynomial of f in the point $a := (0,0,0)$.

```
LIB "standardweyl.lib";
ring R = 0,(x,y,z),dp;
poly f = x2+y2z2+z3;
bfct(f);
vector v=[0,0,0];
number n=-1;
checkrootlocal(f,v,n);
//Singular checks whether -1 is a root and calculates its multiplicity.
//[1]:
//   1
//[2]:
//   2
```

This procedure can be replicated to check the other roots of the global Bernstein-Sato polynomial. By Lemma 7.12 it is clear that this suffices because $a \in \mathrm{Sing}(f)$. Finally, the result is:

$$b_{f,0} = (s+1)^2 \cdot (s+\frac{4}{3}) \cdot (s+\frac{3}{2}) \cdot (s+\frac{5}{3}).$$

Now apply the same procedure to $u \cdot f$ and check if $-\frac{5}{3}$ is a root.

```
LIB "standardweyl.lib";
ring R = 0,(x,y,z),dp;
poly f = (x2+y2z2+z3)*(1+x+y+z);
bfct(f);
vector v=[0,0,0];
number n=-5/3;
checkrootlocal(f,v,n);
//[1]:
//   1
//[2]:
//   1
```

Repeating this again one gets the same local Bernstein-Sato polynomial.

§7.2 Connections between the global and the local world

There are some important facts that connect the global and the local case. First of all consider the set

$$S := K[\mathbf{x}, s] \setminus \langle \mathbf{x}, s \rangle_{K[\mathbf{x},s]} .$$

This set is a multiplicatively closed set in $\mathcal{D}_n[s]$.

(7.19) Lemma
The set S is a left-Ore set in the n-th polynomial Weyl algebra.

Proof
Let $u \in S$ and $p \in \mathcal{D}_n[s]$. One has to prove that there exist elements $v \in S$ and $q \in \mathcal{D}_n[s]$ such that

$$q \cdot u = v \cdot p.$$

An equivalent assignment is to find $v \in S$ and $q \in \mathcal{D}_n[s]$ such that

$$q = v \cdot p \cdot u^{-1}.$$

Without loss of generality let $u = 1 + f(\mathbf{x}, s)$, $f \in K[\mathbf{x}, s]$ and $f(0) = 0$. Consider the following calculation:

$$\partial_i \cdot \frac{1}{u} = \frac{1}{u} \cdot \partial_i - \frac{1}{u^2} \cdot \frac{\partial f}{\partial x_i}.$$

The partial derivative of f is an element of $K[\mathbf{x}, s]$. Given a general differential operator $\boldsymbol{\partial}^\alpha$, $\alpha \in \mathbb{N}_0^n$, all denominators of the element $\boldsymbol{\partial}^\alpha \cdot \frac{1}{u}$ will have the form u^m where $m \in \mathbb{N}_0$ is bounded by $|\alpha| + 1$. In conclusion, an appropriate choice of v is $u^{|\alpha|+1}$. Thus $v \cdot p \cdot u^{-1} \in \mathcal{D}_n[s]$ which finishes the proof. $\square$

Consider the following example.

(7.20) Example
Let $n = 2$ and $p = \partial_x \partial_y$ and $u = 1 + xy + s$, i.e. $f(x, y, s) = xy + s$. Based on the previous lemma one has to compute

$$\begin{aligned} p\frac{1}{u} &= \partial_x\left(\frac{1}{u}\partial_y - \frac{1}{u^2} \cdot x\right) \\ &= \frac{1}{u}\partial_y\partial_x - \frac{1}{u^2} \cdot y\partial_y - \frac{1}{u^2} \cdot x\partial_x + 2\frac{1}{u^3} \cdot xy - \frac{1}{u^2}. \end{aligned}$$

Now choose $v = u^3$ and the result is $q = u^2\partial_y\partial_x - u \cdot y\partial_y - u \cdot x\partial_x + 2 \cdot xy - u$.

Define the set
$$\tilde{S} := K[x_1, ..., x_n] \setminus \langle x_1, ..., x_n \rangle_{K[x_1,...,x_n]} .$$
In the local case a possible approach computing the local Bernstein-Sato polynomial is to compute $\operatorname{ann}_{\tilde{S}^{-1}\mathcal{D}_n[s]}(f^s)$ which is well-defined using the previous lemma. The next lemma will reduce the problem to the global case, also see [19].

(7.21) Lemma
$\operatorname{ann}_{\tilde{S}^{-1}\mathcal{D}_n[s]}(f^s) = \tilde{S}^{-1} \operatorname{ann}_{\mathcal{D}_n[s]}(f^s)$

Proof
Let $u^{-1}p \in \operatorname{ann}_{\hat{S}^{-1}\mathcal{D}_n[s]}(f^s)$, i.e. $(u^{-1}p) \bullet f^s = 0$. Therefore, there exists $v \in \tilde{S}$ such that
$$v \cdot (p \bullet f^s) = 0.$$
By definition $v \neq 0$ and $p \bullet f^s \in K[\mathbf{x}, s]$, i.e $p \bullet f^s$ is equal to zero. This implies $u^{-1}p \in \tilde{S}^{-1} \operatorname{ann}_{\mathcal{D}_n[s]}(f^s)$. The other inclusion is a similar calculation. □

One step in the algorithm to compute the b-function is the elimination of differential operators. In fact, there is the possibility to eliminate the variables $x_1, ..., x_n$ in the polynomial Weyl algebra.

(7.22) Lemma
Let I be a left ideal in $\mathcal{D}_n[s]$ and $S := \{f \in K[\mathbf{x}, s] \mid f(0) \neq 0\}$. Then the following equality is true:
$$S^{-1}\mathcal{D}_n[s]I \cap S^{-1}K[\mathbf{x}, s] = S^{-1}(\mathcal{D}_n[s]I \cap K[\mathbf{x}, s]).$$

Proof
First of all, let $t^{-1}h \in S^{-1}(\mathcal{D}_n[s]I \cap K[\mathbf{x}, s])$, $h \in \mathcal{D}_n[s]I \cap K[\mathbf{x}, s]$ and $t \in S$. This implies $t^{-1}h \in S^{-1}\mathcal{D}_n[s]I$ and $t^{-1}h \in S^{-1}K[\mathbf{x}, s]$.
On the other hand, let $u^{-1}g \in S^{-1}\mathcal{D}_n[s]I \cap S^{-1}K[\mathbf{x}, s]$, i.e. there exist $r \in I$, v_1, $v_2 \in S$ and $h \in K[\mathbf{x}, s]$ such that
$$u^{-1}g = v_1^{-1}r = v_2^{-1}h \in S^{-1}K[\mathbf{x}, s].$$
Consequently, there can not be a differential operator in the representation of r. Otherwise, this would contradict the fact
$$v_1^{-1}r \in S^{-1}K[\mathbf{x}, s]$$
Therefore, $r \in K[\mathbf{x}, s]$ and that finishes the proof. □

Consider the next example:

(7.23) Example
Let $n = 2$ and $I = \mathcal{D}_2 \langle (1 - xy) \cdot \partial_x, \partial_x + y \rangle$. In order to eliminate the differential operators choose the following ordering.

$$M = \begin{array}{c} \begin{array}{cccc} x & y & Dx & Dy \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ & dp & & \end{pmatrix} \end{array}.$$

Using the computer algebra system Singular a Gröbner basis of I with respect to the ordering $<_M$ is given by

$$\{y, y + Dy\}.$$

Obviously, the reduced Gröbner basis is given by

$$\{y, Dy\}.$$

This shows that $S^{-1}(\mathcal{D}_2 I \cap K[x,y]) = \langle y \rangle$.
On the other hand, take the ordering

$$M = \begin{array}{c} \begin{array}{cccc} x & y & Dx & Dy \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ & ds & & \end{pmatrix} \end{array}.$$

Now the variables x, y are local. The following set is a standard basis of I:

$$\{y, y + Dy\}.$$

The reduced standard basis is given by

$$\{y, Dy\}.$$

Therefore, one gets $S^{-1}\mathcal{D}_2 I \cap S^{-1}K[x,y]) = \langle y \rangle$.

