The Cyclicity Problem for Polynomial Systems of ODEs

16th Hilbert's problem (the second part)

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y), \qquad (A)$$

$P_n(x, y)$, $Q_n(x, y)$, are polynomials of degree n.
Let $h(P_n, Q_n)$ be the number of limit cycles of system (A) and let $H(n) = \sup h(P_n, Q_n)$.
The question of the second part of the 16th Hilbert's problem:

- find a bound for $H(n)$ as a function of n.
  (The problem is still unresolved even for $n = 2$.)

The cyclicity problem

The cyclicity problem

Find an upper bound for the number of limit cycles in a neighborhood of elementary singular point. This problem is called the *cyclicity problem* or the local Hilbert's 16th problem.

The center problem

$$\dot{u} = \lambda u - v + \sum_{j+l=2} \alpha_{jl} u^j v^l, \quad \dot{v} = u + \lambda v + \sum_{j+l=2} \beta_{jl} u^j v^l \qquad (1)$$

Trajectories are either ovals (solutions are periodic) or spirals (solutions are not periodic).

In the first case the origin is a center, in the second case it is a focus.

The center problem

$$\dot{u} = \lambda u - v + \sum_{j+l=2} \alpha_{jl} u^j v^l, \quad \dot{v} = u + \lambda v + \sum_{j+l=2} \beta_{jl} u^j v^l \qquad (1)$$

Trajectories are either ovals (solutions are periodic) or spirals (solutions are not periodic).
In the first case the origin is a center, in the second case it is a focus.

The Poincaré center problem

- Find all systems with a center at the origin within the family (1).

Poincaré (return) map

$$\dot{u} = \lambda u - v + \sum_{j+l=2} \alpha_{jl} u^j v^l, \quad \dot{v} = u + \lambda v + \sum_{j+l=2} \beta_{jl} u^j v^l$$

Poincare map

$$\mathcal{P}(\rho) = e^{2\pi\lambda}\rho + \eta_2(\alpha, \beta, \alpha_{ij}, \beta_{ij})\rho^2 + \eta_3(\alpha, \beta, \alpha_{ij}, \beta_{ij})\rho^3 + \dots.$$

Limit cycles $\longleftrightarrow$ isolated fixed points of $\mathcal{P}(\rho)$.
α changes the sign $- >$ Hopf bifurcation
W.l.o.g. we assume that $\alpha = 0, \beta = 1$. Then $\eta_k(\alpha_{ij}, \beta_{ij})$ are polynomials.

$$\dot{u} = \lambda u - v + \sum_{j+l=2}^{m} \alpha_{jl} u^j v^l, \quad \dot{v} = u + \lambda v + \sum_{j+l=2}^{m} \beta_{jl} u^j v^l \qquad (2)$$

$$A = (\alpha_{20}, \beta_{20}, \ldots, \beta_{0m}).$$

Definition

For parameters (λ, A) let $n_{(\lambda,A),\epsilon}$ denote the number of limit cycles of the corresponding system (2) that lie wholly within an ϵ-neighborhood of the origin. The singularity at the origin for system (2) with fixed coefficients $(\lambda^*, A^*) \in E(\lambda, A)$ has *cyclicity c with respect to the space $E(\lambda, A)$* if there exist positive constants δ_0 and ϵ_0 such that for every pair ϵ and δ satisfying $0 < \epsilon < \epsilon_0$ and $0 < \delta < \delta_0$

$$\max\{n_{(\lambda,A),\epsilon} : |(\lambda, A) - (\lambda^*, A^*)| < \delta\} = c\,.$$

To study limit cycles in a system

$$\dot{u} = -v + \sum_{j+l=2} \alpha_{jl} u^j v^l, \quad \dot{v} = u + \sum_{j+l=2} \beta_{jl} u^j v^l \tag{3}$$

we compute the Poincare map:

$$\mathcal{P}(\rho) = \rho + \eta_2(\alpha_{ij}, \beta_{ij})\rho^2 + \eta_3(\alpha_{ij}, \beta_{ij})\rho^3 + \cdots + \eta_k(\alpha_{ij}, \beta_{ij})\rho^k + \vdots$$

Let $\mathcal{B} = \langle \eta_3, \eta_4, \ldots \rangle \subset \mathbb{R}[\alpha_{ij}, \beta_{ij}]$ be the ideal generated by all focus quantities η_i.

$$\mathcal{B} = \langle \eta_{u_1}, \eta_{u_2}, \ldots, \eta_{u_k} \rangle$$

and $u_1 < \cdots < u_k$.
Then for any s

$$\eta_s = \eta_{u_1}\theta_1^{(s)} + \eta_{u_2}\theta_2^{(s)} + \cdots + \eta_{u_k}\theta_k^{(k)},$$

$$\mathcal{P}(\rho) - \rho = \eta_{u_1}(1 + \mu_1\rho + \ldots)\rho^{u_1} + \cdots + \eta_{u_k}(1 + \mu_k\rho + \ldots)\rho^{u_k}.$$

