Limit cycles, centers and time-reversibility in systems of polynomial differential equations

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Lotka-Volterra equations

Consider a biological system in which two species interact, one a predator and one its prey. They evolve in time according to the pair of the equations:

$$\frac{dx}{dt} = x(\alpha - \beta y), \frac{dy}{dt} = -y(\gamma - \delta x)$$

where,
y is the number of some predator;
x is the number of its prey;
$\frac{dx}{dt} = \dot{x}$ and $\frac{dy}{dt} = \dot{y}$ represent the growth of the two populations against time t;

The prey equation:

$$\frac{dx}{dt} = \alpha x - \beta xy.$$

The prey are assumed to have an unlimited food supply, and to reproduce exponentially unless subject to predation; this exponential growth is represented by the term αx. The rate of predation upon the prey is assumed to be proportional to the rate at which the predators and the prey meet; this is represented by βxy.

The predator equation:

$$\frac{dy}{dt} = \delta xy - \gamma y.$$

δxy - the growth of the predator population. γy represents the loss rate of the predators due to either natural death or emigration; it leads to an exponential decay in the absence of prey.

The equation expresses the change in the predator population as growth fueled by the food supply, minus natural death.

16th Hilbert's problem and related problems

$$\dot{x} = P_n(x,y), \quad \dot{y} = Q_n(x,y), \qquad (A)$$

$P_n(x,y)$, $Q_n(x,y)$, are polynomials of degree n.
Let $h(P_n, Q_n)$ be the number of limit cycles of system (A) and let $H(n) = \sup h(P_n, Q_n)$.
The question of the second part of the 16th Hilbert's problem:

- find a bound for $H(n)$ as a function of n.
  (The problem is still unresolved even for $n = 2$.)

A simpler problem: is $H(n)$ finite? Unresolved.

16th Hilbert's problem and related problems

An even simpler problem: is $h(P_n, Q_n)$ finite?

- Chicone and Shafer (1983) proved that for $n = 2$ a fixed system (A) has only finite number of limit cycles in any bounded region of the phase plane.
- Bamòn (1986) and V. R (1986) proved that $h(P_2, Q_2)$ is finite.
- Il'yashenko (1991) and Ecalle (1992): $h(P_n, Q_n)$ is finite for any n.

Local Hilbert's 16th problem

Find an upper bound for the number of limit cycles in a neighborhood of elementary singular point. This problem is called the *cyclicity problem* or the local Hilbert's 16th problem.

Poincare (return) map

$$\dot{u} = \alpha u - \beta v + \sum_{j+l=2} \alpha_{jl} u^j v^l, \quad \dot{v} = \beta u + \alpha v + \sum_{j+l=2} \beta_{jl} u^j v^l$$

Poincare map

$$\mathcal{P}(\rho) = e^{2\pi \frac{\alpha}{\beta}} \rho + \eta_2(\alpha, \beta, \alpha_{ij}, \beta_{ij})\rho^2 + \eta_3(\alpha, \beta, \alpha_{ij}, \beta_{ij})\rho^3 + \dots .$$

Limit cycles $\longleftrightarrow$ isolated fixed points of $\mathcal{P}(\rho)$.
α changes the sign $->$ Hopf bifurcation
W.l.o.g. we assume that $\alpha = 0, \beta = 1$. Then $\eta_k(\alpha_{ij}, \beta_{ij})$ are polynomials.

The Bautin ideal and Bautin's theorem

To study limit cycles in a system

$$\dot{u} = -v + \sum_{j+l=2} \alpha_{jl} u^j v^l, \quad \dot{v} = u + \sum_{j+l=2} \beta_{jl} u^j v^l \qquad (1)$$

we compute the Poincare map:

$$\mathcal{P}(\rho) = \rho + \eta_2(\alpha_{ij}, \beta_{ij})\rho^2 + \eta_3(\alpha_{ij}, \beta_{ij})\rho^3 + \cdots + \eta_k(\alpha_{ij}, \beta_{ij})\rho^k.$$