§8 Localization

§8.1 Another approach to compute the local b-function

In the literature on can already find algorithms which compute global Bernstein-Sato respectively local Bernstein-Sato polynomials (see for instance [2],[1],[5],[21]). However, all approaches assume that the variable s is a global variable. In this chapter we will investigate the geometric localization with respect to s. Let $A := K[x_1, ..., x_n] = K[\mathbf{x}]$, where $n \in \mathbb{N}$ and K a field with $\mathrm{char}(K) = 0$, be the polynomial ring in the variables $x_1, ..., x_n$. In most applications the choice will be $K \in \{\mathbb{R}, \mathbb{C}\}$. Moreover, let $a := (a_1, ..., a_n) \in K^n$ be a point which can be associated with the maximal ideal $\mathfrak{m}_a := \langle x_1 + a_1, ..., x_n + a_n \rangle_{K[\mathbf{x}]} \subseteq K[\mathbf{x}]$, if $K = \mathbb{C}$. The fact that $\mathfrak{m}_a$ is a prime ideal leads to the fact that the set $S := \{f \in K[\mathbf{x}] \mid f \notin \mathfrak{m}_a\} = \{f \in K[\mathbf{x}] \mid f(0) \neq 0\}$ is a multiplicatively closed set having the property $1 \in S$. From now on the study of the localization

$$R := S^{-1}K[\mathbf{x}] = \left\{g^{-1}f \mid f, g \in K[\mathbf{x}],\ g(0) \neq 0\right\} = K[\mathbf{x}]_{\mathfrak{m}_a}$$

is an important objective. The elements of the localization R will be the coefficients of the differential operators $\partial_1, ..., \partial_n$. More precisely, one is interested in the algebra

$$\mathcal{D}_{n,a} := R\left\langle \partial_1, ..., \partial_n \mid \partial_i f = f\partial_i + \frac{\partial f}{\partial x_i},\ f \in R \right\rangle$$

which is nothing else but a different presentation of $\mathcal{D}_{\mathfrak{m}_a}$. Next $\mathcal{D}_{n,a}$ will be tensored with the polynomial ring in the variable s and one defines the algebra $\mathcal{D}_{n,a}[s] := \mathcal{D}_{n,a} \otimes_K K[s]$. The following question arises:
Let $I \subseteq \mathcal{D}_{n,a}[s]$ be an ideal. What is $I \cap K[s]$?
At first, one observes that $K[s]$ is a principal ideal domain and, consequently, one has $I \cap K[s] = \{0\}$ or there exists a unique monic generator $b \in K[s]$ having the property $I \cap K[s] = \langle b \rangle_{K[s]}$. Let $b = \prod_{j=1}^{m} (s + \alpha_j)^{\mu_{\alpha_j}(b)}$ be the factorization of b in $\mathbb{C}$, i.e. $\alpha_j \in \mathbb{C}$ and $\mu_{\alpha_j}(b) \in \mathbb{N}$. The following steps will supply a solution
Step 1:
The ring $\mathcal{D}_{n,a}[s]$ is not commutative whilst the ring $K[\mathbf{x}]_{\mathfrak{m}_a}[s]$ is still commutative. For this reason it is natural to eliminate the differential operators $\partial_1, ..., \partial_n$. The variables $\partial_1, ..., \partial_n$ are global, i.e. $\partial_i > 1$, and the variables $x_1, ..., x_n$ are local, i.e. $x_i < 1$. First of all, it would be interesting to compute $I \cap K[\mathbf{x}]_{\mathfrak{m}_a}[s]$. For this purpose it makes sense to choose the following monomial ordering:

$$M := \begin{array}{c} \begin{array}{ccccccccc} x_1 & x_2 & \cdots & x_n & \partial_1 & \partial_2 & \cdots & \partial_n & s \end{array} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 \\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ & & & dp & & & & & \end{pmatrix} \end{array} \in \mathbb{Q}^{(2n+4)\times(2n+1)}.$$

The abbreviation dp denotes the graded reverse lexicographical ordering. By this choice it is obvious that all monomials $x_j^l \partial_i^k$ are global if $k > 0$. Moreover, one has the relation $x_j^l \partial_i^k > s$ if $k > 0$. Thus this monomial ordering provides the opportunity to eliminate the variables $\partial_1, ..., \partial_n$. The first three rows of M are important. The other rows can be different, but the matrix M must have full column rank. In order to compute $I \cap K[\mathbf{x}]_{\mathfrak{m}_a}[s]$ it is possible to use a standard basis algorithm with respect to $<_M$, the monomial ordering which is induced by M. One possibility to compute a standard basis is the method of Lazard introduced in chapter 3.1. Of course, the standard basis has finitely many generators $f_1, ..., f_r$, and one has to filter with respect to the leading monomials to get a generating system concerning $I \cap K[\mathbf{x}]_{\mathfrak{m}_a}[s]$. Now define the ideal

$$J := \langle f_1, ..., f_r \rangle_{K[\mathbf{x}]_{\mathfrak{m}_a}[s]} := \langle F \rangle_{K[\mathbf{x}]_{\mathfrak{m}_a}[s]} .$$

Step 2:
If $I \cap K[s] \neq \{0\}$ there exists $1 \leq i \leq r$ such that

$$\mathrm{lm}_{<_M}(f_i) \in \{s^\gamma \mid \gamma \in \mathbb{N}_0\} := \langle s \rangle .$$

Without loss of generality this element is unique by the standard basis property. However, the following problem still persists:
Let $f_k \in F$ such that $\mathrm{lm}_<(f_k) \in \langle s \rangle$ and $f_k = p(s) + q(x_1, ..., x_n, s)$. By the special choice of the monomial ordering the inclusion

$$I \cap K[s] \subseteq \langle p(s) \rangle_{K[s]}$$

holds.
To see this let $\phi(s) \in K[s]$ and $\phi(s) \notin \langle p(s) \rangle_{K[s]}$. If $\mathrm{tdeg}(\phi) < \mathrm{tdeg}(p)$ the normal form of ϕ with respect to I is not zero because I is a standard basis. Assume that $\mathrm{tdeg}(p) \leq \mathrm{tdeg}(\phi)$ holds and let

$$\phi = a \cdot p + r$$

where $a, r \in K[s]$, $r \neq 0$ and $\mathrm{tdeg}(r) < \mathrm{tdeg}(p)$. By the choice of $<_M$ all monomials of the form $s^m x_l$ where $m \in \mathbb{N}_0$ and $l \in \mathbb{N}$ are local, i.e. $s^m x_l < 1$. Therefore, one has the equality

$$\delta := \mathrm{NF}_{<_M}(\phi(s)|I) = r - a \cdot q(\mathbf{x}, s) \neq 0$$

where $\mathrm{lm}_{<_M}(\delta) = \mathrm{lm}_{<_M}(r)$.Furthermore, $\mathrm{tdeg}(\mathrm{lm}_{<_M}(r)) < \mathrm{tdeg}(p)$ and consequently it is impossible to reduce δ by I. Therefore, $\phi \notin I$.
However, equality is not to be expected.
Choosing the monomial ordering $<_M$ the variable s is global. Another approach is to choose an ordering, treating s as a local variable. This changes the situation because now one has to consider the algebra

$$\mathcal{D}_{n,a}[s]_{\langle s \rangle} := T^{-1}\mathcal{D}_{n,a}[s].$$

The set $T \subseteq \mathcal{D}_{n,a}[s]$ is multiplicatively closed and has the form
$T := \{q \in K[s] \mid q(0) \neq 0\}$. Observe that $1 \in T$. A possible choice concerning the monomial ordering is for example:

$$N := \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & \cdots & x_n & \partial_1 & \partial_2 & \cdots & \partial_n & s \end{array} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 \\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -1 \\ & & & dp & & & & & \end{pmatrix} \end{array} \in \mathbb{R}^{(2n+4)\times(2n+1)}.$$

A standard basis with respect to $<_N$ is again computable by using the method of Lazard. Hereby, one gets a generating system of $I\mathcal{D}_{n,a}[s]_{\langle s\rangle} \cap T^{-1}\left(K[\mathbf{x}]_{\mathfrak{m}_a}[s]\right)$, i.e. elimination of the differential operators is possible. However, it is necessary to have a closer look at $T^{-1}\left(A_{\mathfrak{m}_a}[s]\right)$.

(8.1) Lemma
One has the isomorphism

$$T^{-1}\left(K[\mathbf{x}]_{\mathfrak{m}_0}[s]\right) \cong (S \cdot T)^{-1} K[\mathbf{x}, s]$$

with $S \cdot T := \{f \in K[\mathbf{x}, s] \mid f(x,s) = g(x) \cdot h(s),\ f(0) \neq 0\}$.

Proof
The mapping

$$\psi : T^{-1}\left(K[\mathbf{x}]_{\mathfrak{m}_0}[s]\right) \to (S \cdot T)^{-1} K[\mathbf{x}, s]$$

with

$$\psi\left(\frac{\left(\frac{f}{g}\right)}{h}\right) = \frac{f}{g \cdot h},$$

$f \in K[\mathbf{x}]$, $g \in K[\mathbf{x}] \setminus \mathfrak{m}_0$, $h \in T$, is well-defined and bijective because K is a field implying that $K[\mathbf{x}, s]$ is a domain. □

Instead of calculating the polynomial $b \in K[s]$, it is also possible to verify if a complex number $\beta \in \mathbb{C}$ is a root of b. In Lemma 8.1 the choice of $\mathfrak{m}_0$ is not restrictive because the mapping $x_i \mapsto x_i - a_i$ is bijective. For the same reason, one can set $\beta = 0$. The following theorem gives some important facts by the calculation of $\mathcal{D}_{n,a}[s]_{\langle s\rangle} \cap K[s]_{\langle s\rangle}$.