Bautin's Theorem

If $\mathcal{B} = \langle \eta_{u_1}, \eta_{u_2}, \ldots, \eta_{u_k} \rangle$ then the cyclicity of system (3) (i.e. the maximal number of limit cycles which appear from the origin after small perturbations) is less or equal to k.

$$\mathcal{P}(\rho) = \rho + \eta_3(\alpha_{ij}, \beta_{ij})\rho^3 + \eta_4(\alpha_{ij}, \beta_{ij})\rho^4 + \dots.$$

Center: $\eta_3 = \eta_4 = \eta_5 = \dots = 0$.

Poincaré center problem

Find all systems with a center at the origin within a given polynomial family

$$\mathcal{P}(\rho) = \rho + \eta_3(\alpha_{ij}, \beta_{ij})\rho^3 + \eta_4(\alpha_{ij}, \beta_{ij})\rho^4 + \dots .$$

Center: $\eta_3 = \eta_4 = \eta_5 = \cdots = 0$.

Poincaré center problem

Find all systems with a center at the origin within a given polynomial family

Algebraic counterpart

Find the variety of the Bautin ideal $\mathcal{B} = \langle \eta_3, \eta_4, \eta_5 \dots \rangle$. (This variety is called the center variety.)

The cyclicity problem

Find an upper bound for the maximal number of limit cycles in a neighborhood of a center or a focus

The cyclicity problem

Find an upper bound for the maximal number of limit cycles in a neighborhood of a center or a focus

By Bautin's theorem:

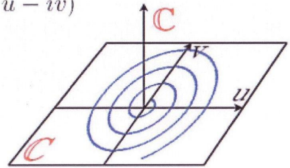
Algebraic counterpart

Find a basis for the Bautin ideal $\langle \eta_3, \eta_4, \eta_5, \ldots \rangle$ generated by all coefficients of the Poincaré map

Complexification: $x = u + iv \quad (\bar{x} = u - iv)$

$$\dot{x} = \; i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} \bar{x}^q)$$

$$\dot{\bar{x}} = -\, i(\bar{x} - \sum_{p+q=1}^{n-1} \bar{a}_{pq} \bar{x}^{p+1} x^q)$$



$$\dot{x} = i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \quad \dot{y} = -i(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}) \quad (4)$$

The change of time $d\tau = idt$ transforms (4) to the system

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \; \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}). \quad (5)$$

Poincaré-Lyapunov Theorem

The system

$$\frac{du}{dt} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \quad \frac{dv}{dt} = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j \qquad (6)$$

has a center at the origin (equivalently, all coefficients of the Poincaré map are equal to zero) if and only if it admits a first integral of the form

$$\Phi = u^2 + v^2 + \sum_{k+l \geq 2} \phi_{kl} u^k v^l.$$

Definition of a center for complex systems

System

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q) = P, \ \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}) = Q, \tag{7}$$

has a center at the origin if it admits a first integral of the form

$$\Phi(x, y; a_{10}, b_{10}, \ldots) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j} x^j y^{s-j}$$

For the complex system

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q) = P, \ \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}) = Q,$$

one looks for a function of the form
$\Phi(x, y; a_{10}, b_{10}, \ldots) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j} x^j y^{s-j}$ such that

$$\frac{\partial \Phi}{\partial x} P + \frac{\partial \Phi}{\partial y} Q = g_{11}(xy)^2 + g_{22}(xy)^3 + \cdots , \tag{8}$$

and $g_{11}, g_{22}, \ldots$ are polynomials in a_{pq}, b_{qp}. These polynomials are called *focus quantities*.

The Bautin ideal

The ideal $\mathcal{B} = \langle g_{11}, g_{22}, \ldots \rangle$ generated by the focus quantities is called the *Bautin ideal*.

The center problem

Find the variety $\mathbf{V}(\mathcal{B})$ of the Bautin ideal $\mathcal{B} = \langle g_{11}, g_{22}, g_{33} \ldots \rangle$.
$\mathbf{V}(\mathcal{B})$ is called the center variety of the system.

The cyclicity of the quadratic system

Generalized Bautin's theorem

If the ideal $\mathcal{B}$ of all focus quantities of system

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \quad \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1})$$

is generated by the m first f. q., $\mathcal{B} = \langle g_{11}, g_{22}, \ldots, g_{mm} \rangle$, then at most m limit cycles bifurcate from the origin of the corresponding real system

$$\dot{u} = \lambda u - v + \sum_{j+l=2}^{n} \alpha_{jl} u^j v^l, \quad \dot{v} = u + \lambda v + \sum_{j+l=2}^{n} \beta_{jl} u^j v^l,$$

that is the cyclicity of the system is less or equal to m.

