Lyapunov functions

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$$\dot{x} = \mathbf{f}(t, \mathbf{x}) \tag{1}$$

Definition

- The equilibrium solution $\mathbf{x}(t) \equiv 0$ of (1) is *stable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\mathbf{x}_0$ satisfies $||\mathbf{x}_0|| < \delta$ then $||\mathbf{x}(t, \mathbf{x}_0)|| < \epsilon$ for all $t \geq t_0$.
- The equilibrium sulution $\mathbf{x}(t) \equiv 0$ of (1) is *asymptotically stable* if it is stable and if there exists $\delta_1 > 0$ such that if $\mathbf{x}_0$ satisfies $||\mathbf{x}_1|| < \delta_1$ then $\lim_{t\to\infty} \mathbf{x}(t, \mathbf{x}_0) = 0$.
- An equilibrium of (1) is *unstable* if it is not stable.

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A Lyapunov function is any continuous function $V : G \to \mathbb{R}$, such that $V(t, 0) \equiv 0$ for all $t \in [t_0, +\infty]$

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$\forall V \in C^1(G)$ we define the derivative with respect to the system:

$$D(V) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\dot{\mathbf{x}} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, \mathbf{x})$$

Definition

• Lyapunov function $V(t, \mathbf{x})$ is positive defined (PD) if $V(t, \mathbf{x}) \geq 0$ on G; Lyapunov function $V(t, \mathbf{x})$ is negative defined (ND) if $V(t, \mathbf{x}) \leq 0$ on G.

• A Lyapunov function $V(\mathbf{x})$ is *strict positive definite* (SPD) if $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$.

• A Lyapunov function $V(t, \mathbf{x})$ is *strict positive definite* if there exists a SPD Lyapunov function $W(\mathbf{x})$ such that $V(t, \mathbf{x}) \geq W(\mathbf{x})$ on G.

• A Lyapunov function $V(t, \mathbf{x})$ is *strict negative definite* (SND) if the function $-V(t, \mathbf{x})$ is SPD.

Lemma

Let the Lyapunov function $V(t, \mathbf{x})$ be SPD (or SND). If $||x|| \leq a_1 < a$, then

$$||x|| \to \mathbf{0} \Leftrightarrow V(t, \mathbf{x}) \to 0.$$

Lyapunov stability theorem

If there exists a SPD Lyapunov function $V(\mathbf{x}) \in C^1(G)$ such that $D(V)$ is ND, then the solution $\mathbf{x}(t) \equiv 0$ of system (1) is stable.

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Lyapunov asymptotic stability theorem

If there exists a SPD Lyapunov function $V(\mathbf{x}) \in C^1(G)$ such that $D(V)$ is SND, then the solution $\mathbf{x}(t) \equiv 0$ of system (1) is asymptotically stable.

Stability of linear systems

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad A \ \text{ is } n \times n \text{ matrix}$$

Lemma

Let $U(\mathbf{x})$ be a (S)ND quadratic form. If all eigenvalues of the matrix A have negative real parts, then the differential equation

$$\frac{\partial V}{\partial \mathbf{x}} A\mathbf{x} = U(\mathbf{x})$$

has a unique solution $V(\mathbf{x})$ and V is a (S)PD quadratic form.

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Corollary

If all eigenvalues of the matrix A have negative real parts, then the solution $\mathbf{x} \equiv 0$ is asymptotically stable.

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{g}(t, \mathbf{x}),$$

$$\frac{||\mathbf{g}(t, \mathbf{x})||}{||\mathbf{x}||} \to 0 \text{ when } ||\mathbf{x}|| \to 0, \tag{2}$$

uniformly for $t \in [t_0, \infty)$.

Lyapunov Theorem on Stability by the First Approximation

If all eigenvalues of the matrix A have negative real parts and (2) holds, then the solution $\mathbf{x} \equiv 0$ is asymptotically stable.