Normal forms

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$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{X}(\mathbf{x}) \qquad (1)$$

where $\mathbf{x} \in \mathbb{C}^n$, A is a possibly complex $n \times n$ matrix, and each component $X_k(\mathbf{x})$ of $\mathbf{X}$, $1 \leq k \leq n$, is a formal or convergent power series, possibly with complex coefficients, that contains no constant or linear terms.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, $\mathbf{x}^\alpha$ denotes $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$,
$\mathcal{H}_s$ denotes the vector space of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ ($\mathbb{C}^n$ to $\mathbb{C}^n$) each of whose components is a homogeneous polynomial of degree s; elements of $\mathcal{H}_s$ will be termed vector homogeneous functions. If $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$, $\mathbf{e}_j = (0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0)^T$, then a basis for $\mathcal{H}_s$ is the collection of vector homogeneous functions

$$\mathbf{v}_{j,\alpha} = \mathbf{x}^\alpha \mathbf{e}_j \tag{2}$$

for all j such that $1 \le j \le n$ and all α such that $|\alpha| = s$. For example, a basis for $\mathcal{H}_2$ in the case $\mathbf{X}(\mathbf{x}) = (X_1(x_1, x_2), X_2(x_1, x_2))^T$ is

$$\left\{ \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix} \right\}.$$

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ and $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{C}^n$ we will let (α, κ) denote the scalar product $(\alpha, \kappa) = \sum_{j=1}^n \alpha_j \kappa_j$.

Lemma

Let A be an $n \times n$ matrix with eigenvalues $\kappa_1, \dots, \kappa_n$, and let $\mathbf{L}$ be the corresponding homological operator on $\mathcal{H}_s$, that is, the linear operator on $\mathcal{H}_s$ defined by

$$\mathbf{L}\mathbf{h}(\mathbf{y}) = d\mathbf{h}(\mathbf{y})A\mathbf{y} - A\mathbf{h}(\mathbf{y})\,. \tag{3}$$

Let $\kappa = (\kappa_1, \dots, \kappa_n)$. Then the eigenvalues λ_j, $i = j, \dots, N$, of $\mathbf{L}$ are

$$\lambda_j = (\alpha, \kappa) - \kappa_m,$$

where m ranges over $\{1, \dots, n\} \subset \mathbb{N}$ and α ranges over $\{\beta \in \mathbb{N}_0^n : |\beta| = s\}$. $(N = nC(s + n - 1, s))$.

We say that

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{X}(\mathbf{x}) \quad \mathbf{X} = \sum_{|\alpha|>2} X_\alpha \mathbf{x}^\alpha \tag{4}$$

is *formally equivalent* to

$$\dot{\mathbf{y}} = A\mathbf{y} + \mathbf{Y}(\mathbf{y}) \tag{5}$$

if there is a change of variables

$$\mathbf{x} = \mathbf{H}(\mathbf{y}) = \mathbf{y} + \mathbf{h}(\mathbf{y}) \tag{6}$$

that transforms (4) into (5), where $\mathbf{Y}$ and $\mathbf{h}$, Y_j and h_j, $j = 1, \ldots, n$, are formal power series.

Theorem

Let $\kappa_1, \ldots, \kappa_n$ be the eigenvalues of the $n \times n$ matrix A, set $\kappa = (\kappa_1, \ldots, \kappa_n)$, and suppose that

$$(\alpha, \kappa) - \kappa_m \neq 0 \tag{7}$$

for all $m \in \{1, \ldots, n\}$ and for all $\alpha \in \mathbb{N}_0^n$ for which $|\alpha| \geq 2$. Then systems (4) and (5) are formally equivalent for all **X** and **Y**, and the equivalence transformation (6) is uniquely determined by **X** and **Y**.