The center variety of the quadratic system

$$\dot{x} = x - a_{10}x^2 - a_{01}xy - a_{-12}y^2,\ \dot{y} = -(y - b_{10}xy - b_{01}y^2 - b_{2,-1}x^2). \tag{9}$$

Theorem (H. Dulac 1908, C. Christopher & C. Rouseeau, 2001)

The variety of the Bautin ideal of system (9) coincides with the variety of the ideal $\mathcal{B}_3 = \langle g_{11}, g_{22}, g_{33} \rangle$ and consists of four irreducible components:

1) $\mathbf{V}(J_1)$, where $J_1 = \langle 2a_{10} - b_{10}, 2b_{01} - a_{01} \rangle$,
2) $\mathbf{V}(J_2)$, where $J_2 = \langle a_{01}, b_{10} \rangle$,
3) $\mathbf{V}(J_3)$, where $J_3 = \langle 2a_{01} + b_{01}, a_{10} + 2b_{10},\ a_{01}b_{10} - a_{-12}b_{2,-1} \rangle$,
4) $\mathbf{V}(J_4) = \langle f_1, f_2, f_3, f_4, f_5 \rangle$, where
$f_1 = a_{01}^3 b_{2,-1} - a_{-12}b_{10}^3,\ f_2 = a_{10}a_{01} - b_{01}b_{10},$
$f_3 = a_{10}^3 a_{-12} - b_{2,-1}b_{01}^3,$
$f_4 = a_{10}a_{-12}b_{10}^2 - a_{01}^2 b_{2,-1}b_{01},\ f_5 = a_{10}^2 a_{-12}b_{10} - a_{01}b_{2,-1}b_{01}^2.$

Proof. Computing the first three focus quantities we have

$g_{11} = a_{10}a_{01} - b_{10}b_{01}$,

$g_{22} = a_{10}a_{-12}b_{10}^2 - a_{01}^2 b_{01}b_{2,-1} - \frac{2}{3}(a_{-12}b_{10}^3 - a_{01}^3 b_{2,-1}) - \frac{2}{3}(a_{01}b_{01}^2 b_{2,-1} - a_{10}^2 a_{-12}b_{10})$,

$g_{33} = -\frac{5}{8}(-a_{01}\, a_{-12} b_{10}^4 + 2\, a_{-12} b_{01} b_{10}^4 + a_{01}^4 b_{10}\, b_{2,-1} - 2\, a_{01}^3\, b_{01}\, b_{10}\, b_{2,-1} - 2\, a_{10}\, a_{-12}^2\, b_{10}^2\, b_{2,-1} + a_{-12}^2\, b_{10}^3\, b_{2,-1} - a_{01}^3\, a_{-12}\, b_{2,-1}^2 + 2\, a_{01}^2\, a_{-12}\, b_{01}\, b_{2,-1}^2)$.

Using the radical membership test we see that

$$g_{22} \notin \sqrt{\langle g_{11} \rangle}, \quad g_{33} \notin \sqrt{\langle g_{11}, g_{22} \rangle}, \; g_{44}, g_{55}, g_{66} \in \sqrt{\langle g_{11}, g_{22}, g_{33} \rangle},$$

i.e., $\mathbf{V}(\mathcal{B}_1) \supset \mathbf{V}(\mathcal{B}_3) \supset \mathbf{V}(\mathcal{B}_3) = \mathbf{V}(\mathcal{B}_4) = \mathbf{V}(\mathcal{B}_5)$. We expect that

$$\mathbf{V}(\mathcal{B}_3) = \mathbf{V}(\mathcal{B}). \tag{10}$$

The inclusion $\mathbf{V}(\mathcal{B}) \subseteq \mathbf{V}(\mathcal{B}_3)$ is obvious, therefore in order to check that (16) indeed holds we only have to prove that

$$\mathbf{V}(\mathcal{B}_3) \subseteq \mathbf{V}(\mathcal{B}). \tag{11}$$

To do so, we first look for a decomposition of the variety $\mathbf{V}(\mathcal{B}_3)$. To verify that (11) holds there remains to show that every system (9) with coefficients from one of the sets $\mathbf{V}(J_1), \mathbf{V}(J_2), \mathbf{V}(J_3), \mathbf{V}(J_4)$ has a center at the origin, that is, there is a first integral $\Psi(x, y) = xy + h.o.t.$

The problem has been solved for:

- The quadratic system ($\dot{x} = P_n,\ \dot{y} = Q_n,\ n = 2$) - Bautin (1952) (Żoładek (1994), Yakovenko (1995), Françoise and Yomdin (1997), Han, Zhang & Zhang (2007)).
- The system with homogeneous cubic nonlinearities - Sibirsky (1965) (Żołądek (1994))

In both cases the analysis is relatively simple because the Bautin ideal is a radical ideal.