Let $\mathcal{B} = \langle \eta_3, \eta_4, \ldots \rangle \subset \mathbb{R}[\alpha_{ij}, \beta_{ij}]$ be the ideal generated by all focus quantities η_i. There is k such that

$$\mathcal{B} = \langle \eta_{u_1}, \eta_{u_2}, \ldots, \eta_{u_k} \rangle.$$

The Bautin ideal and Bautin's theorem

Then for any s

$$\eta_s = \eta_{u_1}\theta_1^{(s)} + \eta_{u_2}\theta_2^{(s)} + \cdots + \eta_{u_k}\theta_k^{(k)},$$

$$\mathcal{P}(\rho) - \rho = \eta_{u_1}(1 + \mu_1\rho + \dots)\rho^{u_1} + \cdots + \eta_{u_k}(1 + \mu_k\rho + \dots)\rho^{u_k}.$$

Bautin's Theorem

If $\mathcal{B} = \langle \eta_{u_1}, \eta_{u_2}, \dots, \eta_{u_k} \rangle$ then the cyclicity of system (1) (i.e. the maximal number of limit cycles which appear from the origin after small perturbations) is less or equal to k.

Proof. Bautin N.N. Mat. Sb. (1952) v.30, 181-196 (Russian); Trans. Amer. Math. Soc. (1954) v.100
Roussarie R. Bifurcations of planar vector fields and Hilbert's 16th problem (1998), Birkhauser.

$$\mathcal{P}(\rho) = \rho + \eta_3(\alpha_{ij}, \beta_{ij})\rho^3 + \eta_4(\alpha_{ij}, \beta_{ij})\rho^4 + \dots.$$

Center: $\eta_3 = \eta_4 = \eta_5 = \cdots = 0.$

Poincaré center problem

Find all systems with a center at the origin within a given polynomial family

Algebraic counterpart

Find the variety of the Bautin ideal $\mathcal{B} = \langle \eta_3, \eta_4, \eta_5 \dots \rangle$. (This variety is called the center variety.)

An algebraic point of view

The cyclicity problem

Find an upper bound for the maximal number of limit cycles in a neighborhood of a center or a focus

By Bautin's theorem:

Algebraic counterpart

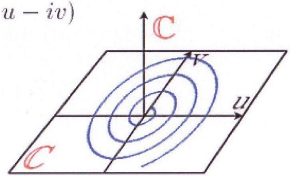
Find a basis for the Bautin ideal $\langle \eta_3, \eta_4, \eta_5, \dots \rangle$ generated by all coefficients of the Poincaré map

Complexification

Complexification: $x = u + iv \quad (\bar{x} = u - iv)$

$$\dot{x} = \ i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} \bar{x}^q)$$

$$\dot{\bar{x}} = -\, i(\bar{x} - \sum_{p+q=1}^{n-1} \bar{a}_{pq} \bar{x}^{p+1} x^q)$$



$$\dot{x} = i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \quad \dot{y} = -i(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}) \quad (2)$$

The change of time $d\tau = idt$ transforms (2) to the system

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \ \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}). \quad (3)$$

Poincaré-Lyapunov Theorem

The system

$$\frac{du}{dt} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \quad \frac{dv}{dt} = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j \tag{4}$$

has a center at the origin (equivalently, all coefficients of the Poincaré map are equal to zero) if and only if it admits a first integral of the form

$$\Phi = u^2 + v^2 + \sum_{k+l \geq 2} \phi_{kl} u^k v^l.$$

Definition of center for complex systems

System

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q) = P, \ \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}) = Q, \tag{5}$$

has a center at the origin if it admits a first integral of the form

$$\Phi(x, y; a_{10}, b_{10}, \dots) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j} x^j y^{s-j}$$

For system (4) always there exists a Lyapunov function $V(u,v)=u^2+v^2+\sum_{k+j>3} V_{kj}u^k v^j$ such that

$$\frac{dV}{dt}=\xi_2(u^2+v^2)+\xi_4(u^2+v^2)^2+\xi_6(u^2+v^2)^6+\dots .$$

Let the first different from zero coefficient be $\xi_{2k}<0$, i.e. $\frac{dV}{dt}=\xi_{2k}(u^2+v^2)^{2k}+\dots .$
We slightly change the coefficients $\alpha_{ij},\ \beta_{ij}$ of the system such that $|\xi_{2k-2}|\ll|\xi_{2k}|$, but $\xi_{2k-2}>0$. In such way $k-1$ limit cycle bifurcate from the origin.