(8.2) Theorem
Let $I \subseteq \mathcal{D}_{n,a}[s]$ such that

$$I \cap K[s] = K[s]\langle b\rangle$$

where $b \neq 0$. Furthermore, let

$$J := I\mathcal{D}_{n,a}[s]_{\langle s \rangle} \cap T^{-1}\left(A_{\mathfrak{m}_a}[s]\right) = \langle g_1, ..., g_t \rangle_{T^{-1}(A_{\mathfrak{m}_a}[s])}$$

where $g_i = s_i^{-1} r_i \in I\mathcal{D}_{n,a}[s]_{\langle s \rangle}$.
Then one has the following statements:

$$(i)\ b(0) = 0 \Longleftrightarrow J \cap K[s]_{\langle s \rangle} = \left\langle s^{\mu_0(b)} \right\rangle_{K[s]_{\langle s \rangle}}$$

$$(ii)\ b(0) \neq 0 \Longleftrightarrow 1 \in J$$

Proof
Let $b(0) = 0$ and $\mu_0(b) \in \mathbb{N}$ be the multiplicity of the root. By factorization one has

$$b(s) = \prod_{j=1}^{m} (s + \alpha_j)^{\mu_{\alpha_j}(b)} \in J.$$

If one treats b as an element of $K[s]_{\langle s \rangle}$ all factors except $s^{\mu_0(b)}$ are units and consequently $s^{\mu_0(b)} \in J$. The ring $K[s]$ is a principal ideal domain and so is $K[s]_{\langle s \rangle}$. There exists $\frac{f}{g} \in J$, $f, g \in K[s]$ and $g(0) \neq 0$ such that $J \cap K[s]_{\langle s \rangle} = \left\langle \frac{f}{g} \right\rangle_{K[s]_{\langle s \rangle}}$. If $f(0) \neq 0$ the element f is a unit and this implies $1 \in J$. Otherwise, $f(0) = 0$ and there exists $\gamma \in \mathbb{N}$ such that $J = \langle s^\gamma \rangle_{K[s]_{\langle s \rangle}}$. Independently of both cases there exists $\gamma \in \mathbb{N}_0$, with the property $J = \langle s^\gamma \rangle_{K[s]_{\langle s \rangle}}$. By using $s^{\mu_0(b)} \in J$ one concludes that $\gamma \leq \mu_0(b)$. Assume that $\gamma < \mu_0(b)$, i.e. $s^\gamma \in J$. Then there exist elements $\frac{p_1}{q_1}, ..., \frac{p_t}{q_t}$, $p_i \in K[\mathbf{x}, s]_{\langle x_1, ..., x_n \rangle}$ and $q_i \in T$ such that:

$$s^\gamma = \sum_{j=1}^{t} \frac{p_j}{q_j} g_j.$$

Multiplying both sides by $\mathcal{Q} := \prod_{j=1}^{t} q_j \cdot s_j$ one gets

$$\mathcal{Q} \cdot s^\gamma \in I \cap K[s] = \langle b \rangle_{K[s]}. \tag{7}$$

Obviously, one has $\mathcal{Q} \in K[s]$ and $\mathcal{Q} \neq 0$. By using (7) there exists $q \in K[s]$ such that:

$$\mathcal{Q} \cdot s^\gamma = q \cdot b = q \cdot \prod_{j=1}^{m} (s + \alpha_j)^{\mu_{\alpha_j}(b)}.$$

Comparing the prime factor decomposition this implies $\gamma \geq \mu_{\alpha_j}(b)$ which is a contradiction because the assumption was $\gamma < \mu_{\alpha_j}(b)$. Using contraposition of the first statement the second one is obvious. $\square$

Next, one is interested in computing $J \cap K[s]_{\langle s \rangle}$, i.e. the task is:
Given: an ideal $I = \langle f_1, ..., f_r \rangle \subseteq T^{-1}\left(A_{\mathfrak{m}_a}[s]\right)$, $f_i \in A[s]$.
Objective: calculate $I \cap K[s]_{\langle s \rangle}$.
The next lemma will show that it suffices to find a polynomial $f \in I$ with a special structure.

(8.3) Lemma
Let $I \cap K[s]_{\langle s \rangle} \neq \{0\}$ and let $\gamma \in \mathbb{N}_0$ be minimal with the property: There exists $p \in K[\mathbf{x}, s]$ such that

$$(i)\;\; p(0) \neq 0$$
$$(ii)\;\; s^\gamma \cdot p \in I$$

This implies: $I \cap K[s]_{\langle s \rangle} = \langle s^\gamma \rangle K[s]_{\langle s \rangle}$

Proof
By assumption there exists a $\beta \in \mathbb{N}_0$ such that $I \cap K[s]_{\langle s \rangle} = \left\langle s^\beta \right\rangle$. Assume that $\beta > \gamma$. Without loss of generality the element p is monic and its representation looks as follows:

$$p(\mathbf{x}, s) = 1 + \sum_{(\alpha,\delta)\in\mathbb{N}_0^{n+1}\setminus\{0\}} a_{\alpha,\delta} \cdot x^\alpha s^\delta$$

and the set

$$\operatorname{supp}(p-1) := \left\{(\alpha, \delta) \in \mathbb{N}^{n+1} \mid a_{\alpha,\delta} \neq 0\right\}$$

is finite. The fact $p(0) = 1$ implies that $(0, 0) \notin \operatorname{supp}(p - 1)$. The inequality $\beta > \gamma$ implies that there exist $\alpha^* \in \mathbb{N}_0^n$ and $\delta^* \in \mathbb{N}$ such that $a_{\alpha,\delta} \neq 0$. Otherwise, there would not be any monomial in the representation of p such that $\delta > 0$. But this implies $1 \in I$ which is a contradiction. Furthermore, the inequality $\beta > \gamma$ implies that $s^{\beta-1} \cdot p \in I$. One gets the following considerations:

$$\begin{aligned} s^{\beta-1} \cdot p &= s^{\beta-1} \cdot \left(1 + \sum_{\alpha\in\mathbb{N}_0^n, \delta\in\mathbb{N}_0} a_{\alpha,\delta} \cdot x^\alpha s^\delta\right) \\ &= s^{\beta-1} \cdot \left(1 + \sum_{(\alpha,\delta)\in\operatorname{supp}(p-1),\delta=0} a_{\alpha,0} \cdot x^\alpha\right) + s^{\beta-1} \cdot \sum_{(\alpha,\delta)\in\operatorname{supp}(p-1),\delta\neq0} a_{\alpha,\delta} \cdot x^\alpha s^\delta. \end{aligned}$$

Consequently,

$$s^{\beta-1} \cdot \sum_{(\alpha,\delta)\in\operatorname{supp}(p-1),\delta\neq0} a_{\alpha,\delta} \cdot x^\alpha s^\delta \in I$$

because $s^\beta \in I$. One has the relation $s^{\beta-1} \cdot p \in I$ and this yields:

$$s^{\beta-1} \cdot \left(1 + \sum_{(\alpha,\delta)\in\mathrm{supp}(p-1),\delta=0} a_{\alpha,0} \cdot x^\alpha\right) \in I.$$

However, the term in brackets is a unit because one has $p(0) = 1$ by assumption. In conclusion, this implies $s^{\beta-1} \in I$ which is impossible since

$$I \cap K[s]_{\langle s\rangle} = \left\langle s^\beta \right\rangle.$$

Hence $\beta \leq \gamma$ and the equality must take place. □

The result is surprising because the polynomial $p \in K[\mathbf{x}, s]$ does not have to be a unit because applying Lemma 8.1 each unit can be factorized in a term which only depends on $\mathbf{x}$ and another term which only depends on s. The assumptions in Lemma 8.3 are much weaker. From this lemma one can deduce the following corollary.