Bautin's theorem for the quadratic system

The cyclicity of the origin of system

$$\dot{u} = \lambda u - v + \alpha_{20} u^2 + \alpha_{11} uv + \alpha_{02} v^2,\ \ \dot{v} = u + \lambda v + \beta_{20} u^2 + \beta_{11} uv + \beta_{02} v^2$$

equals three.

Proof. We have for all k

$$g_{kk}|_{\mathbf{V}(\mathcal{B}_3)} \equiv 0 \tag{12}$$

where $\mathcal{B}_3 = \langle g_{11}, g_{22}, g_{33} \rangle$.
Hence, if $\mathcal{B}_3$ is a radical ideal then (12) and Hilbert Nullstellensatz yield that $g_{kk} \in \mathcal{B}_3$. Thus, to prove that an upper bound for the cyclicity is equal to three it is sufficient to show that $\mathcal{B}_3$ is a radical ideal.
With help of SINGULAR we check that

$$std(radical(\mathcal{B}_3)) = std(\mathcal{B}_3). \tag{13}$$

Hence, $\mathcal{B}_3 = \mathcal{B}$. This completes the proof.

- Good news:

- Good news:
  Using algorithms of computational algebra the cyclicity of a polynomial system can be easily investigated in the case when the Bautin ideal is a radical ideal (provided we know its variety)
- Bad news:

- Good news:
  Using algorithms of computational algebra the cyclicity of a polynomial system can be easily investigated in the case when the Bautin ideal is a radical ideal (provided we know its variety)
- Bad news:
  It happens very seldom that the Bautin ideal is a radical ideal

Cyclicity of systems with non-radical Bautin ideal

$$\dot{x} = \lambda x + i(x - a_{-12}\bar{x}^2 - a_{20}x^3 - a_{02}x\bar{x}^2) \tag{14}$$

$$\dot{x} = (x - a_{-12}y^2 - a_{20}x^3 - a_{02}xy^2) \quad \dot{y} = -(y - b_{2,-1}x^2 - b_{20}x^2y - b_{02}y^3) \tag{15}$$

Lemma

The variety of the Bautin ideal of system (15) coincides with the variety of the ideal $\mathcal{B}_6 = \langle g_{11}, g_{22}, \ldots, g_{66} \rangle$.

By the Hilbert Basis Theorem $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_k)$ for some k. Using the Radical Membership Test one can easily verify that

$$g_{66} \notin \sqrt{\langle g_{11}, \ldots, g_{55} \rangle} \quad \text{but} \quad g_{77}, g_{88}, g_{99} \in \sqrt{\langle g_{11}, g_{22}, \ldots, g_{66} \rangle},$$

which leads us to expect that

$$\mathbf{V}(\mathcal{B}_6) = \mathbf{V}(\mathcal{B}). \tag{16}$$

It was shown by Y. R. Liu (1990) that (11) holds.

We use the specific structure of the focus quantities.

$$\dot{x} = (x - a_{-12}y^2 - a_{20}x^3 - a_{02}xy^2), \ \dot{y} = -(y - b_{2,-1}x^2 - b_{20}x^2y - b_{02}y^3)$$

We write down the ideal

$$\langle a_{p_k q_k} - t_k, \quad b_{q_k p_k} - \gamma^{p_k - q_k} t_k \rangle.$$

$$\mathcal{J} = \langle 1 - w\gamma, a_{-12} - t_1, \gamma^3 b_{2,-1} - t_1, a_{20} - t_2, \ b_{02} - \gamma^2 t_2, a_{02} - t_3, \gamma^2 b_{20} - t_3 \rangle$$

Computing the Gröbner basis with respect to the lexicographic order with
$w > \gamma > t_1 > t_2 > t_3 > a_{-12} > a_{20} > a_{02} > b_{20} > b_{02} > b_{2,-1}$ we obtain a list of polynomials and pick up the polynomials that do not depend on w, γ, t_1, t_2, t_3 :

$a_{20}a_{02} - b_{20}b_{02},\ a^2_{-12}a_{20}b^2_{20} - a^2_{02}b^2_{2,-1}b_{02},\ a^2_{-12}a^2_{20}b_{20} - a_{02}b^2_{2,-1}b^2_{02},\ -a^3_{02}b^2_{2,-1} - a^2_{-12}b^3_{20},\ a^2_{-12}a^3_{20} - b^2_{2,-1}b^3_{02},$ The monomials of the binomials form a basis of the subalgebra: $c_1 = a_{20}a_{02}, c_2 = b_{20}b_{02}, c_3 = a^3_{02}b^2_{2,-1}, c_4 a^2_{02}b^2_{2,-1}b_{02}, c_5 = a_{02}b^2_{2,-1}b^2_{02}, \dots$
The focus quantities of system (15) belong to the subalgebra $\mathbb{C}[c_1, \dots, c_{15}]$ that is,

$$g_{kk} = g_{kk}(c_1, \dots, c_{13}) \tag{17}$$

We prove that although the ideal of focus quantities is not radical ideal in $\mathbb{C}[a, b]$, it is a radical ideal in $\mathbb{C}[c_1, \dots, c_{15}]$ and use this to resolve the cyclicity problem for system (14).