For the complex system

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q) = P, \ \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}) = Q,$$

one looks for a function of the form
$\Phi(x, y; a_{10}, b_{10}, \dots) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j} x^j y^{s-j}$ such that

$$\frac{\partial \Phi}{\partial x} P + \frac{\partial \Phi}{\partial y} Q = g_{11}(xy)^2 + g_{22}(xy)^3 + \cdots , \tag{6}$$

and $g_{11}, g_{22}, \dots$ are polynomials in a_{pq}, b_{qp}. These polynomials are called *focus quantities*.

The Bautin ideal

The ideal $\mathcal{B} = \langle g_{11}, g_{22}, \dots \rangle$ generated by the focus quantities is called the *Bautin ideal*.

The center problem

Find the variety $\mathbf{V}(\mathcal{B})$ of the Bautin ideal $\mathcal{B} = \langle g_{11}, g_{22}, g_{33} \ldots \rangle$.
$\mathbf{V}(\mathcal{B})$ is called the center variety of the system.

Consider the quadratic system

$$\begin{aligned} \dot{x} &= x - a_{10}x^2 - a_{01}xy - a_{-12}y^2, \\ \dot{y} &= -(y - b_{10}xy - b_{01}y^2 - b_{2,-1}x^2). \end{aligned} \tag{7}$$

Theorem

The variety of the Bautin ideal of system (7) coincides with the variety of the ideal $\mathcal{B}_3 = \langle g_{11}, g_{22}, g_{33} \rangle$ and consists of four irreducible components:

1) $\mathbf{V}(J_1)$, where $J_1 = \langle 2a_{10} - b_{10}, 2b_{01} - a_{01} \rangle$,
2) $\mathbf{V}(J_2)$, where $J_2 = \langle a_{01}, b_{10} \rangle$,
3) $\mathbf{V}(J_3)$, where $J_3 = \langle 2a_{01} + b_{01}, a_{10} + 2b_{10},\ a_{01}b_{10} - a_{-12}b_{2,-1} \rangle$,
4) $\mathbf{V}(J_4) = \langle f_1, f_2, f_3, f_4, f_5 \rangle$, where

$f_1 = a_{01}^3 b_{2,-1} - a_{-12} b_{10}^3,\ f_2 = a_{10}a_{01} - b_{01}b_{10},$
$f_3 = a_{10}^3 a_{-12} - b_{2,-1} b_{01}^3,$
$f_4 = a_{10}a_{-12}b_{10}^2 - a_{01}^2 b_{2,-1} b_{01},\ f_5 = a_{10}^2 a_{-12} b_{10} - a_{01} b_{2,-1} b_{01}^2.$

Proof. Computing the first three focus quantities we have

$g_{11} = a_{10}a_{01} - b_{10}b_{01},$

$g_{22} = a_{10}a_{-12}b_{10}^2 - a_{01}^2 b_{01} b_{2,-1} - \frac{2}{3}(a_{-12}b_{10}^3 - a_{01}^3 b_{2,-1}) - \frac{2}{3}(a_{01}b_{01}^2 b_{2,-1} - a_{10}^2 a_{-12} b_{10}),$

$g_{33} = -\frac{5}{8}(-a_{01}\, a_{-12} b_{10}^4 + 2\, a_{-12} b_{01} b_{10}^4 + a_{01}^4 b_{10}\, b_{2,-1} - 2\, a_{01}^3\, b_{01}\, b_{10}\, b_{2,-1} - 2\, a_{10}\, a_{-12}^2\, b_{10}^2\, b_{2,-1} + a_{-12}^2\, b_{10}^3\, b_{2,-1} - a_{01}^3\, a_{-12}\, b_{2,-1}^2 + 2\, a_{01}^2\, a_{-12}\, b_{01}\, b_{2,-1}^2).$

Using the radical membership test we see that

$$g_{22} \notin \sqrt{\langle g_{11} \rangle}, \qquad g_{33} \notin \sqrt{\langle g_{11}, g_{22} \rangle}, \quad g_{44}, g_{55}, g_{66} \in \sqrt{\langle g_{11}, g_{22}, g_{33} \rangle}. \tag{8}$$