(8.4) Corollary
Let $I \subseteq \mathcal{D}_{n,a}[s]$ such that

$$I \cap K[s] \neq \{0\}.$$

One gets the following equivalence.

$$s^\gamma \in I \Longleftrightarrow s^\gamma \in \left(I + \langle x_1 s^\gamma, ..., x_n s^\gamma\rangle_{K[\mathbf{x}]_{\langle x_1,...,x_n,s\rangle}}\right).$$

Proof
The implication from left to right is obvious. Let $s^\gamma \in (I + \langle x_1 s^\gamma, ..., x_n s^\gamma\rangle)$, i.e. there exist elements $p_1, ..., p_n \in K[\mathbf{x}, s]_{\langle x_1,...,x_n,s\rangle}$ such that

$$s^\gamma \cdot \left(1 + \sum_{j=1}^{n} p_i x_i\right) \in I. \tag{8}$$

Consequently, there exist polynomials $q_1, ..., q_n \in K[\mathbf{x}, s]$ with the property

$$s^\gamma \cdot \left(\underbrace{1 + \sum_{j=1}^{n} q_i x_i}_{=:p}\right) \in I$$

and $p(0) \neq 0$. To see that let $\mathfrak{p}$ the product of all denominators of the elements p_1. Obviously, one has $\mathfrak{p}(0) \neq 0$. Multiplying equation (8) by $\mathfrak{p}$ yields:

$$s^{\gamma} \cdot \left(\mathfrak{p} + \sum_{j=1}^{n} \underbrace{\mathfrak{p} p_i}_{\in K[\mathbf{x},s]} x_i \right) \in I.$$

However, $\mathfrak{p} \neq 0$ and consequently $\mathfrak{p}$ has a constant term not equal to zero which can be chosen as 1. In conclusion, one gets $s^{\gamma} \in I$ by the previous lemma. □

After eliminating all differential operators it suffices to work with the ideal

$$I + \langle x_1 s^{\gamma}, ..., x_n s^{\gamma} \rangle_{K[\mathbf{x}]_{\langle x_1,...,x_n,s \rangle}}$$

which has a more simple structure. Furthermore, there is a procedure which only checks whether $\beta \in \mathbb{C}$ is a root or not. The disadvantage hereof is that the algorithm does not compute the multiplicity. However, the advantage is that one has an additional assumption which provides an easier structure.

(8.5) Lemma
Let $I = \langle f_1, ..., f_r \rangle \subseteq (\mathcal{D}_n[s])_{\langle \mathbf{x},s \rangle}$ be an ideal and assume that $I \cap K[s] \neq \{0\}$. Then:

$$1 \in (I + \langle s \rangle) \Longleftrightarrow 1 \in I.$$

Proof
The direction "$\Leftarrow$" is clear. Proving the other equation requires again the assumption that $I \cap K[s] \neq \{0\}$, i.e. there exist $a_i, b \in (\mathcal{D}_n[s])_{\langle \mathbf{x},s \rangle}$ such that

$$1 = \left(\sum_i a_i f_i \right) + b \cdot s \tag{9}$$

and there exists a $\gamma \in \mathbb{N}_0$ such that $I \cap K[s] = \langle s^{\gamma} \rangle$. If $\gamma = 0$ the result is clear. So assume that $\gamma \neq 0$. Multiplying equation (9) by $s^{\gamma-1}$ one gets

$$s^{\gamma-1} = \underbrace{\left(\sum_i s^{\gamma-1} a_i f_i \right)}_{\in I} + \underbrace{b \cdot s^{\gamma}}_{\in I}$$

and consequently $s^{\gamma-1} \in I$ which contradicts the choice of γ. Therefore, γ has to be zero. □

Consider the following example.

(8.6) Example

Let $f = x^2yz + y^3z - x^2z^2 + xy^2 + y^3 \in K[x,y,z]$ and $p := (-4,0,0) \in \mathrm{Sing}(f)$. The global Bernstein-Sato polynomial is

$$b_f = (s+1)^2 \cdot (s+\frac{7}{6}) \cdot (s+\frac{5}{6}) \cdot (s+\frac{3}{2}).$$

Now verify whether $-\frac{5}{6}$ is a root of the local Bernstein-Sato polynomial in p.

```
LIB "standardweyl.lib";
poly f=x2yz+y3z-x2z2+xy2+y3;
vector v=[-4,0,0];
number n=-5/6;
checkrootlocal(f,v,n,1);
//[1]:
//   0
//[2]:
//   0
```

In conclusion, $-\frac{5}{6}$ is not a root.

The following algorithm provides a possible procedure to decide whether the equality $b(0) = 0$ holds under the assumption $I \cap K[s] \neq \{0\}$. Let $\mathcal{D}_n$ be the n-th polynomial Weyl algebra.

(8.7) Algorithm

Given:

$I := \langle f_1, ..., f_r \rangle \, \mathcal{D}_{n,a}[s]_{\langle s \rangle}$, $f_i \in \mathcal{D}_n[s]$.

First step:

Compute a standard basis $F := \{g_1, ..., g_t\}$ of the ideal $I \cap T^{-1}\left(A_{\mathfrak{m}_a}[s]\right)$ by using the method of Lazard with respect to the monomial ordering $<_N$.

Second step:

Find $g_l \in F$ with the property $\mathrm{lm}_{<_N}(g_l) \in \langle s \rangle$ with a minimal exponent. In the case that $\mathrm{lm}_{<_N}(g_l) = 1$, one concludes that $1 \in I$ and, as a consequence, $b(0) \neq 0$.

In the case that $\mathrm{lm}_{<_N}(g_l) = s^\gamma$, $\gamma > 0$ one checks whether one has

$$s^\gamma \in K[\mathbf{x}, s]_{\langle x_1,...,x_n,s \rangle}$$

by using a normal form algorithm. If $\mathrm{NF}_{<_N}(s^\gamma | I) = 0$ the algorithm terminates.

If that is not the case one increases the exponent. The algorithm terminates after finitely many steps because there exists a $\beta \in \mathbb{N}$ such that $s^\beta \in I$.

The described algorithm is already implemented in the computer algebra system Singular and will be discussed next. For a given polynomial $f \in \mathbb{C}[\mathbf{x}]$, a point

$$v \in \operatorname{Sing}(f) := \mathcal{V}(f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$$

and a number $\beta \in \mathbb{C}$ the procedure '*checkrootlocal*' computes whether β is a root of the local Bernstein-Sato polynomial in the point v. In the case that β is a root the procedure also returns the multiplicity. Now consider some important steps in the implementation. The algorithm requires four input arguments, a polynomial f, a vector v, a complex number n and a number $m \in \mathbb{C}$. The first step in the algorithm is to compute a Gröbner basis of the ideal $\operatorname{ann}_{\mathcal{D}_n[s]}(f^s)$. For this purpose one has to specify the ring and one uses the following commands:

```
LIB "dmod.lib";
def D=Sannfs(f);
setring D;
LD=groebner(LD);
```

Next, one has to define an algebra with variables $x_1, ..., x_n, dx_1, ..., dx_n, s$ and make sure that the relations of the Weyl algebra are met as well. The variable s commutes with all variables.

```
int nn=nvars(BB);
ring r=0,(x(1..nn),dx(1..nn),s),dp;
  matrix cc[2*nn+1][2*nn+1];
  int bb;
  for(bb=1;bb<=nn;bb++){
    cc[bb,bb+nn]=1;
  }
  def Z=nc_algebra(1,cc);
  setring Z;
```

Now transform all calculations to the origin by using the following commands:

```
map trans;
int j;
for(j=1;j<=nn;j++){
  trans[j]=x(j)+v[j];
```

```
}
for(j=1;j<=nn;j++){
  trans[j+nn]=dx(j);
}
trans[2*nn+1]=s+n;
```

If the user chooses $m = 1$ one wants to test whether the complex number n is a root independent of the multiplicity. The procedure also takes less time. Taking this into consideration, the element s is added to the ideal $I = \mathrm{ann}_{\mathcal{D}_n[s]}(f^s) + \langle f \rangle$.

```
if(m==1){
  I=I,s;
}
```

Now the procedure *'eliminateNC'*eliminates the differential operators.

```
I=eliminateNC(I,nn+1..2*nn);
```

The result is a Gröbner basis that only contains the variables $x_1, ..., x_n, s$. The last step is to find out the multiplicity by adding elements of the form $x_1 s^j, ..., x_n s^j, s^{j+1}$ to the ideal I. For better understanding consider the following example.