More precisely, consider the ideal

$J = \langle c_1 - a_{-12}b_{2,-1}, c_2 - a_{20}b_{02}, c_3 - a_{02}b_{20}, c_4 - b_{20}b_{02}, c_5 - a_{02}^3 b_{2,-1}^2, c_6 - a_{02}^2 b_{2,-1}^2 b_{02}, c_7 - a_{02} b_{2,-1}^2 b_{02}^2, c_8 - b_{2,-1}^2 b_{02}^3, c_9 - a_{20}a_{02}, c_{10} - a_{-12}^2 b_{20}^3, c_{11} - a_{-12}^2 a_{20} b_{20}^2, c_{-1,2} - a_{-12}^2 a_{20}^2 b_{20}, c_{13} - a_{-12}^2 a_{20}^3 \rangle$ and the corresponding map

$$F : E(a,b) = \mathbb{A}_{\mathbb{C}}^6 = \mathbb{C}^6 \longrightarrow \mathbb{A}_{\mathbb{C}}^{13} = \mathbb{C}^{13},$$

that is,

$F(a,b) = (a_{-12}b_{2,-1}, a_{20}b_{02}, a_{02}b_{20}, b_{20}b_{02}, a_{02}^3 b_{2,-1}^2, \ldots, a_{-12}^2 a_{20}^3)$.

Let W be the image of $E(a,b)$ under F and $\mathbb{C}[c] := \mathbb{C}[c_1, \ldots, c_{13}]$. F induces the $\mathbb{C}$–algebra homomorphism

$$F^* : \mathbb{C}[c] \longrightarrow \mathbb{C}[a,b].$$

Let $\prec_{(a,b)}$ be an elimination monomial ordering for (a, b) in the algebra $\mathbb{C}[a, b] \otimes_{\mathbb{C}} \mathbb{C}[c] = \mathbb{C}[a, b, c]$. Computing the Gröbner basis J_G of J with respect to $\prec_{(a,b)}$, we find that $J \cap \mathbb{C}[c]$ is the ideal R, generated by

$c_{11}c_{13} - c_{12}^2$, $c_{10}c_{13} - c_{11}c_{12}$, $c_{10}c_{12} - c_{11}^2$, $c_6c_8 - c_7^2$, $c_5c_8 - c_6c_7$, $c_5c_7 - c_6^2$, $c_4c_7c_{13} - c_8c_9c_{12}$, $c_4c_7c_{12} - c_8c_9c_{11}$, $c_4c_7c_{11} - c_8c_9c_{10}$, $c_4c_6c_{13} - c_7c_9c_{12}$, $c_4c_6c_{12} - c_7c_9c_{11}$, $c_4c_6c_{11} - c_7c_9c_{10}$, $c_4c_5c_{13} - c_6c_9c_{12}$, $c_4c_5c_{12} - c_6c_9c_{11}$, $c_4c_5c_{11} - c_6c_9c_{10}$, $c_3c_{13} - c_9c_{12}$, $c_3c_{12} - c_9c_{11}$, $c_3c_{11} - c_9c_{10}$, $c_3c_8 - c_4c_7$, $c_3c_7 - c_4c_6$, $c_3c_6 - c_4c_5$, $c_2c_{12} - c_4c_{13}$, $c_2c_{11} - c_4c_{12}$, $c_2c_{10} - c_4c_{11}$, $c_2c_7 - c_8c_9$, $c_2c_6 - c_7c_9$, $c_2c_5 - c_6c_9$, $c_2c_3 - c_4c_9$, $c_1^2c_9^3 - c_5c_{13}$, $c_1^2c_4c_9^2 - c_6c_{12}$, $c_1^2c_4^2c_9 - c_7c_{11}$, $c_1^2c_4^3 - c_8c_{10}$, $c_1^2c_3c_9^2 - c_5c_{12}$, $c_1^2c_3c_4c_9 - c_6c_{11}$, $c_1^2c_3c_4^2 - c_7c_{10}$, $c_1^2c_3^2c_9 - c_5c_{11}$, $c_1^2c_3^2c_4 - c_6c_{10}$, $c_1^2c_3^3 - c_5c_{10}$, $c_1^2c_2c_9^2 - c_6c_{13}$, $c_1^2c_2c_4c_9 - c_7c_{12}$, $c_1^2c_2c_4^2 - c_8c_{11}$, $c_1^2c_2^2c_9 - c_7c_{13}$, $c_1^2c_2^2c_4 - c_8c_{12}$, $c_1^2c_2^3 - c_8c_{13}$.

R is the kernel of F^* (can be computed with the routine *preimage* of SINGULAR).