Proof. Differentiating (6) with respect to t yields the condition

$$\mathrm{d}\mathbf{h}(\mathbf{y})A\mathbf{y} - A\mathbf{h}(\mathbf{y}) = \mathbf{X}(\mathbf{y} + \mathbf{h}(\mathbf{y})) - \mathrm{d}\mathbf{h}(\mathbf{y})\mathbf{Y}(\mathbf{y}) - \mathbf{Y}(\mathbf{y}), \qquad (8)$$

that $\mathbf{h}$ must satisfy. That is,

$$\mathbf{L}\mathbf{h}(\mathbf{y}) = \mathbf{X}(\mathbf{y} + \mathbf{h}(\mathbf{y})) - \mathrm{d}\mathbf{h}(\mathbf{y})\mathbf{Y}(\mathbf{y}) - \mathbf{Y}(\mathbf{y}).$$

Decomposing **X**, **Y**, and **h** as the sum of their homogeneous parts,

$$\mathbf{X} = \sum_{s=2}^{\infty} \mathbf{X}^{(s)} \qquad \mathbf{Y} = \sum_{s=2}^{\infty} \mathbf{Y}^{(s)} \qquad \mathbf{h} = \sum_{s=2}^{\infty} \mathbf{h}^{(s)} \tag{9}$$

where $\mathbf{X}^{(s)}, \mathbf{Y}^{(s)}, \mathbf{h}^{(s)} \in \mathcal{H}_s$, (8) decomposes into the infinite sequence of equations

$$\mathbf{L}(\mathbf{h}^{(s)}) = \mathbf{g}^{(s)}(\mathbf{h}^{(2)}, \dots, \mathbf{h}^{(s-1)}, \mathbf{Y}^{(2)}, \dots, \mathbf{Y}^{(s-1)}, \mathbf{X}^{(2)}, \dots, \mathbf{X}^{(s)}) - \mathbf{Y}^{(s)}, \tag{10}$$

for $s = 2, 3, \dots$, where $\mathbf{g}^{(s)}$ denotes the function that is obtained after the substitution into $\mathbf{X}(\mathbf{y} + \mathbf{h}(\mathbf{y})) - d\mathbf{h}(\mathbf{y})\mathbf{Y}(\mathbf{y})$ of the expression $\mathbf{y} + \sum_{i=1}^{s} \mathbf{h}^{(i)}$ in the place of $\mathbf{y} + \mathbf{h}(\mathbf{y})$ and the expression $\sum_{i=1}^{s} \mathbf{Y}^{(i)}(\mathbf{y})$ in the place of $\mathbf{Y}(\mathbf{y})$, and maintaining only terms that are of order s.

For $s = 2$ the right-hand side of (10) is

$$\mathbf{X}^{(2)}(\mathbf{y}) - \mathbf{Y}^{(2)}(\mathbf{y}),$$

which is known. For $s > 2$ the right-hand side of (10) is known if $\mathbf{h}^{(2)}, \ldots, \mathbf{h}^{(s-1)}$ have already been computed. By the Lemma the operator $\mathbf{L}$ is invertible. Thus for any $s \geq 2$ there is a unique solution $\mathbf{h}^{(s)}$ to (10). Therefore a unique solution $\mathbf{h}(\mathbf{y})$ of (8) is determined recursively.

Corollary

If condition (7) holds then system (1) is formally equivalent to its linear approximation $\dot{\mathbf{y}} = A\mathbf{y}$. The (possibly formal) coordinate transformation that transforms (1) into $\dot{\mathbf{y}} = A\mathbf{y}$ is unique.

Definition

Let $\kappa_1, \ldots, \kappa_n$ be the eigenvalues of the matrix A and let $\kappa = (\kappa_1, \ldots, \kappa_n)$. Suppose $m \in \{1, \ldots, n\}$ and $\alpha \in \mathbb{N}_0^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n \geq 2$, are such that

$$(\alpha, \kappa) - \kappa_m = 0 \, .$$

Then m and α are called a *resonant pair*, the corresponding coefficient $X_m^{(\alpha)}$ of the monomial $\mathbf{x}^\alpha$ in the mth component of $\mathbf{X}$ is called a *resonant coefficient*, and the corresponding term is called a *resonant term* of $\mathbf{X}$.

$\exists$ a non-singular $n \times n$ matrix S such that $SAS^{-1} = J$ is the Jordan normal form J of A.

$$\mathbf{y} = S\,\mathbf{x}, \tag{11}$$

then in the new coordinates

$$\dot{\mathbf{y}} = J\mathbf{y} + \mathbf{Y}(\mathbf{y}). \tag{12}$$

A "normal form" for system (1) should be one that is as simple as possible. The first step in the simplification process is to apply (11) to change the linear part A in (1) into its Jordan normal form. We begin with (1) in the form

$$\dot{\mathbf{x}} = J\mathbf{x} + \mathbf{X}(\mathbf{x}), \tag{13}$$

where J is a lower triangular Jordan matrix.