From (8) we expect that

$$\mathbf{V}(\mathcal{B}_3) = \mathbf{V}(\mathcal{B}). \tag{9}$$

The inclusion $\mathbf{V}(\mathcal{B}) \subseteq \mathbf{V}(\mathcal{B}_3)$ is obvious, therefore in order to check that (9) indeed holds we only have to prove that

$$\mathbf{V}(\mathcal{B}_3) \subseteq \mathbf{V}(\mathcal{B}). \tag{10}$$

To do so, we first look for a decomposition of the variety $\mathbf{V}(\mathcal{B}_3)$. To verify that (10) holds there remains to show that every system (7) with coefficients from one of the sets $\mathbf{V}(J_1), \mathbf{V}(J_2), \mathbf{V}(J_3), \mathbf{V}(J_4)$ has a center at the origin, that is, there is a first integral $\Psi(x, y) = xy + h.o.t.$

Systems corresponding to the points of $\mathbf{V}(J_1)$ are Hamiltonian with the Hamiltonian

$$H = -\left(xy - \frac{a_{-12}}{3}y^3 - \frac{b_{2,-1}}{3}x^3 - a_{10}x^2y - b_{01}xy^2\right)$$

and, therefore, have centers at the origin (since $D(H) \equiv 0$).
To show that for the systems corresponding to the components $\mathbf{V}(J_2)$ and $\mathbf{V}(J_3)$ the origin is a center we use the Darboux method.

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \quad x,y \in \mathbb{C} \quad P,Q \text{ are polynomials.} \tag{11}$$

The polynomial $f(x,y) \in \mathbb{C}[x,y]$ defines an *algebraic invariant curve* $f(x,y) = 0$ of system (11) if there exists a polynomial $k(x,y) \in \mathbb{C}[x,y]$ such that

$$D(f) := \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q = kf. \tag{12}$$

The polynomial $k(x,y)$ is called *cofactor* of f.

Suppose that the curves defined by

$$f_1 = 0, \dots, f_s = 0$$

are invariant algebraic curves of system (11) with the cofactors $k_1, \dots, k_s$. If

$$\sum_{j=1}^{s} \alpha_j k_j = 0\,, \tag{13}$$

then $H = f_1^{\alpha_1} \cdots f_s^{\alpha_s}$ is a (Darboux) first integral of the system (11) and if

$$\sum_{j=1}^{s} \alpha_j k_j = -P'_x - Q'_y\,, \tag{14}$$

then $\mu = f_1^{\alpha_1} \cdots f_s^{\alpha_s}$ is an integrated factor of (11).

Systems from $\mathbf{V}(J_2)$ and $\mathbf{V}(J_3)$ admit Darboux integrals.
Consider the variety $\mathbf{V}(J_3)$. In this case the system is

$$\begin{aligned} \dot{x} &= x - a_{10}x^2 + \frac{b_{01}}{2}xy - \frac{a_{10}b_{01}}{4b_{2,-1}}y^2, \\ -\dot{y} &= (y - b_{01}y^2 + \frac{a_{10}}{2}xy - b_{2,-1}x^2). \end{aligned} \tag{15}$$

- $f = \sum_{i+j=0}^{n} c_{ij} x^i y^j, \quad k = \sum_{i+j=0}^{m-1} d_{ij} x^i y^j$. ($m$ is the degree of the system; in our case $m = 1$). To find a bound for n is the Poincaré problem (unresolved).
- Equal the coefficients of the same terms in $D(f) = kf$.
- Solve the obtained system of polynomial equations for unknown variables c_{ij}, d_{ij}.

We look for an algebraic invariant curves in the form

$$f = c_{00}+c_{10}x+c_{01}y+c_{20}x^2+c_{11}xy+c_{02}y^2+c_{30}x^3+c_{21}x^2y+c_{12}xy^2+c_{03}y^3$$

and find

$$\ell_1 = 1 + 2\, b_{10}\, x - a_{01}\, b_{2,-1}\, x^2 + \ 2\, a_{01}\, y + 2\, a_{01}\, b_{10}\, x\, y - \frac{a_{01}\, b_{10}^2}{b_{2,-1}}\, y^2,$$