Let C be the subalgebra of $\mathbb{C}[a, b]$, generated by the monomials, corresponding to the components of the map F (that is, by $a_{-12}b_{2,-1}, a_{20}b_{02}, a_{02}b_{20}$ etc.). For a polynomial $f(a, b) \in \mathbb{C}[c_1(a, b), \dots, c_{13}(a, b)] \subset \mathbb{C}[a, b]$ we denote by $f^F \in \mathbb{C}[c]$ the preimage of $f(a, b)$ under F^*. Then, $f^F \in \mathbb{C}[c_1(a, b), \dots, c_{13}(a, b)]$ can be computed via the normal form, that is $f^F =$NF(f, J_G), where J_G is a Gröbner basis of J with respect to an elimination ordering $\prec_{(a,b)}$.

$$
\begin{aligned}
g_{22} &= -i(3a_{20}a_{02} - 3b_{20}b_{02}),\\
g_{44} &= -i(2160a_{20}^3a_{12}^2 + 5760a_{20}^2b_{20}a_{12}^2 + 2160a_{20}b_{20}^2a_{12}^2 - 1440b_{20}^3a_{12}^2\\
&\quad + 1440a_{02}^3b_{21}^2 - 2160a_{02}^2b_{02}b_{21}^2 - 5760a_{02}b_{02}^2b_{21}^2 - 2160b_{02}^3b_{21}^2)\\
g_{55} &= -i(-340200a_{20}^2b_{20}a_{12}^3b_{21} - 226800a_{20}b_{20}^2a_{12}^3b_{21} + 113400b_{20}^3a_{12}^3b_2\\
&\quad + 226800a_{02}^2b_{02}a_{12}b_{21}^3 + 340200a_{02}b_{02}^2a_{12}b_{21}^3)\\
g_{66} &= -i(102060000a_{20}^2b_{20}^2b_{02}a_{12}^2 + 68040000a_{20}b_{20}^3b_{02}a_{12}^2 - 34020000b_2^4\\
&\quad + 34020000a_{02}^3b_{20}b_{02}b_{21}^2 - 68040000a_{02}^2b_{20}b_{02}^2b_{21}^2 - 102060000a_{02}
\end{aligned}
$$

$$
\begin{aligned}
g_{11}^F &= 0,\ \ g_{22}^F = c_9 - c_4, g_{33}^F = 0,\\
g_{44}^F &= \tfrac{2}{3}c_5 - c_6 - \tfrac{8}{3}c_7 - c_8 - \tfrac{2}{3}c_{10} + c_{11} + \tfrac{8}{3}c_{12} + c_{13}\ ,\\
g_{55}^F &= -\tfrac{7}{24}c_1c_5 + \tfrac{7}{12}c_1c_6 + \tfrac{7}{8}c_1c_7 + \tfrac{7}{24}c_1c_{10} - \tfrac{7}{12}c_1c_{11} - \tfrac{7}{8}c_1c_{12},\\
g_{66}^F &= -\tfrac{5}{3}c_3c_5 + \tfrac{5}{3}c_3c_{10} + \tfrac{10}{3}c_4c_5 + 5c_4c_6 - \tfrac{10}{3}c_4c_{10} - 5c_4c_{11}.
\end{aligned}
$$

Let W denote the image of $\mathbb{C}^6$ under F, $\overline{W}$ its Zariski closure, and $\mathbb{C}[\overline{W}]$ the ring of polynomial mappings from $\overline{W}$ to $\mathbb{C}$, which is isomorphic to $\mathbb{C}[c]/\mathbf{I}(\overline{W})$. Then, $\overline{W} = \mathbf{V}(R)$.
Denote by V the variety $\mathbf{V}(\mathcal{B})$ and by V_c the image of V under F, $V_c = F(V)$. V_c is a subset of W and its Zariski closure $\overline{V}_c$ is a subvariety of $\overline{W}$. Let U be the subvariety $U = \mathbf{V}(\langle g^F_{kk} : k \in \mathbb{N}\rangle)$ of $\overline{W}$ and let $U_6 = \mathbf{V}(G_6)$ for $G_6 := \langle g^F_{11}, g^F_{22}, \ldots, g^F_{66}\rangle \subset \mathbb{C}[c]$. We claim that

$$U_6 = U = \overline{V}_c. \tag{18}$$

It is clear that $g_{kk} \in \mathbf{I}(V)$ implies that $g^F_{kk} \in \mathbf{I}(V_c)$, which in turn implies that $g^F_{kk} \in \mathbf{I}(\overline{V}_c)$, so that