Definition

A *normal form* for system (1) is a system (13) in which every non-resonant coefficient is equal to zero. A *normalizing transformation* for system (1) is any (possibly formal) change of variables (6) transforms (1) to a normal form.

Theorem

Any system

$$\dot{\mathbf{x}} = J\mathbf{x} + \mathbf{X}(\mathbf{x}),$$

is formally equivalent to a normal form.

Proof.

$$\dot{\mathbf{x}} = J\mathbf{x} + \mathbf{X}(\mathbf{y})$$

$$\mathbf{x} = \mathbf{H}(\mathbf{y}) = \mathbf{y} + \mathbf{h}(\mathbf{y})$$

$$\dot{\mathbf{y}} = J\mathbf{y} + \mathbf{Y}(\mathbf{y})$$

$$\mathbf{L}(\mathbf{h}^{(s)}) = \mathbf{g}^{(s)}(\mathbf{h}^{(2)}, \dots, \mathbf{h}^{(s-1)}, \mathbf{Y}^{(2)}, \dots, \mathbf{Y}^{(s-1)}, \mathbf{X}^{(2)}, \dots, \mathbf{X}^{(s)}) - \mathbf{Y}^{(s)},$$

for $s = 2, 3, \dots$.

We look for $\mathbf{x} = \mathbf{y} + \mathbf{h}(\mathbf{y})$ that transforms the system into $\dot{\mathbf{y}} = J\mathbf{y} + \mathbf{Y}(\mathbf{y})$. By Lemma the matrix of the operator $\mathbf{L}$ is lower triangular with the eigenvalues $(\alpha, \kappa) - \kappa_m$ on the main diagonal. Therefore any coefficient $h_m^{(\alpha)}$ of $\mathbf{h}^{(s)}$ is determined by the equation

$$[(\alpha, \kappa) - \kappa_m] h_m^{(\alpha)} = g_m^{(\alpha)} - Y_m^{(\alpha)}, \tag{14}$$

where $g_m^{(\alpha)}$ is a known expression depending on the coefficients of $\mathbf{h}^{(i)}$ satisfying $j < s$. Suppose that for $i = 2, \ldots s - 1$ the homogeneous terms $h^{(j)}$ and $Y^{(j)}$ have been determined. Then for any $m \in \{1, \ldots, n\}$, α with $|\alpha| = s$, if the pair m and α is non-resonant, that is, if $(\alpha, \kappa) - \kappa_m \neq 0$, then we choose $Y_m^{\alpha} = 0$ so that $\mathbf{Y}$ will be a normal form, and choose $h_m^{(\alpha)}$ as uniquely determined by equation (14). If $(\alpha, \kappa) - \kappa_m = 0$, then we may choose $h_m^{(\alpha)}$ arbitrarily ($h_m^{(\alpha)} = 0$), but the resonant coefficient $Y_m^{(\alpha)}$ must be chosen to be $g_m^{(\alpha)}$, $Y_m^{(\alpha)} = g_m^{(\alpha)}$. The process can be started because for $s = 2$ $X_m^{(\alpha)} - Y_m^{(\alpha)}$.

Another description of the normalizing process

For $k \geq 2$ let $\mathcal{H}_k$ denote the vector space of functions from $\mathbb{C}^n$ to $\mathbb{C}^n$ all of whose components are homogeneous polynomial functions of degree k, let

$$\mathbf{L}\mathbf{h}(\mathbf{y}) = \mathrm{d}\mathbf{h}(\mathbf{y})J\mathbf{y} - J\mathbf{h}(\mathbf{y}),$$

and let $\mathcal{K}_k$ be any complement to $\mathrm{Image}(\mathbf{L})$ in $\mathcal{H}_k$, so that $\mathcal{H}_k = \mathrm{Image}(\mathbf{L}) \oplus \mathcal{K}_k$. Then there is a formal change of coordinates $\mathbf{x} = \mathbf{H}(\mathbf{y}) = \mathbf{y} + \mathbf{h}(\mathbf{y})$ such that in the new coordinates system is

$$\dot{\mathbf{y}} = J\mathbf{y} + \mathbf{f}^{(2)}(\mathbf{y}) + \cdots + \mathbf{f}^{(r)}(\mathbf{y}) + \ldots,$$

where for $k \geq 2$, $\mathbf{f}^{(k)} \in \mathcal{K}_k$.