$$\ell_2 = (2\, b_{10}\, b_{2,-1}^2+6\, b_{10}^2\, b_{2,-1}^2\, x+3\, {b_{10}}^3\, b_{2,-1}^2\, x^2-3\, a_{01}\, b_{10}\, b_{2,-1}^3\, x^2-a_{01}\, b_{10}^2\, b_{2,-\cdots}$$
$$6\, a_{01}\, b_{10}\, b_{2,-1}^2\, y-3\, b_{10}^4\, b_{2,-1}\, x\, y+6\, a_{01}\, b_{10}^2\, b_{2,-1}^2\, x\, y-3\, a_{01}^2\, b_{2,-1}^3\, x\, y+3\, a_{01}\, b_{10}^3\, b_{2,-\cdots}$$
$$3\, a_{01}^2\, b_{10}\, b_{2,-1}^3 x^2\, y-3\, a_{01}\, b_{10}^3\, b_{2,-1}\, y^2+3\, a_{01}^2\, b_{10}\, b_{2,-1}^2\, y^2-3\, a_{01}\, b_{10}^4\, b_{2,-1}\, x\, y^2+3\, a_0^2\cdots$$
$$a_{01}\, b_{10}^5\, y^3 - a_{01}^2\, b_{10}^3\, b_{2,-1}\, y^3)/(2\, b_{10}\, b_{2,-1}^2)$$

with the cofactors $k_1 = 2\,(b_{10}\, x - a_{01}\, y)$ and $k_2 = 3\,(b_{10}\, x - a_{01}\, y)$, respectively. The equation $\alpha_1 k_1 + \alpha_2 k_2 = 0$ has a solution $\alpha_1 = -3, \alpha_2 = 2$, therefore the corresponding system has a Darboux first integral $\ell_1^{-3}\ell_2^2 \equiv c$. The integral is defined when $b_{10}b_{2,-1} \neq 0$. However

$$\overline{\mathbf{V}(J_3) \setminus \mathbf{V}(b_{10}b_{2,-1})} = \mathbf{V}(J_3).$$

Therefore every system from $\mathbf{V}(J_3)$ has a center at the origin.

Generalized Bautin's theorem

If the ideal $\mathcal{B}$ of all focus quantities of system

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \quad \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1})$$

is generated by the m first f. q., $\mathcal{B} = \langle g_{11}, g_{22}, \dots, g_{mm} \rangle$, then at most m limit cycles bifurcate from the origin of the corresponding real system

$$\dot{u} = \lambda u - v + \sum_{j+l=2}^{n} \alpha_{jl} u^j v^l, \quad \dot{v} = u + \lambda v + \sum_{j+l=2}^{n} \beta_{jl} u^j v^l,$$

that is the cyclicity of the system is less or equal to m.

- The quadratic system ($\dot{x} = P_n, \ \dot{y} = Q_n, \ n = 2$) - Bautin (1952) (Żoladek (1994), Yakovenko (1995), Françoise and Yomdin (1997), Han, Zhang & Zhang (2007)).
- The system with homogeneous cubic nonlinearities Sibirsky (1965) (Żołądek (1994))

In both cases the analysis is relatively simple because the Bautin ideal is a radical ideal.

Bautin's theorem for the quadratic system

The cyclicity of the origin of system

$$\dot{u} = \lambda u - v + \alpha_{20} u^2 + \alpha_{11} uv + \alpha_{02} v^2, \ \ \dot{v} = u + \lambda v + \beta_{20} u^2 + \beta_{11} uv + \beta_{02} v^2$$

equals three.

Methods to treat the systems with non-radical Bautin ideal have been developed recently

- V. Levandovskyy, V. R., D. S. Shafer (2009) J. Differential Equations, **246** 1274-1287.
- V. Levandovskyy, A. Logar and V. R. (2009) Open Systems & Information Dynamics, **16**, No. 4, 429-439.
- M. Han, V. R. (2010) J. Mathematical Analysis and Applications, **368**, 491-497.

These studies exploit special properties and the structure of g_{ii}.

Time-reversible systems

$$\frac{d\mathbf{z}}{dt} = F(\mathbf{z}) \quad (\mathbf{z} \in \Omega), \tag{16}$$

$F : \Omega \mapsto T\Omega$ is a vector field and Ω is a manifold.