$$\overline{V}_c \subset U \subset U_6. \tag{19}$$

Applying the ideas in §1.8.3 of (Greuel, G.-M. and Pfister, G. A SINGULAR Introduction to Commutative Algebra, 2002) if we form the ideal $N = \langle J_G \cap \mathbb{C}[c], \mathcal{B}_6, J\rangle = \langle R, \mathcal{B}_6, J\rangle$ in $\mathbb{C}[a,b,c]$ and compute $H = N \cap \mathbb{C}[c]$, then $\overline{V}_c = \mathbf{V}(H)$. We also checked that the ideals H and G_6 are the same ideal in $\mathbb{C}[c]$, so that $\overline{V}_c = \mathbf{V}(G_6) = U_6$. Together with (19) this yields

$$U_6 = U = \overline{V}_c.$$

Let $\widetilde{G}_6$ denote the ideal $\langle g_{11}^F, \ldots, g_{66}^F\rangle$ in $\mathbb{C}[\overline{W}]$. By the natural isomorphism of $\mathbb{C}[\overline{W}]$ with $\mathbb{C}[c]/R$ this ideal is radical if and only if the ideal $\langle g_{11}^F + R, \ldots, g_{66}^F + R\rangle$ in $\mathbb{C}[c]/R$ is radical. Letting r_j, $1 \le j \le 44$, denote the generators of R as listed above, it is easy to check that this is true if the ideal $H = \langle g_{11}^F, \ldots, g_{66}^F, r_1, \ldots, r_{44}\rangle$ is a radical ideal in $\mathbb{C}[c]$. Computing the radical of H with SINGULAR we find that it is. Thus $\widetilde{G}_6$ is radical in $\mathbb{C}[\overline{W}]$.

The equality $U = U_6$ tells us that for every $k \in \mathbb{N}$ $g^F_{kk} \in \mathbf{I}(U_6) = \mathbf{I}(\mathbf{V}(G_6))$. But then viewed as a polynomial mapping on $\overline{W}$, i.e., as an element of $\mathbb{C}[\overline{W}]$, we have that $g^F_{kk} \in \mathbf{I}_{\overline{W}}(\mathbf{V}_{\overline{W}}(\widetilde{G}_6))$. This means that $g^F_{kk} \in \sqrt{\widetilde{G}_6} = \widetilde{G}_6$ in $\mathbb{C}[\overline{W}]$. Thus there exist polynomials $f_{j,k}$ such that for $c \in \overline{W}$

$$g^F_{kk}(c) = g^F_{11}(c) f_{1,k}(c) + \cdots + g^F_{66}(c) f_{6,k}(c).$$

Applying F^* and that by the Generalized Bautin Theorem the cyclicity of a center at the origin is at most four (since $g_{11} = g_{33} = 0$.).

$$\dot{x} = \lambda x + i(x - a_{-12}\bar{x}^2 - a_{20}x^3 - a_{11}x^2\bar{x} - a_{02}x\bar{x}^2). \qquad (20)$$

With system (20) we associate the complex system

$$\begin{aligned} \dot{x} &= i(x - a_{-12}y^2 - a_{20}x^3 - a_{11}x^2y - a_{02}xy^2) \\ \dot{y} &= -i(y - b_{2,-1}x^2 - b_{20}x^2y - b_{11}xy^2 - b_{02}y^3) \end{aligned} \qquad (21)$$

The first seven focus quantities define the variety of the Bautin ideal of (21), that is, $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_7)$, where $\mathcal{B}_7 = \langle g_{11}, \ldots, g_{77} \rangle$ (Y.-R. Liu, 1990).

$\mathcal{B}_7$ is not a radical ideal.

Computations show that the ideals $\mathcal{B}_7,\ \mathcal{B}_8, \mathcal{B}_9$ of system (20) are nonradical ideals also in $\mathbb{C}[\mathcal{M}]$.
Nevertheless, $\mathcal{B}_9$ has a relatively simple primary decomposition in $\mathbb{C}[\mathcal{M}]$ and it allows to obtain a bound for cyclicity of system (20) for "almost all" values of parameters a_{kj}.

The cyclicity of the system

Theorem

The center at the origin of system (20) with all parameters a_{kj} different from zero has cyclicity at most eight.

To prove the theorem it is sufficient to show that for any (a^*, b^*) with all coordinates different from zero and $b_{jk} = \bar{a}_{kj}$, and $k > 9$

$$g_{kk} = g_{11} f_1 + g_{22} f_2 + g_{44} f_4 + g_{55} f_4 + \cdots + g_{99} f_9 \qquad (22)$$

in $\mathcal{G}_{(a^*,b^*)}$, where $\mathcal{G}_{(a^*,b^*)}$ is the ring of germs of complex analytic functions at (a^*, b^*).