Convergence Theorem

Let $\kappa_1, \ldots, \kappa_n$ be the eigenvalues of the matrix J in (13) and set $\kappa = (\kappa_1, \ldots, \kappa_n)$. Suppose $\mathbf{X}$ is analytic, that is, that each component X_m is given by a convergent power series, and that for each resonant coefficient $Y_j^{(\alpha)}$ in the normal form $\mathbf{Y}$ of $\mathbf{X}$, $\alpha \in \mathbb{N}^n$ (that is, every entry in the multi-index α is positive). Suppose further that there exist positive constants d and ϵ such that the following conditions hold:

(a) for all $\alpha \in \mathbb{N}_0^n$ and all $m \in \{1, \ldots, n\}$ such that $(\alpha, \kappa) - \kappa_m \neq 0$,

$$|(\alpha, \kappa) - \kappa_m| \geq \epsilon\,; \tag{15}$$

(b) for all α and β in $\mathbb{N}_0^n$ for which $2 \leq |\beta| \leq |\alpha| - 1$, $\alpha - \beta + e_m \in \mathbb{N}_0^n$ for all $m \in \{1, \ldots, n\}$, and

$$(\alpha - \beta, \kappa) = 0, \tag{16}$$

the following inequality holds:

$$\left| \sum_{j=1}^{n} \beta_j Y_j^{(\alpha-\beta+e_j)} \right| \leq d|(\beta, \kappa)| \sum_{j=1}^{n} \left| Y_j^{(\alpha-\beta+e_j)} \right| . \tag{17}$$

Then the normalizing transformation $\mathbf{x} = \mathbf{H}(\mathbf{y})$ is analytic, that is, each component $h_m(\mathbf{y})$ of $\mathbf{h}$ is given by a convergent power series, so that system (13) is analytically equivalent to its normal form.

$$\begin{aligned}\dot{x}_1 &= 2x_1 + \ ax_1^2 + \ bx_1x_2 + \ cx_2^2 + \cdots \\ \dot{x}_2 &= \ \ x_2 + a'x_1^2 + b'x_1x_2 + c'x_2^2 + \cdots\ .\end{aligned}$$

$$\dot{x}_1 = 2x_1 + \ ax_1^2 + \ bx_1x_2 + \ cx_2^2 + \cdots$$
$$\dot{x}_2 = \ \ x_2 + a'x_1^2 + b'x_1x_2 + c'x_2^2 + \cdots \, .$$

The normal form is

$$\dot{y}_1 = 2y_1 + Y_1^{(0,2)}y_2^2$$
$$\dot{y}_2 = \ \ y_2 \, .$$

$$\begin{aligned} \dot{x}_1 &= 2x_1 + \ ax_1^2 + \ bx_1x_2 + \ cx_2^2 + \cdots \\ \dot{x}_2 &= \ \ x_2 + a'x_1^2 + b'x_1x_2 + c'x_2^2 + \cdots . \end{aligned}$$

The normal form is

$$\begin{aligned} \dot{y}_1 &= 2y_1 + Y_1^{(0,2)} y_2^2 \\ \dot{y}_2 &= \ \ y_2 . \end{aligned}$$

For $|\alpha| = 2$, $g_m^{(\alpha)} = X_m^{(\alpha)}(\mathbf{y})$, so $Y_1^{(0,2)} = c$. Thus, the system is equivalent to

$$\begin{aligned} \dot{y}_1 &= 2y_1 + cy_2^2 \\ \dot{y}_2 &= \ \ y_2 . \end{aligned}$$

$$\dot{x}_1 = x_1 + ax_1^2 + bx_1x_2 + cx_2^2$$
$$\dot{x}_2 = -x_2 + a'x_1^2 + b'x_1x_2 + c'x_2^2 \,.$$

$$\dot{x}_1 = x_1 + \ ax_1^2 + \ bx_1x_2 + \ cx_2^2$$
$$\dot{x}_2 = -x_2 + a'x_1^2 + b'x_1x_2 + c'x_2^2\,.$$

The normal form is

$$\dot{y}_1 = y_1 + y_1 \sum_{k=1}^{\infty} Y_1^{(k+1,k)} (y_1 y_2)^k,$$
$$\dot{y}_2 = -y_2 + y_2 \sum_{k=1}^{\infty} Y_2^{(k,k+1)} (y_1 y_2)^k.$$

