de Rham Cohomology for the Complement of an Affine Variety

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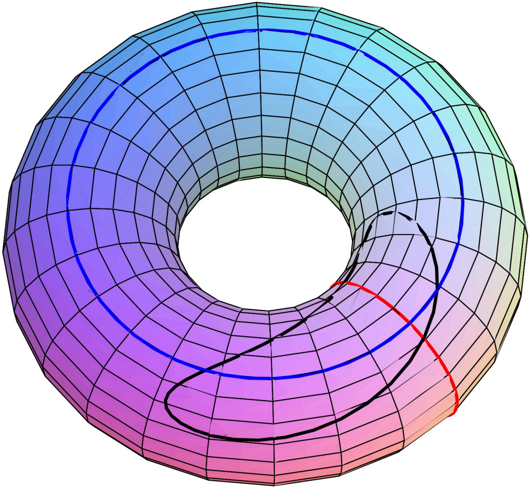
Motivation – de Rham Cohomology



Notation

- $R_n = \mathbb{C}[x_1, \ldots, x_n]$
- $X = \mathbb{C}^n$, $Y = \mathsf{Var}(f_0, \ldots, f_r)$, $U = X \backslash Y$
- $D_n = R_n \langle \partial_1, \ldots, \partial_n \rangle$ n-th Weyl algebra
- $\mathcal{O}_X$ structure sheaf on X
- $\mathcal{D}_X = \mathcal{O}_X \otimes_{R_n} D_n$ sheaf version of D_n
- Fourier transform $\mathcal{F}$:

$$\mathcal{F}(x_i) = \partial_i, \qquad \mathcal{F}(\partial_i) = -x_i$$

with $\tilde{M} = \mathcal{F}(M)$.

Algebraic de Rham Complex

The de Rham complex $\Omega^\bullet(\mathcal{M})$ of a $\mathcal{D}_X$-module $\mathcal{M}$:

$$0 \longrightarrow \Omega^0(\mathcal{M}) \xrightarrow{\delta} \Omega^1(\mathcal{M}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Omega^n(\mathcal{M}) \longrightarrow 0$$

with

$$\Omega^k(\mathcal{M}) = \mathcal{M} \otimes_{\mathbb{Z}} \bigwedge^k \mathbb{Z}^n,$$

$$\delta(u \otimes dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = \sum_{j=1}^{n} (\partial_j u) \otimes dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

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de Rham complex on a variety V:

$$\Omega^\bullet(V) = \Omega^\bullet(\mathcal{D}_V).$$

Goal: Compute $H^i_{dR}(U, \mathbb{C}) = \mathbb{H}^i(\Omega^\bullet(\mathcal{D}_U))$ for $U = X \backslash Y$.

Complement of a hypersurface

Theorem (Oaku and Takayama [2])

For $X = \mathbb{C}^n$, $Y = \mathrm{Var}(f_0)$, $U = X \backslash Y$ the de Rham cohomology is the cohomology of

$$\Omega^\bullet(D_n) \otimes_{D_n} R_n[f_0^{-1}].$$

In other words, it is computable by means of Gröbner bases.

Problem: This only works for $\mathrm{Var}(f_0)$.

Main Theorem

Theorem (Walther [5], Thm 6.1)

For $X = \mathbb{C}^n$, $Y = \mathrm{Var}(f_0, \ldots, f_r)$, $U = X \backslash Y$ the de Rham cohomology is

$$H^i_{dR}(U, \mathbb{C}) = H^{i-n}(\Omega \otimes^L_{D_n} MV^\bullet)$$

with $\Omega = \Omega(D_n)$.

The computation of $H^i_{dR}(U, \mathbb{C})$ will be the main topic of this talk.