Primary Decomposition of $\mathcal{B}_6$ in $\mathbb{C}[c_1, \ldots, c_{15}]$

Output produced by *primdecSY* (Shimoyama and Yokoyama algorithm) of SINGULAR

```
[1]:
   [1]:
      _[1]=c14-c15         _[2]=c12-3*c13           _[3]=c11-9*c13
      _[4]=c10-27*c13      _[5]=c7-3*c8             _[6]=c6-9*c8
      _[7]=c5-27*c8        _[8]=c4-c9               _[9]=c3-3*c9
      _[10]=3*c2-c9        _[11]=c1^2*c9^3-27*c8*c13
   [2]:
      _[1]=c14-c15         _[2]=c12-3*c13           _[3]=c11-9*c13
      _[4]=c10-27*c13      _[5]=c7-3*c8             _[6]=c6-9*c8
      _[7]=c5-27*c8        _[8]=c4-c9               _[9]=c3-3*c9
     _[10]=3*c2-c9         _[11]=c1^2*c9^3-27*c8*c13
```

Primary Decomposition of $\mathcal{B}_6$ in $\mathbb{C}[c_1, \ldots, c_{15}]$

```
[2]:
   [1]:
      _[1]=c15          _[2]=c14          _[3]=c12+c13
      _[4]=c11-c13      _[5]=c10+c13      _[6]=c7+c8
      _[7]=c6-c8        _[8]=c5+c8        _[9]=c4-c9
      _[10]=c3+c9       _[11]=c2+c9       _[12]=c1^2*c9^3+c8*c13
   [2]:
      _[1]=c15          _[2]=c14          _[3]=c12+c13
      _[4]=c11-c13      _[5]=c10+c13      _[6]=c7+c8
      _[7]=c6-c8        _[8]=c5+c8        _[9]=c4-c9
      _[10]=c3+c9       _[11]=c2+c9       _[12]=c1^2*c9^3+c8*c13
```

and so on.

Primary Decomposition of $\mathcal{B}_6$ in $\mathbb{C}[c_1, \ldots, c_{15}]$

```
[6]:
   [1]:
      _[1]=c14-c15
      _[2]=2*c5-3*c6-8*c7-3*c8-2*c10+3*c11+8*c12+3*c13
      ........................................
      _[158]=3*c2^4*c9^8*c13^2-6*c2^3*c9^9*c13^2+5*c2^2*c9
      _[159]=27*c2^6*c9^7*c13-54*c2^5*c9^8*c13+45*c2^4*c9^
      _[160]=c3^8*c9^6
      _[161]=27*c2^8*c9^6-54*c2^7*c9^7+45*c2^6*c9^8-20*c2^
      _[162]=c2^3*c9^10*c13^2
      _[163]=c2^5*c9^9*c13-2*c2^4*c9^10*c13
      _[164]=3*c2^7*c9^8-6*c2^6*c9^9+5*c2^5*c9^10
      _[165]=c2^6*c9^10
```

Primary Decomposition of $\mathcal{B}_6$ in $\mathbb{C}[c_1, \ldots, c_{15}]$

```
[2]:
   _[1]=c15
   _[2]=c14
   _[3]=c13
   _[4]=c12
   _[5]=c11
   _[6]=c10
   _[7]=c9
   _[8]=c8
   _[9]=c7
   _[10]=c6
   _[11]=c5
   _[12]=c4
   _[13]=c3          _[14]=c2          _[15]=c1
```

Proposition

Let $I = \langle g_1, \ldots, g_t \rangle$ be an ideal in $\mathbb{C}[x_1, \ldots, x_n]$ such that the primary decomposition of I is given as

$$I = P_1 \cap \cdots \cap P_k \cap Q,$$

where P_s is prime for $s = 1, \ldots, k$, and $Q \neq \sqrt{Q} = \langle x_1, \ldots, x_n \rangle$. Let g be a polynomial vanishing on $\mathbf{V}(I)$ and let $x^* = (x_1^*, \ldots, x_n^*)$ be an arbitrary point of $V(I)$ different from the origin $(0, \ldots, 0)$. Then in a small neighborhood of x^*

$$g = g_1 f_1 + \cdots + g_t f_t,$$

where $f_1, \ldots, f_t$ are power series convergent at x^*.

The ideal $\langle g_{11}^F, \dots, g_{99}^F \rangle \subset \mathbb{C}[c_1, \dots, c_{15}]$ has the structure as in the Proposition.
Therefore, by the proposition, there exist rational functions $f_{j,k}$ such that for

$$g_{kk}^F(c) = g_{11}^F(c) f_{1,k}(c) + g_{22}^F(c) f_{2,k}(c) + g_{44}^F(c) f_{4,k}(c) + \dots \\ + g_{99}^F(c) f_{9,k}(c) + \sum_{j=1}^{44} r_j(c) s_{j,k}(c), \quad (23)$$

and $f_{j,k}$ (resp. $s_{j,k}(c)$) are of the form $f_{j,k}(c) = \hat{f}_{j,k}/c_l^j$ (resp. $s_{j,k}(c) = \hat{s}_{j,k}/c_l^j$), with $\hat{f}_{j,k}, \hat{s}_{j,k}$ being polynomials.
After some technical work it can be proved that the cyclicity of a center at the origin of the corresponding system of ODEs is at most eight.