For any system of our interest

$$\dot{x}_1 = x_1 + \sum_{k+m\geq 2} X_{km} x_1^k x_2^m, \qquad \dot{x}_2 = -x_2 + \sum_{k+m\geq 2} X_{km} x_1^k x_2^m$$

The normal form is

$$\dot{y}_1 = y_1 + y_1 \sum_{k=1}^{\infty} Y_1^{(k+1,k)} (y_1 y_2)^k,$$

$$\dot{y}_2 = -y_2 + y_2 \sum_{k=1}^{\infty} Y_2^{(k,k+1)} (y_1 y_2)^k.$$

The center problem

The linear approximation does not necessarily determine the geometric behavior of the trajectories of the nonlinear system in a neighborhood of the origin.

$$\dot{u} = -v - u(u^2 + v^2) \qquad \dot{v} = \quad u - v(u^2 + v^2)\,. \qquad (18)$$

In polar coordinates system (18) is $\dot{r} = -r^3,\ \dot{\varphi} = 1\,.$ Thus whereas the origin is a center for the corresponding linear system, every trajectory of (18) spirals towards the origin, which is thus a stable focus. On the other hand, one can just as easily construct examples in which the addition of higher order terms does not destroy the center.

$$\begin{aligned} \dot{u} &= \alpha u - \beta v + P(u,v) \\ \dot{v} &= \beta u + \alpha v + Q(u,v)\,, \end{aligned} \tag{19}$$

where $P(u,v) = \sum_{k=2}^{\infty} P^{(k)}(u,v)$ and $Q(u,v) = \sum_{k=2}^{\infty} Q^{(k)}(u,v)$, and $P^{(k)}(u,v)$ and $Q^{(k)}(u,v)$ (if nonzero) are homogeneous polynomials of degree k.

In polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ the system becomes

$$\begin{aligned}
\dot{r} &= \alpha r + P(r\cos\varphi, r\sin\varphi)\cos\varphi + Q(r\cos\varphi, r\sin\varphi)\sin\varphi \\
&= \alpha r + r^2\left[P^{(2)}(\cos\varphi, \sin\varphi)\cos\varphi + Q^{(2)}(\cos\varphi, \sin\varphi)\sin\varphi + \cdots\right] \\
\dot{\varphi} &= \beta - r^{-1}[P(r\cos\varphi, r\sin\varphi)\sin\varphi - Q(r\cos\varphi, r\sin\varphi)\cos\varphi] \\
&= \beta - r\left[P^{(2)}(\cos\varphi, \sin\varphi)\sin\varphi - Q^{(2)}(\cos\varphi, \sin\varphi)\cos\varphi + \cdots\right].
\end{aligned} \tag{20}$$

For $|r|$ sufficiently small, if $\beta > 0$ then the polar angle φ increases as t increases, while if $\beta < 0$ then the angle decreases as t increases.

The equation of its trajectories

$$\frac{dr}{d\varphi} = \frac{\alpha r + r^2 F(r, \sin\varphi, \cos\varphi)}{\beta + rG(r, \sin\varphi, \cos\varphi)} = R(r, \varphi)\,. \tag{21}$$

The function $R(r, \varphi)$ is a 2π-periodic function of φ and is analytic for all φ and for $|r| < r^*$, for some sufficiently small r^*. The fact that the origin is an singularity for (19) corresponds to the fact that $R(0, \varphi) \equiv 0$, so that $r = 0$ is a solution of (21). We can expand $R(r, \varphi)$ in a power series in r,

$$\frac{dr}{d\varphi} = R(r, \varphi) = rR_1(\varphi) + r^2 R_2(\varphi) + \cdots = \frac{\alpha}{\beta} r + \cdots \tag{22}$$

where $R_k(\varphi)$ are 2π-periodic functions of φ. The series is convergent for all φ and for all sufficiently small r.

Denote by $r = f(\varphi, \varphi_0, r_0)$ the solution of system (22) with initial conditions $r = r_0$ and $\varphi = \varphi_0$. The function $f(\varphi, \varphi_0, r_0)$ is an analytic function of all three variables φ, φ_0, and r_0, and has the property that

$$f(\varphi, \varphi_0, 0) \equiv 0 \tag{23}$$

(because $r = 0$ is a solution of (22)). Equation (23) and continuous dependence of solutions on parameters yield the following proposition.