Definition

A time-reversible symmetry of (16) is an invertible map $R : \Omega \mapsto \Omega$, such that

$$\frac{d(R\mathbf{z})}{dt} = -F(R\mathbf{z}). \tag{17}$$

Example

$$\dot{u} = v + vf(u, v^2), \qquad \dot{v} = -u + g(u, v^2), \tag{18}$$

The transformation $u \to u$, $v \to -v$, $t \to -t$ leaves the system unchanged $\Rightarrow$ the u–axis is a line of symmetry for the orbits $\Rightarrow$ no trajectory in a neighborhood of $(0,0)$ can be a spiral $\Rightarrow$ the origin is a center.

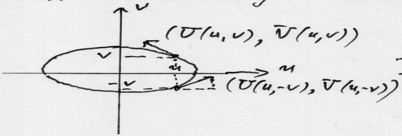
Here

$$R : u \mapsto u, \;\; v \mapsto -v. \tag{19}$$

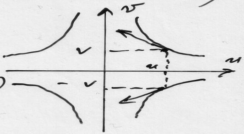
$$\dot{u} = U(u,v) \qquad x = u + iv \qquad \dot{x} = P(x, \bar{x})$$
$$\dot{v} = V(u,v) \qquad\qquad (P = U + iV)$$

$$u \to u, \quad v \to -v$$
$$\Updownarrow$$
$$x \to \bar{x}, \quad \bar{x} \to x \qquad (A)$$

Time-reversibility



Reversibility



$$U(u,v) = -U(u,-v) \qquad\qquad U(u,v) = U(u,-v)$$
$$V(u,v) = V(u,-v) \qquad\qquad V(u,v) = V(u,-v)$$

Note that,

$$P(\bar{x}, x) = U(u,-v) + i V(u,-v) = -U(u,v) + iV(u,v) = -\overline{P(x,\bar{x})}$$

$$P(\bar{x}, x) = U(u,-v) + iV(u,-v) = +U(u,v) - iV(u,v) = \overline{P(x,\bar{x})}$$

(A) yields $\dot{\bar{x}} = \overline{P(x,\bar{x})}$. Therefore

$$\dot{\bar{x}} = -P(\bar{x}, x) \qquad\qquad \dot{\bar{x}} = P(\bar{x}, x)$$

Complexification

$\dot{u} = U(u, v), \quad \dot{v} = V(u, v) \quad x = u + iv$

$$\dot{x} = \dot{u} + i\dot{v} = U + iV = P(x, \bar{x}) \tag{20}$$

We add to (20) its complex conjugate to obtain the system

$$\dot{x} = P(x, \bar{x}), \ \dot{\bar{x}} = \overline{P(x, \bar{x})}. \tag{21}$$

The condition of time-reversibility with respect to $Ou = Im\, x$:
$P(\bar{x}, x) = -\overline{P(x, \bar{x})}$.

Time-reversibility with respect to $y = \tan\varphi\, x$:

$$e^{2i\varphi}\overline{P(x,\bar{x})} = -P(e^{2i\varphi}\bar{x}, e^{-2i\varphi}x). \tag{22}$$

Consider $\bar{x}$ as a new variable y and allow the parameters of the second equation of (21) to be arbitrary. Then (21) yields the complex system $\dot{x} = P(x,y), \quad \dot{y} = Q(x,y)$. which is is time–reversible with respect to a transformation

$$R : x \mapsto \gamma y,\ \ y \mapsto \gamma^{-1} x$$

if and only if for some γ

$$\gamma Q(\gamma y, x/\gamma) = -P(x,y), \quad \gamma Q(x,y) = -P(\gamma y, x/\gamma)\,. \tag{23}$$

In the particular case when $\gamma = e^{2i\varphi}$, $y = \bar{x}$, and $Q = \bar{P}$ the equality (23) is equivalent to the reflection with respect a line and the reversion of time.