(8.8) Example
Consider the polynomial $f = x^2 + y^2 \cdot z^2 + z^3 \in \mathbb{C}[x, y, z]$ and the point $p := (0, 0, 0)$, see Example 7.18. Using the previous algorithm it is possible to compute the local Bernstein-Sato polynomial in p. The global Bernstein-Sato polynomial of f is

$$b_f = (s + 1)^3 \cdot (s + \frac{3}{2}) \cdot (s + \frac{4}{3}) \cdot (s + \frac{5}{3})$$

First, the algorithm checks the root -1. By using Singular a Gröbner basis of the ideal $\operatorname{ann}_{\mathcal{D}_3[s]}(f^s) \subseteq \mathcal{D}_3[s]$ with respect to the monomial ordering dp is given by:

$$
\begin{aligned}
I[1] =& 3xDx + yDy + 2zDz - 6s \\
I[2] =& 2y^2Dy - 2yzDz + 3zDy \\
I[3] =& 3z^3Dx + 2xyDy - 2xzDz \\
I[4] =& yz^2Dx - xDy \\
I[5] =& 2y^2zDx + 3z^2Dx - 2xDz \\
I[6] =& 2yz^3Dz - z^3Dy - 4yz^2s + 2x^2Dy \\
I[7] =& 2y^2z^2Dz - 3xz^2Dx - yz^2Dy - 4y^2zs + 2x^2Dz \\
I[8] =& yz^3Dy + 2z^4Dz - 2x^2yDy + 2x^2zDz - 6z^3s \\
I[9] =& 4z^4Dz^2 + z^3Dy^2 - 4x^2yDyDz + 4x^2zDz^2 + 4yz^2Dys + 6yz^2Dy \\
& - 2x^2Dy^2 - 12z^3Dzs + 14z^3Dz + 4x^2Dz - 32z^2s.
\end{aligned}
$$

In the next step the polynomial f is added to the set of generators and one has to eliminate the differential operators. Again using Singular a generating system of $(I + \langle f \rangle) \cap K[x,y,z,s] =: J$ is:

$$
\begin{aligned}
& y^2z^2 + z^3 + x^2, \\
& xs, \\
& yz^2s, \\
& 2y^2zs + 3z^2s, \\
& z^3s, \\
& 3z^2s^2 + 2z^2s, \\
& 4y^3s^2 + 6yzs^2 + yzs, \\
& 6yzs^3 + 7yzs^2 + 2yzs, \\
& 12y^2s^3 + 8y^2s^2 + 18zs^3 + 15zs^2 + 2zs, \\
& 18zs^4 + 27zs^3 + 13zs^2 + 2zs, \\
& 18ys^5 + 27ys^4 + 13ys^3 + 2ys^2, \\
& 18s^6 + 27s^5 + 13s^4 + 2s^3.
\end{aligned}
$$

It is known that the multiplicity of -1 is at least one. Therefore, the algorithm adds the elements xs, ys, zs, s^2 to check whether the multiplicity is equal to one. Now Singular computes a standard basis of $J + \langle xs, ys, zs, s^2 \rangle$ with respect to the ordering

ds. The result is:

$$\begin{aligned} &x^2 + z^3 + y^2 z^2, \\ &xs, \\ &ys, \\ &zs, \\ &s^2. \end{aligned}$$

Consequently, the multiplicity is greater than one because

$$s \notin J + \left\langle xs, ys, zs, s^2 \right\rangle.$$

Now the algorithm computes a standard basis of $J + \langle xs^2, ys^2, zs^2, s^3 \rangle$, i.e. it verifies if the multiplicity is two. Now the result is

$$\begin{aligned} &x^2 + z^3 + y^2 z^2, \\ &xs, \\ &ys, \\ &zs, \\ &s^2, \end{aligned}$$

which proves that the multiplicity is equal to two because

$$s^2 \in J + \left\langle xs^2, ys^2, zs^2, s^3 \right\rangle.$$

which differs from the global multiplicity.

At the end of this subsection, it might be interesting to see that the localization at the variable s is natural. To see this one has to consider the '*subcentral character decomposition*', see [19]. Let $f \in K[\mathbf{x}]$ be a polynomial,

$$b_f = \prod_{i=1}^{k} (s - \alpha_i)^{\mu(\alpha_i)}$$

be the b-function and

$$M := \frac{\mathcal{D}_n[s]}{\operatorname{ann}_{\mathcal{D}_n[s]}(f^s) + \mathcal{D}_n[s] f}$$

be a $\mathcal{D}_n[s]$-module. Consider the exact sequence

$$0 \longrightarrow \frac{\mathcal{D}_n[s]}{\operatorname{ann}_{\mathcal{D}_n[s]}(f^{s+1})} \xrightarrow{\cdot f} \frac{\mathcal{D}_n[s]}{\operatorname{ann}_{\mathcal{D}_n[s]}(f^s)} \xrightarrow{\pi} M \longrightarrow 0.$$

In [19, THEOREM 3.5.18.] it is proven that

$$\operatorname{ann}_{\mathcal{D}_n[s]}(M) = \mathcal{D}_n[s]b_f(s)$$

and M has a decomposition of the form

$$M = \bigoplus_{\chi_\alpha, \alpha \in C(M)} M^{\chi_\alpha}$$

where

$$\chi_\beta : K[s] \to K,\ s \mapsto \beta$$

for $\beta \in K$,

$$C(M) := \left\{\beta \in K \mid b_f(\beta) = 0\right\}$$

and

$$M^\chi := \left\{m \in M \mid \forall p \in K[s]\ \exists k \in \mathbb{N} : (q - \chi(q))^k \cdot m = 0\right\} \subseteq M$$

which is a $\mathcal{D}_n[s]$-submodule of M.

(8.9) Lemma
Let $\beta \in \mathbb{C}$ and moreover let

$$S_\beta^{-1} := \{f \in K[s] \setminus \{0\} \ \mid f(\beta) \neq 0\} = K[s] \setminus \langle s - \beta \rangle .$$

Then:

$$S_\beta^{-1}M = \begin{cases} S_\beta^{-1}M^{\chi_\beta}, & \beta \in C(M) \\ 0, & \beta \notin C(M). \end{cases}$$

Proof
Let $\beta, \alpha \in \mathbb{C}$ and $\alpha \neq \beta$. Furthermore, let $m \in M^{\chi_\alpha}$, i.e. there exists $k \in \mathbb{N}$ such that

$$\langle s - \alpha \rangle^k m = 0.$$

In conclusion, this yields

$$\left\langle (s-\alpha)^k \right\rangle \subseteq \operatorname{ann}^M_{K[s]}(m)$$

and hence an exact sequence

$$0 \longrightarrow \left\langle (s-\alpha)^k \right\rangle \longrightarrow \operatorname{ann}^M_{K[s]}(m).$$

However, Localization is exact and consequently

$$S_\beta^{-1} \left\langle (s-\alpha)^k \right\rangle \subseteq S_\beta^{-1} \operatorname{ann}^M_{K[s]}(m) = \operatorname{ann}^{S_\beta^{-1}M}_{S_\beta^{-1}K[s]}(1^{-1}m).$$

By assumption, one has $\alpha \neq \beta$ and therefore the element $s - \alpha$ is a unit in $S_\beta^{-1}K[s]$ and hence

$$S_\beta^{-1}M^{\chi_\alpha} = 0.$$

Now let $\beta \in C(M)$. This yields the following calculations:

$$\begin{aligned} S_\beta^{-1}M &= S_\beta^{-1} \bigoplus_{\chi_\alpha, \alpha \in C(M)} M^{\chi_\alpha} \\ &= \bigoplus_{\chi_\alpha, \alpha \in C(M)} S_\beta^{-1}M^{\chi_\alpha} = S_\beta^{-1}M^{\chi_\beta}. \end{aligned}$$

If $\beta \notin C(M)$ one gets by similar calculations

$$S_\beta^{-1}M = \bigoplus_{\chi_\alpha, \alpha \in C(M)} \underbrace{S_\beta^{-1}M^{\chi_\alpha}}_{=0} = 0$$

and the claim follows. □

(8.10) Remark
In the situation of Lemma 8.9 one has the equation

$$S_\beta^{-1}M = S_\beta^{-1}M^{\chi_\beta}$$

if $\beta \in C(M)$. Therefore, one has

$$\operatorname{ann}_{S_\beta^{-1}K[s]}(S_\beta^{-1}M) = \operatorname{ann}_{S_\beta^{-1}K[s]}(S_\beta^{-1}M^{\chi_\beta}).$$

It is proven that $\operatorname{ann}_{K[s]}(M^{\chi_\beta}) = K[s]\left\langle (s-\beta)^{\mu(\beta)} \right\rangle$, see [19]. Therefore, one has

$$S_\beta^{-1}K[s]\left\langle (s-\beta)^{\mu(\beta)} \right\rangle \subseteq \operatorname{ann}_{S_\beta^{-1}K[s]}(S_\beta^{-1}M).$$

It is not clear whether the other inclusion is true.

§8.2 Localization in an algebraic, non-rational point

The reason for dealing with algebraic localization is illustrated in the following example.