Proposition

Every trajectory of system (19) in a sufficiently small neighborhood of the origin crosses every ray $\varphi = c$, $0 \le c < 2\pi$.

The proposition implies that in order to investigate all trajectories in a sufficiently small neighborhood of the origin it is sufficient to consider all trajectories passing through a segment $\Sigma = \{(u, v) : v = 0, 0 \le u \le r^*\}$ for r^* sufficiently small, that is, all solutions $r = f(\varphi, 0, r_0)$. We can expand $f(\varphi, 0, r_0)$ in a power series in r_0,

$$r = f(\varphi, 0, r_0) = w_1(\varphi) r_0 + w_2(\varphi) r_0^2 + \cdots , \qquad (24)$$

which is convergent for all $0 \le \varphi \le 2\pi$ and for $|r_0| < r^*$.

This function is a solution of (22), hence

$$w_1' r_0 + w_2' r_0^2 + \cdots \equiv$$
$$R_1(\varphi)(w_1(\varphi)r_0 + w_2(\varphi)r_0^2 + \cdots) + R_2(\varphi)(w_1(\varphi)r_0 + w_2(\varphi)r_0^2 + \cdots)^2 + \cdots$$

where the primes denote differentiation with respect to φ.
Equating the coefficients of like powers of r_0 in this identity we obtain recurrence differential equations for the functions $w_j(\varphi)$:

$$\begin{aligned} w_1' &= R_1(\varphi)w_1\,, \\ w_2' &= R_1(\varphi)w_2 + R_2(\varphi)w_1^2\,, \\ w_3' &= R_1(\varphi)w_3 + 2R_2(\varphi)w_1w_2 + R_3(\varphi)w_1^3\,, \\ &\vdots \end{aligned} \tag{25}$$

The initial condition $r = f(0,0,r_0) = r_0$ yields

$$w_1(0) = 1, \qquad w_j(0) = 0 \quad \text{for} \quad j > 1. \tag{26}$$

Using these conditions we can consequently find the functions $w_j(\varphi)$ by integrating the equations (25). In particular,

$$w_1(\varphi) = e^{\frac{\alpha}{\beta}\varphi}. \tag{27}$$

Setting $\varphi = 2\pi$ in the solution $r = f(\varphi, 0, r_0)$ we obtain the value $r = f(2\pi, 0, r_0)$, corresponding to the point of Σ where the trajectory $r = f(\varphi, 0, r_0)$ first intersects Σ again.

Definition

Fix a system of the form (19).

- The function

$$\mathcal{R}(r_0) = f(2\pi, 0, r_0) = \tilde{\eta}_1 r_0 + \eta_2 r_0^2 + \eta_3 r_0^3 + \cdots \tag{28}$$

  (defined for $|r_0| < r^*$), where $\tilde{\eta}_1 = w_1(2\pi)$ and $\eta_j = w_j(2\pi)$ for $j \geq 2$, is called the *Poincaré first return map* or just the *return map*.

- The function

$$\mathcal{P}(r_0) = \mathcal{R}(r_0) - r_0 = \eta_1 r_0 + \eta_2 r_0^2 + \eta_3 r_0^3 + \cdots \tag{29}$$

  is called the *difference function*.

- The coefficient η_j, $j \in \mathbb{N}$, is called the j-th *Lyapunov number*.

The first Lyapunov number η_1 is $\eta_1 = \tilde{\eta}_1 - 1 = e^{2\pi\alpha/\beta} - 1$. Zeros of the difference function correspond to *cycles* (closed orbits, that is, orbits that are ovals) of system (19); *isolated* zeros correspond to *limit cycles* (isolated closed orbits).

System (19) has a center at the origin if and only if all the Lyapunov numbers are zero. Moreover if $\eta_1 \neq 0$, or if for some $k \in \mathbb{N}$

$$\eta_1 = \eta_2 = \cdots = \eta_{2k} = 0, \; \eta_{2k+1} \neq 0, \tag{30}$$

then all trajectories in a neighborhood of the origin are spirals and the origin is a focus, which is stable if $\eta_1 < 0$ or (30) holds with $\eta_{2k+1} < 0$ and is unstable if $\eta_1 > 0$ or (30) holds with $\eta_{2k+1} > 0$.