Main Theorem – computations

The cohomology of $H^{i-n}(\Omega \otimes^L_{D_n} MV^\bullet)$ is calculated as follows:

1. $MV^\bullet$: $0 \longrightarrow \check{C}^1 \longrightarrow \cdots \longrightarrow \check{C}^{r+1} \longrightarrow 0$,
2. V_d-strict D_n-free complex quasi-isomorphic to $M\tilde{V}^\bullet$:

$$\cdots \longrightarrow A^{r-1}[\mathfrak{m}_{r-1}] \longrightarrow A^r[\mathfrak{m}_r] \longrightarrow 0,$$

3. b-function $b(s) = b_{A^\bullet[\mathfrak{m}_\bullet]}(s)$ for $A^\bullet[\mathfrak{m}_\bullet]$ restricted to the origin.
4. Find $k_0, k_1 \in \mathbb{Z}$: $b(k) = 0$ for $k \in \mathbb{Z} \Longrightarrow k_0 \leq k \leq k_1$.
5. $\Omega \otimes^L_{D_n} MV^\bullet$ is quasi-isomorphic to a complex of finite-dimensional $\mathbb{C}$-vector spaces.

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Čech complex

Algorithm (Walther [4], Alg 5.1)

The Čech complex for $f_0, \ldots, f_r \in R_n$ *in terms of finitely generated* D_n*-modules is computed as follows:*

- *Compute annihilators* $J^{\Delta}((F_I)^s)$ *and* $b^{\Delta}_{F_I}(s)$ *for all increasing* $I \subseteq \{0, \ldots, r\}$ *and* $F_I = f_{I_1} \cdots f_{I_k}$.
- a_I *smallest integer root of* $b^{\Delta}_{F_I}(s)$, $a := \min_I\{a_I\}$.

Čech complex

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- *a_I smallest integer root of $b^{\Delta}_{F_I}(s)$, $a := \min_I \{a_I\}$.*
- $\check{C}^k := \bigoplus_{|I|=k} D_n / J^{\Delta}((F_I)^s)|_{s=a}$
- *Compute matrices M_k for $\check{C}^k \to \check{C}^{k+1}$.*

Main Theorem – computations

The cohomology of $H^{i-n}(\Omega \otimes^L_{D_n} MV^\bullet)$ is calculated as follows:

1. $MV^\bullet$: $0 \longrightarrow \check{C}^1 \longrightarrow \cdots \longrightarrow \check{C}^{r+1} \longrightarrow 0,$
2. V_d-strict D_n-free complex quasi-isomorphic to $\tilde{MV}^\bullet$:
$$\cdots \longrightarrow A^{r-1}[\mathfrak{m}_{r-1}] \longrightarrow A^r[\mathfrak{m}_r] \longrightarrow 0,$$
3. b-function $b(s) = b_{A^\bullet[\mathfrak{m}_\bullet]}(s)$ for $A^\bullet[\mathfrak{m}_\bullet]$ restricted to the origin.
4. Find $k_0, k_1 \in \mathbb{Z}$: $b(k) = 0$ for $k \in \mathbb{Z} \Longrightarrow k_0 \leq k \leq k_1$.
5. $\Omega \otimes^L_{D_n} MV^\bullet$ is quasi-isomorphic to a complex of finite-dimensional $\mathbb{C}$-vector spaces.

V_d-filtrations

For $0 \leq d \leq n$ set $H = \mathrm{Var}(x_1, \ldots, x_d)$.

- $\alpha \in \mathbb{Z}^n \quad \rightsquigarrow \quad \alpha_H = (\alpha_1, \ldots, \alpha_d, 0, \ldots, 0)$.
- V_d-filtration of D_n:

$$F_H^k(D_n) = \{x^\alpha \partial^\beta \,|\, |\alpha_H| + k \geq |\beta_H|\}$$

- $V_d \deg(x^\alpha \partial^\beta) = \min_k \{x^\alpha \partial^\beta \in F_H^k(D_n)\}$

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- $V_d \deg(x^\alpha \partial^\beta) = \min_k \{x^\alpha \partial^\beta \in F_H^k(D_n)\}$
- V_d-filtration of D_n-modules $A = D_n^t[\mathfrak{m}]$:

$$F_H^k(A[\mathfrak{m}]) = \sum_{j=1}^{t} F_H^{k-\mathfrak{m}(j)}(D_n) \cdot e_j,$$

analogous for A/I.