(8.11) Example
Consider the polynomial

$$f := (x^2 + \frac{9}{4} \cdot y^2 + z^2 - 1)^3 - x^2 \cdot z^3 - \frac{9}{80} \cdot y^2 \cdot z^3.$$

The partial derivatives of f are given by

$$\begin{aligned}
\partial_x f &= (6x \cdot (x^2 + \frac{9}{4} \cdot y^2 + z^2 - 1)^2 - 2xz^3 \\
\partial_y f &= \frac{27}{2} \cdot (x^2 + \frac{9}{4} \cdot y^2 + z^2 - 1)^2) - \frac{9}{40} yz^3 \\
\partial_z f &= 6z \cdot (x^2 + \frac{9}{4} \cdot y^2 + z^2 - 1)^2 - 3x^2z^2 - \frac{27}{80} y^2z^2.
\end{aligned}$$

By calculation, the relation $p := (\sqrt{\frac{1}{19}} \cdot i, \sqrt{\frac{80}{171}}, 0) \in \mathrm{Sing}(f)$ is true. Furthermore, if $z = 0$ one has the ellipse

$$x^2 + \frac{9}{4} \cdot y^2 + z^2 - 1 = 0$$

and consequently p is not an isolated singularity. Now it would be worthwhile to compute the local Bernstein-Sato polynomial in p. The difficulty now is to calculate with algebraic, non-rational numbers. In this chapter the objective is to develop an algorithm computing the local Bernstein-Sato polynomial in such cases. Finally, it will be possible to prove that the local Bernstein-Sato polynomial of f in p is

$$b_{f,p} = (s+1)^2 \cdot (s + \frac{2}{3}) \cdot (s + \frac{4}{3}) \cdot (s + \frac{5}{3}).$$

Actually, all components of a point are complex numbers. However, it is possible treating them like variables. Some computer algebra systems like Singular can not directly compute with non-rational, algebraic numbers. Consider some important points in the following example.

(8.12) Example
Let $f = i^2 \cdot x + x + y \in \mathbb{Q}[i, x, y]$. One wants to simulate a calculation with the complex number i, i.e. one has the relation $i^2 + 1 = 0$. Perform some computations with Singular.

```
ring r=0,(i,x,y),dp;
poly f=i^2*x+x+y;
leadmonom(f);
//i^2x
```

Of course, that computation does not simulate the calculation with i. If one takes $f \in \mathbb{C}[x,y]$ the leading monomial is y. Consider the following calculations:

```
ring r=0,(i,x,y),dp;
ideal I=i^2*x+x+y,i^2+1;
std(I);
//_[1]=y
//_[2]=i2+1
```

Consequently, one has in mind a possible strategy to simulate the calculation with i by adding the minimal polynomial to a given ideal. Obviously, one needs a compatible monomial ordering which guarantees that the leading monomial of $f \in \mathbb{Q}[i,\mathbf{x}]$ is equal to the leading monomial of $f \in \mathbb{Q}(i)[\mathbf{x}]$.

In the following let $K[\mathbf{x},\mathbf{a}]$ the polynomial ring in the variables $x_1,...,x_n,a_1,...,a_m$ and $K_{\mathbf{a}}[\mathbf{x}]$ the polynomial ring in the variables $x_1,...,x_n$ with parameteres $a_1,...,a_m$ which satisfy algebraic relations. Moreover, let $f_{\mathbf{a}}$ denotes a polynomial where $\mathbf{a}$ are parameters and $f := f(\mathbf{a},\mathbf{x})$ a polynomial where $\mathbf{a}$ are variables. In order to get a compatible monomial ordering consider the following lemma.

(8.13) Lemma
Let $f_{\mathbf{a}} \in K_{\mathbf{a}}[\mathbf{x}]$, i.e. $a_1,...,a_m$ are parameters. The monomial ordering $<_M$ induced by the matrix

$$M := \begin{array}{c} \begin{array}{cc} \mathbf{x} & \mathbf{a} \end{array} \\ \begin{pmatrix} X & 0 \\ 0 & A \end{pmatrix} \end{array},$$

where $<_A$ a global monomial ordering has the properties:

- $a_i > 1$ for all i
- $(\mathrm{lm}_{<_M}(f))_{\mathbf{a}} = \mathrm{lm}_{<_X}(f_{\mathbf{a}})$.

Proof
By assumption the monomial ordering $<_A$ is a global ordering, i.e. all variables are global and therefore $a_i > 1$ for all i. To prove the second statement let

$$f_{\mathbf{a}} = \sum_{\mu \in \mathbb{N}_0^n} c_\mu \mathbf{x}^\mu,$$

where $c_\mu \in K_{\mathbf{a}}$. Denote the leading monomial by $\mathbf{x}^{\mu^*}$ and the leading coefficient by $c^*_{\mathbf{a}}$. By the block structure of the matrix M one first compares the part which only depends on $\mathbf{x}$. Consequently,

$$\mathbf{x}^{\mu^*} >_X \mathbf{x}^{\mu} \tag{10}$$

for all $\mu \in \mathbb{N}_0^n$ such that $c_\mu \neq 0$. In conclusion, if one treats the parameters a_i as variables one has

$$(\mathrm{lm}_{<_M}(f))_{\mathbf{a}} = \mathbf{x}^{\mu^*}.$$

□

Obviously, the leading term can be totally different because the inequality (10) is not strict. In the following let $R \subseteq K[\mathbf{a}]$ be the ideal which contains all relations of the given algebraic numbers, i.e. the generators of R are the minimal polynomials.

(8.14) Remark
The ideal $R \subseteq K[\mathbf{a}]$ is zero dimensional and a prime ideal by construction. As an example let $a(1) := i$ and $a(2) := \sqrt{2}$, $K = \mathbb{Q}$. As mentoined above the ideal R looks as follows.

$$R := \left\langle a(1)^2 + 1, a(2)^2 - 2 \right\rangle \subseteq K[a(1), a(2)]$$

This ideal is of course a prime ideal and zero-dimensional, since

$$\dim_K(K[a(1), a(2)]/R) = 4.$$

Obviously the ring $K_{\mathbf{a}}$ is nothing other than the field $K[\mathbf{a}]/R$.

By using the previous lemma one gets the following result.

(8.15) Lemma
Let $I \subseteq K[\mathbf{x}, \mathbf{a}]$ an ideal which contains the ideal R. Let G be a standard basis of $I = \langle G \rangle$ with respect to $<_M$. This implies that $G_{\mathbf{a}}$ is a standard basis of $I_{\mathbf{a}} \subseteq K_{\mathbf{a}}[\mathbf{x}]$ with respect to $<_X$.

Proof
Let $G = \{g_1, ..., g_t\}$. By assumption the set G is a standard basis, i.e. for all $f \in I$ there exists $g \in G$ such that $\mathrm{lm}_{<_M}(g) | \mathrm{lm}_{<_M}(f)$. One has to show that for all $f \in I_{\mathbf{a}}$ there exists $g \in G_{\mathbf{a}}$ such that $\mathrm{lm}_{<_X}(g) | \mathrm{lm}_{<_X}(f)$. Let $0 \neq f \in I_{\mathbf{a}}$, i.e. $f \notin R$ and let $\mathrm{lm}_{<_X}(f) = \mathbf{x}^{\mu^*}$ the leading monomial and $\mathrm{lc}_{<_X}(f) = b_{\mathbf{a}}$ the leading coefficient. Furthermore, denote by $\mathbf{a}^{\nu^*}$ the leading monomial in the representation of $b_{\mathbf{a}}$ with respect to $<_A$. By assumption $\mathbf{a}^{\nu^*} \notin R$. All generators of R are irreducible because all relations of algebraic numbers are given by their minimal polynomial. In conclusion,

$K_{\mathbf{a}}$ is a field and consequently the element $\mathbf{a}^{\nu^*}$ is invertible. Moreover, the inverse has a representation in $K_{\mathbf{a}}$ and therefore the element f is also an element of I, since $R \subseteq I$. One may assume without loss of generality that $\mathrm{lc}_{<_M}(f)$ is equal to one, since all elements except zero have an inverse with a polynomial representation and $R \subseteq I$. For this reason, by the choice of $<_M$ and the fact that G is a standard basis of I there exists a $g \in G$ such that $\mathrm{lm}_{<_M}(g) \in \mathrm{Mon}(\mathbf{x})$ and $\mathrm{lm}_{<_M}(g)|\mathbf{x}^{\mu^*}$. All variables $a_1, ..., a_m$ are global and by Lemma 8.13 $\mathrm{lm}_{<_X}(g_{\mathbf{a}}) = \mathrm{lm}_{<_M}(g)$. This implies that there exists a $g \in G_{\mathbf{a}}$ such that $\mathrm{lm}_{<_X}(g)|\,\mathrm{lm}_{<_X}(f)$. This finishes the proof. □

Consider the following example.