Systems of our interest are of the form

$$\begin{aligned} \dot{x} = &\ \ x - \sum_{(p,q)\in S} a_{pq} x^{p+1} y^q = P(x,y), \\ \dot{y} = &\ -y + \sum_{(p,q)\in S} b_{qp} x^q y^{p+1} = Q(x,y), \end{aligned} \tag{24}$$

where S is the set $S = \{(p_j, q_j)\,|\,p_j + q_j \geq 0, j = 1, \dots, \ell\} \subset (\{-1\} \cup \mathbb{N}_0) \times \mathbb{N}_0$, and $\mathbb{N}_0$ denotes the set of nonnegative integers. We will assume that the parameters $a_{p_j q_j}$, $b_{q_j p_j}$ $(j = 1, \dots, \ell)$ are from $\mathbb{C}$ or $\mathbb{R}$. Denote by $(a, b) = (a_{p_1 q_1}, \dots, a_{p_\ell q_\ell}, b_{q_\ell p_\ell} \dots, b_{q_1 p_1})$ the ordered vector of coefficients of system (24), by $E(a, b)$ the parameter space of (24) (e.g. $E(a, b)$ is $\mathbb{C}^{2\ell}$ or $\mathbb{R}^{2\ell}$), and by $k[a, b]$ the polynomial ring in the variables a_{pq}, b_{qp} over the field k.

The condition of time-reversibility

$$\gamma Q(\gamma y, x/\gamma) = -P(x,y), \quad \gamma Q(x,y) = -P(\gamma y, x/\gamma).$$

yields that system (24) is time–reversible if and only if

$$b_{qp} = \gamma^{p-q} a_{pq}, \qquad a_{pq} = b_{qp}\gamma^{q-p}. \tag{25}$$

We rewrite (25) in the form

$$a_{p_k q_k} = t_k, \quad b_{q_k p_k} = \gamma^{p_k - q_k} t_k \tag{26}$$

for $k = 1, \dots, \ell$. From a geometrical point of view equations (26) define a surface in the affine space
$\mathbb{C}^{3\ell+1} = (a_{p_1 q_1}, \dots, a_{p_\ell q_\ell}, b_{q_\ell p_\ell}, \dots, b_{q_1 p_1}, t_1, \dots, t_\ell, \gamma)$. Thus the set of all time-reversible systems is the projection of this surface onto $\mathbb{C}^{2\ell} = E(a,b)$.

Theorem (e.g. Cox D, Little J and O'Shea D 1992 *Ideals, Varieties, and Algorithms*)

Let k be an infinite field, $f_1, \ldots, f_n$ be elements of $k[t_1, \ldots, t_m]$,

$$x_1 = f_1(t_1, \ldots, t_m), \ \ldots \ x_n = f_n(t_1, \ldots, t_m),$$

and let $F : k^m \to k^n$, be the function defined by

$$F(t_1, \ldots, t_m) = (f_1(t_1, \ldots, t_m), \ldots, f_n(t_1, \ldots, t_m)).$$

Let $J = \langle f_1 - x_1, \ldots, f_n - x_n \rangle \subset k[y, t_1, \ldots, t_m, x_1, \ldots, x_n]$, and let $J_{m+1} = J \cap k[x_1, \ldots, x_n]$. Then $\mathbf{V}(J_{m+1})$ is the smallest variety in k^n containing $F(k^m)$.

Let

$$H = \langle a_{p_k q_k} - t_k, b_{q_k p_k} - \gamma^{p_k - q_k} t_k \mid k = 1, \dots, \ell \rangle, \qquad (27)$$

Let $\mathcal{R}$ be the set of all time-reversible systems in the family (24). From the previous theorem we obtain

Theorem

$\overline{\mathcal{R}} = \mathbf{V}(\mathcal{I})$ where $\mathcal{I} = k[a, b] \cap H$, that is, the Zariski closure of the set $\mathcal{R}$ of all time-reversible systems is the variety of the ideal $\mathcal{I}$.

Computation of $\mathcal{I} = k[a, b] \cap H$

Elimination Theorem

Fix the lexicographic term order on the ring $k[x_1, \ldots, x_n]$ with $x_1 > x_2 > \cdots > x_n$ and let G be a Groebner basis for an ideal I of $k[x_1, \ldots, x_n]$ with respect to this order. Then for every ℓ, $0 \le \ell \le n-1$, the set $G_\ell := G \cap k[x_{\ell+1}, \ldots, x_n]$ is a Groebner basis for the ideal $I_\ell = I \cap k[x_{\ell+1}, \ldots, x_n]$ (the ℓ–th elimination ideal of I).