(8.16) Example
Let $f = (x^2 + y)^3 \cdot x^2 \in K[x, y]$. Using Singular one computes $\mathrm{ann}_{\mathcal{D}_n[s]}(f^s)$ in the following way:

```
LIB "dmod.lib";
ring l=0,(x,y),dp;
poly f=(x^2+y)^3*x^2;
def D=Sannfs(f);
setring D;
LD=groebner(LD);
LD;
//LD[1]=x*Dx+2*y*Dy-8*s
//LD[2]=x^2*Dy+y*Dy-3*s
//LD[3]=2*x*y*Dy^2-y*Dx*Dy-8*x*Dy*s+3*Dx*s
//LD[4]=4*y^2*Dy^3+y*Dx^2*Dy+8*x*Dx*Dy*s
//       -16*y*Dy^2*s+6*y*Dy^2-3*Dx^2*s+8*Dy*s
```

Next, one has to construct the tensor algebra.

```
ring r=0,(a,b),dp;
ring rr=0,(x,y,Dx,Dy,s),dp;
matrix mm[5][5];
mm[1,3]=1;mm[2,4]=1;
def A=nc_algebra(1,mm);
def B=A+r;
setring B;
B;
//   characteristic : 0
```

```
//   number of vars : 7
//        block   1 : ordering dp
//                  : names    x y Dx Dy s
//        block   2 : ordering dp
//                  : names    a b
//        block   3 : ordering C
//   noncommutative relations:
//    Dxx=x*Dx+1
//    Dyy=y*Dy+1
```

One has

$$\frac{\partial f}{\partial x} = 6(x^2+y)^2 \cdot x^3 + 2x \cdot (x^2+y)^3$$
$$\frac{\partial f}{\partial y} = 3(x^2+y)^2 \cdot x^2.$$

Now, $(\sqrt{2}i)^2 + 2 = 0$ and hence point $p := (\sqrt{2}i, 2) \in \mathrm{Sing}(f)$. Moreover, point p is not an isolated singularity because each point $(\alpha_1, \alpha_2) \in \mathbb{C}^2$ that satisfies $x^2 + y = 0$ is an element of $\mathrm{Sing}(f)$. By using Lemma 8.15 it is possible to compute the local Bernstein-Sato polynomial in p calculating only with polynomials. The global Bernstein-Sato polynomial is

$$b_f = (s+\frac{1}{3}) \cdot (s+\frac{1}{2}) \cdot (s+\frac{2}{3}) \cdot (s+1)^2.$$

Next, the procedure *'eliminateNC '* eliminates the differential operators $\partial_1, ..., \partial_n$. The result is a generating system of $I \cap \mathbb{Q}[\sqrt{2}][x,y]$. First, calculate the multiplicity of $s = -1$.

```
poly f=imap(l,f);
ideal I=imap(D,LD),f,a^2+1,b^2-2;
map trans =B,x+a*b,y+2,Dx,Dy,s-1,a,b;
I=trans(I);
I=eliminateNC(I,3..4);
ring R=0,(x,y,s),ds;
ring S=0,(a,b),dp;
def T=R+S;
setring T;
ideal I=imap(B,I);
ideal J=I,x,y,s;
```

```
std(J);
//_[1]=a2+1
//_[2]=b2-2
//_[3]=x
//_[4]=y
//_[5]=s
J=I,xs,ys,s^2;
std(J);
//_[1]=a2+1
//_[2]=b2-2
//_[3]=s
//_[4]=64x3-48x2yab+12xy2a2b2+2y3ab-112x4ab+72x3ya2b2+30x2y2ab
      -2xy3a2b2+76x5a2b2+78x4yab-12x3y2a2b2-x2y3ab+50x6ab
      -18x5ya2b2-3x4y2ab-8x7a2b2-3x6yab-x8ab
```

The previous considerations reveal that the multiplicity is one. Repeating this procedure with the other candidates the local Bernstein-Sato polynomial in p is

$$b_{f,p} = (s+\frac{1}{3})\cdot(s+\frac{2}{3})\cdot(s+1).$$

(8.17) Corollary
Let $f \in \mathbb{Q}[\mathbf{x}]$ and $p \in \mathcal{V}(f) \subseteq \overline{\mathbb{Q}}^n$. Let $I \subseteq \mathbb{Q}[\mathbf{x}]$ be the maximal ideal such that $p \in \mathcal{V}$. Then for all $q \in \mathcal{V}(I)$ one has

$$b_{f,q}(s) = b_{f,p}(s).$$

(8.18) Remark
The Corollary 8.17 induces a certain symmetry. For a better understanding consider again the polynomial

$$f := (x^2+\frac{9}{4}\cdot y^2+z^2-1)^3 - x^2\cdot z^3 - \frac{9}{80}\cdot y^2\cdot z^3 \in \mathbb{Q}[x,y,z].$$

By Example 8.11 it is known that $p := (\sqrt{\frac{1}{19}}\cdot i, \sqrt{\frac{80}{171}}, 0) \in \mathrm{Sing}(f)$. The ideal containing all important relations is represented by

$$R := \left\langle a(1)^2+\frac{1}{19} =: \mu_1, a(2)^2-\sqrt{\frac{80}{171}} =: \mu_2 \right\rangle \subseteq \mathbb{Q}[a(1),a(2)].$$

Furthermore, the number of solutions of R, i.e. the number of solutions of the system

$$\mu_1 = 0$$
$$\mu_2 = 0,$$

is equal to 4. That is why, one has four symmetric points which are located on the ellipse defined by the equation

$$x^2 + \frac{9}{4}y^2 = 1.$$

In order to determine those points one has to compute the roots of μ_1 respectively μ_2 and the result is

$$\begin{aligned} p_1 &:= (\sqrt{\frac{1}{19}} \cdot i, \sqrt{\frac{80}{171}}, 0) \\ p_2 &:= (-\sqrt{\frac{1}{19}} \cdot i, \sqrt{\frac{80}{171}}, 0) \\ p_3 &:= (\sqrt{\frac{1}{19}} \cdot i, -\sqrt{\frac{80}{171}}, 0) \\ p_4 &:= (-\sqrt{\frac{1}{19}} \cdot i, -\sqrt{\frac{80}{171}}, 0) \end{aligned}$$

.

In general one has to consider the zero-dimensional prime ideal mentioned in Corollary 8.17 which describes all relations. Consequently, the number of symmetric points is equal to the number of solutions.

Consider another example in Singular.

(8.19) Example
Let $f = (x^3 + 3y^2)^2$ and let $\alpha := \sqrt[3]{3}$. Computing the partial derivatives yields

$$\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2(x^3 + 3y^2) \\ \frac{\partial f}{\partial y} &= 6y(x^3 + 3y^2). \end{aligned}$$

This implies

$$p_1 := (\alpha, i), p_2 := (-\frac{\alpha}{2} + \frac{\sqrt{3}}{2}\alpha i, i) \in \mathrm{Sing}(f).$$

Moreover, point p_1 is not an isolated singularity because each point $(\beta_1, \beta_2) \in \mathbb{C}^2$ that satisfies $x^3 + 3y^2 = 0$ is an element of $\mathrm{Sing}(f)$. The global Bernstein-Sato polynomial is

$$b_f = (s+1)(s + \frac{5}{12})(s + \frac{1}{2})(s + \frac{7}{12})(s + \frac{11}{12})(s + \frac{13}{12}).$$

By using Singular, the procedure '*checkrootalgebraic*', and the library '*standardweyl.lib*' one gets the following computations.

```
ring l=0,(x(1),x(2),a(1),a(2)),dp;
poly f=(x(1)^3+3*x(2)^2)^2;
number n=-5/12;
vector v=[a(1),a(2)];
vector w=[a(1)^3-3,a(2)^2+1];
checkrootalgebraic (f,v, n, w);
//[1]:
//   0
//[2]:
//   0
```

Checking the other roots in the same way the local Bernstein-Sato polynomial is given by

$$b_{f,p_1} = b_{f,p_2} = (s+1)(s+\frac{1}{2}).$$

The number of symmetric points is equal to the number of solutions of

$$R := \left\langle x^2+1, y^3-3 \right\rangle \subseteq K[x,y]$$

which is equal to 6.

§9 Conclusion

The last chapter presents a possibility to compute the local b-function by localizing the variable s. The main objective was to show that this localization is a natural one, especially see 8.9. Moreover, this thesis provides an algorithm to realize the calculations with algebraic, non-rational numbers only by calculating with polynomials and by choosing a special monomial ordering, see chapter 8.2. Of course, there are other ways to realize those calculations which are more sophisticated. The approach presented in this thesis only provides an alternative solution and is realized in the computer algebra system Singular. Furthermore, by applying this alternative solution one has a nice symmetry condition discussed in Remark 8.18 and Example 8.19. Unfortunately, all considerations in chapter 8 are restricted to the case $K[s]$, i.e. only one variable. If one has to work with Bernstein-Sato ideals it is still possible to treat the variables $s_1, \ldots, s_m$ as local ones. However, the property 'principal ideal domain' is lost and consequently Lemma 8.3 is inapplicable.

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§A Procedures

The following procedures are written in the computer algebra system Singular and they need the Singular libraries "ncpreim.lib","nctools.lib","dmod.lib","primdec.lib".

(A.1) Remark
Furthermore, one needs the library 'standardweyl.lib' which is not published yet. However, the library will be available soon.

(A.2) Algorithm (checkrootlocal)
The procedure '*checkrootlocal*' decides whether a rational number is a root of the local b-function in p where p is a vector which contains rational entries, see 8.1.

```
proc checkrootlocal (poly f, vector p, number n, number #)
"USAGE: checkrootlocal (f,p,n,m);
        f a polynomial, p a vector, n a rational number, # a number
ASSUME: f a polynomial in the polynomial ring K[x1,...,xn],
@*      K the field of rational numbers
        p a vector with rational entries which describes the
        localization in p
@*      n a rational number which is a canditate for the root
        of the local b-function in p
        # a number which should be different from 0
        if the user is not interested in the multiplicity
RETURN: list of integers
@*      The first entry of the list is 1 if n is a root and 0 if
        n is not a root of the local b-function in p
@*      If # is not equal to 0 the second entry is 0 because
        the multiplicity is not of interest
@*      If # is equal to 0 the second entry is 0 if n is not a root.
        Otherwise the second entry is the root's multiplicity
PURPOSE: Decides whether a rational number is a root of the local
         b-function in p and computes the multiplicity
"
```

(A.3) Example

```
ring R = 0,(x,y,z),dp;
poly f = 400*(x2y2+y2z2+x2z2)+(x2+y2+z2-1)^3;
vector v=[1,0,0];
```

```
number n=-1;
checkrootlocal(f,v,n,0);
//[1]:
//   1
//[2]:
//   1
```

Consequently $s = -1$ is a root with multiplicity 1.

```
ring R = 0,(x,y,z),dp;
poly f = 400*(x2y2+y2z2+x2z2)+(x2+y2+z2-1)^3;
vector v=[1,0,0];
number n=-1/2;
checkrootlocal(f,v,n,1);
//[1]:
//   0
//[2]:
//   0
```

Consequently $s = -\frac{1}{2}$ is not a root, also see 7.18.

(A.4) Algorithm (checkrootalgebraic)
The procedure '*checkrootalgebraic*' decides whether a rational number is a root of the local b-function in p where p is a vector which contains polynomials in the ring

$$\mathbb{Q}[a_1, ..., a_m],$$

see 8.2. Furthermore, the procedure requires a vector w which contains polynomials in the ring

$$\mathbb{Q}[a_1, ..., a_m].$$

The entries represent the minimal polynomials which are needed to simulate calculations with algebraic numbers. For a better understanding consider the following example.

(A.5) Example
Assume that $p := (\sqrt{2} \cdot i, \sqrt{3})$, i.e. one has three algebraic, non-rational numbers. Therefore one has to define the polynomial ring

$$\mathbb{Q}[a_1, a_2, a_3]$$

where a_1 represents $\sqrt{2}$, a_2 represents i and a_3 represents $\sqrt{3}$. The minimal polynomials of $\sqrt{2}$,i and $\sqrt{3}$ are given by

$$\mu_{\sqrt{2}} := a_1^2 - 2$$
$$\mu_i := a_2^2 + 1$$
$$\mu_{\sqrt{3}} := a_3^2 - 3.$$

Therefore, one has

$$w = [a_1^2 - 2, a_2^2 + 1, a_3^2 - 3]$$

and

$$p = [a_1 \cdot a_2, a_3].$$

```
proc checkrootalgebraic
     (poly f, vector p, number n, vector w, number #)
"USAGE: checkrootalgebraic (f,p,n,w,m);
        f a polynomial, p a vector, n a rational number,
        w a vector, m a number
ASSUME: f a polynomial in the polynomial ring K[x1,...,xn],
        K the field of rational numbers
@*      p a vector with polynomial entries which describes the
        localization in p
@*      n a rational number which is a candidate for a root
        of the local b-function in p
@*      w a vector which contains algebraic relations which
        simulate the calculations with algebraic numbers
        over K
        the ideal generated by these elements should be
        zero-dimensional and prime over K
@*      # a number which should be different from 0
        if the user is not interested in the multiplicity
RETURN: list of integers
@*      The first entry of the list is 1 if n is a root and 0 if
        n is not a root of the local b-function in p
@*      If # is not equal to 0 the second entry is 0 because
        the multiplicity is not of interest
@*      If # is equal to 0 the second entry is 0 if n is not a root.
        Otherwise the second entry is the root's multiplicity
PURPOSE: Decides whether a rational number is a root of the local
```

```
         b-function in p and computes the multiplicity
"
```

(A.6) Example

```
ring l=0,(x(1),x(2),x(3),a(1),a(2)),dp;
poly f=(x(1)^2+9/4*x(2)^2+x(3)^2-1)^3-x(1)^2*x(3)^3
        -9/80*x(2)^2*x(3)^3;
number n=-1;
vector v=[a(1),a(2),0];
vector w=[19*a(1)^2+1,171*a(2)^2-80];
checkrootalgebraic(f,v,n,w,0);
//[1]:
//   1
//[2]:
//   2
```

Consequently $s = -1$ is a root with multiplicity 2.

(A.7) Algorithm (Moradivision)
The following algorithm '*Moradivision*' computes a weak normal form of a polynomial f in the local Weyl algebra

$$D := \mathbb{Q}[\mathbf{x}]_{\mathfrak{m}_a} \langle \boldsymbol{\partial} \rangle$$

where $a \in \mathbb{Q}^n$ is a point and

$$\mathfrak{m}_a = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$$

with respect to an ideal, based on [12]. Let $f \in D$ be a polynomial and $I := \langle f_1, ..., f_l \rangle \subseteq D$ a left ideal. A standard representation of f with respect to I has the structure

$$uf = \sum_{i=1}^{l} a_i f_i + h$$

where u is a unit in D, $a_i \in D$ represent the linear combination and h is a weak normal form, i.e it can not be reduced by the ideal I. The monomial ordering is given by the following matrix:

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & \ldots & x_n & \partial_1 & \partial_2 & \ldots & \partial_n \end{array} \\ \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 \\ -1 & -1 & \ldots & -1 & 0 & 0 & \ldots & 0 \\ & & & & dp & & & \end{pmatrix} \end{array}$$

If $a \neq 0$ one has to shift the variables $x_1, ..., x_n$ to the origin.

```
proc Moradivision (poly f,ideal I)
"USAGE:  Moradivision (f,I); f a polynomial and I a left ideal
ASSUME:  f a polynomial in the local Weyl algebra and
@*       I a left ideal in the local Weyl algebra
@*       K the field of rational numbers
RETURN:  list of polynomials
PURPOSE: The first entry of the list is a weak normal form of f
         with respect to I
@*       The second entry is an ideal where the generators
         represent the linear combination in the
         standard representation of f
@*       The last entry is the unit in the
         standard representation of f
"
```

(A.8) Example

```
intvec w1=0,0,1,1;
intvec w2=-1,-1,0,0;
ring rr=0,(x(1),x(2),dx(1),dx(2)),(a(w1),a(w2),dp);
matrix e[4][4];
e[1,3]=1;e[2,4]=1;
def A=nc_algebra(1,e);
setring A;
poly f=x(1)*dx(1)+x(1)^2*dx(1)-x(2)-x(1)*x(2);
poly g=x(2);
ideal I=f,g;
poly h=x(1)*dx(1)+x(2);
Moradivision(h,I);
//[1]:
//   0
//[2]:
//   _[1]=1
//   _[2]=2+2*x(1)
//[3]:
//  1+x(1)
```

(A.9) Algorithm (LocalCohomology)

The algorithm '*LocalCohomology*' computes a basis of the vector space $\mathrm{MB}(H_F)$, see 4.2. The monomial ordering should be local and F should fulfill condition 2.

```
proc LocalCohomology (ideal F)
```

```
"USAGE: LocalCohomology (F); F an ideal
ASSUME: F is an ideal in the polynomial ring K[x1,...,xn]
@*      K the filed of rational numbers
@*      0 has to be an isolated root of V(<F>)
@*      The monomial ordering should be local
RETURN: ideal
PURPOSE: The procedure LocalCohomology computes a basis
         of the vector space H_{F} w.r.t. a local
         monomial ordering
```

(A.10) Example

```
 ring r=0,(x,y),ds;
 poly f=x^3*y+x*y^4+x^2*y^3;
 ideal I=diff(f,x),diff(f,y);
 LocalCohomology(I);
//_[1]=-4x3-2/3x2y+xy3
//_[2]=-1/3x2y+y4
//_[3]=x3-1/3x2y2+y5
//_[4]=14/33x3+4/3x4+5/33x3y-14/99x2y2-1/3x2y3+7/33xy4+y6
//_[5]=y3
//_[6]=xy2
//_[7]=y2
//_[8]=xy
//_[9]=y
//_[10]=x2
//_[11]=x
//_[12]=1
```

Eigenständigkeitserklärung

Hiermit versichere ich, dass ich die Masterarbeit selbständig und lediglich unter Benutzung der angegebenen Quellen und Hilfsmittel verfasst habe.

Ich versichere außerdem, dass die vorliegende Arbeit noch nicht einem anderen Prüfungsverfahren zugrunde gelegen hat.

Aachen, den