By the theorem, to find a generating set for the ideal $\mathcal{I}$ it is sufficient to compute a Groebner basis for H with respect to a term order with $\{w, \gamma, t_k\} > \{a_{p_k q_k},\ b_{q_k p_k}\}$ and take from the output list those polynomials, which depend only on $a_{p_k q_k},\ b_{q_k p_k}$ $(k = 1, \ldots, \ell)$.

An algorithm for computing the set of all time-reversible systems

Let

$$H = \langle a_{p_k q_k} - t_k, b_{q_k p_k} - \gamma^{p_k - q_k} t_k \mid k = 1, \ldots, \ell \rangle.$$

- Compute a Groebner basis G_H for H with respect to any elimination order with $\{w, \gamma, t_k\} > \{a_{p_k q_k}, b_{q_k p_k} \mid k = 1, \ldots, \ell\}$;
- the set $M = G_H \cap k[a, b]$ is a set of binomials; $\mathbf{V}(\langle M \rangle)$ is the set of all time-reversible systems.

Theorem

Let $\mathbb{C}[M]$ be the polynomial subalgebra generated by the monomials of M. Then the focus quantities belong to $\mathbb{C}[M]$.

The cyclicity of a cubic system

$$\dot{x} = \lambda x + i(x - a_{-12}\bar{x}^2 - a_{20}x^3 - a_{02}x\bar{x}^2) \tag{28}$$

With system (28) we associate the complex system

$$\begin{aligned} \dot{x} &= i(x - a_{-12}y^2 - a_{20}x^3 - a_{02}xy^2) \\ \dot{y} &= -i(y - b_{2,-1}x^2 - b_{20}x^2y - b_{02}y^3) \end{aligned} \tag{29}$$

We compute a Groebner basis of the ideal

$$\mathcal{J} = \langle 1 - w\gamma^4, a_{-12} - t_1, \gamma^3 b_{2,-1} - t_1, a_{20} - t_2,\ b_{02} - \gamma^2 t_2, a_{02} - t_3, \gamma^2 b_{20} - t_3 \rangle$$

with respect to the lexicographic order with $w > \gamma > t_1 > t_2 > t_3 > a_{-12} > a_{20} > a_{02} > b_{20} > b_{02} > b_{2,-1}$ we obtain a list of polynomials and pick up the polynomials that do not depend on w, γ, t_1, t_2, t_3 :

$a_{20}a_{02} - b_{20}b_{02},\ a^2_{-12}a_{20}b^2_{20} - a^2_{02}b^2_{2,-1}b_{02},\ a^2_{-12}a^2_{20}b_{20} - a_{02}b^2_{2,-1}b^2_{02},\ -a^3_{02}b^2_{2,-1} - a^2_{-12}b^3_{20},\ a^2_{-12}a^3_{20} - b^2_{2,-1}b^3_{02},$ The monomials of the binomials form a basis of the subalgebra: $c_1 = a_{20}a_{02}, c_2 = b_{20}b_{02}, c_3 = a^3_{02}b^2_{2,-1}, c_4 a^2_{02}b^2_{2,-1}b_{02}, c_5 = a_{02}b^2_{2,-1}b^2_{02}, \dots$

The focus quantities of system (29) belong to the subalgebra $\mathbb{C}[c_1, \dots, c_{15}]$ that is,

$$g_{kk} = g_{kk}(c_1, \dots, c_{13}). \tag{30}$$

We prove that although the ideal of focus quantities is not radical ideal in $\mathbb{C}[a, b]$, it is a radical ideal in $\mathbb{C}[c_1, \dots, c_{15}]$ and use this to resolve the cyclicity problem for system (28).

Topic to be considered in the course

- Stability of solutions of systems of ODEs. Lyapunov functions.
- Normal forms, their computation, properties and convergence.
- Poincare return map. The center problem. Characterization of centers in polar coordinates and via normal forms. Centers of complex systems.
- Time-reversibility in two-dimensional systems of ODEs. Invariants of the rotation group. Interconnection of invariants and time-reversibility.
- Limit cycle bifurcations in polynomial systems of ODEs. The cyclicity problem and the Bautin ideal.