V_d-strict complexes

- A complex of free D_n-modules

$$A^\bullet : \cdots \longrightarrow A^{i-1}[\mathfrak{m}_{i-1}] \xrightarrow{\phi^{i-1}} A^i[\mathfrak{m}_i] \longrightarrow \cdots$$

is V_d-strict if:

$$\mathrm{im}(\phi^{i-1}) \cap F_H^k(A^i[\mathfrak{m}_i]) = \phi^{i-1}(F_H^k(A^{i-1}[\mathfrak{m}_{i-1}]))$$

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- G is a V_d-strict Gröbner basis for $N \leq D_n^t[\mathfrak{m}]$, if each $n \in N$ can be represented as:

$$n = \sum_i a_i g_i, \qquad V_d\,\mathrm{deg}(n) \geq V_d\,\mathrm{deg}(a_i g_i).$$

- V_d-strict Gröbner bases can be calculated (Oaku and Takayama [3]) using a refining ordering for V_d and homogenisation.

Replace $C^\bullet$ by a quasi-isomorphic V_d-strict complex

Algorithm

Given $C^\bullet: \; 0 \longrightarrow C^0 \longrightarrow \cdots \longrightarrow C^r \longrightarrow 0,$ *a shift vector* $\mathfrak{m}_r$ *for* C^r*, compute quasi-isomorphic* V_d*-strict complex* $A^\bullet[\mathfrak{m}_\bullet]$*:*

1. *Break* $C^\bullet$ *into short exact sequences:*

$$0 \to B^i \to Z^i \to H^i \to 0, \quad 0 \to Z^i \to C^i \to B^{i+1} \to 0.$$

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2. *Make both* V_d*-strict with agreeing shift vectors for* B^i, Z^i.

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2. *Make both* V_d*-strict with agreeing shift vectors for* B^i*,* Z^i*.*
3. *Compute a Cartan-Eilenberg resolution for both, such that the resolutions for* B^i*,* Z^i *coincide.*

Replace $C^\bullet$ by a quasi-isomorphic V_d-strict complex

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4. *From (3), assemble a* V_d*-strict CE resolution of* $C^\bullet$.
5. *Take the total complex.*

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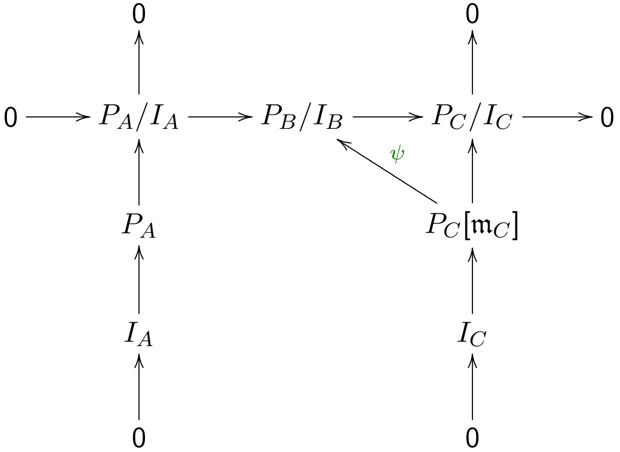
One V_d-strict short exact sequence

$$0 \longrightarrow P_A/I_A \longrightarrow P_B/I_B \longrightarrow P_C/I_C \longrightarrow 0$$

One V_d-strict short exact sequence

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & P_A/I_A & \longrightarrow & P_B/I_B & \longrightarrow & P_C/I_C & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
 & & P_A & & & & P_C[\mathfrak{m}_C] & & \\
 & & \uparrow & & & & \uparrow & & \\
 & & I_A & & & & I_C & & \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

One V_d-strict short exact sequence



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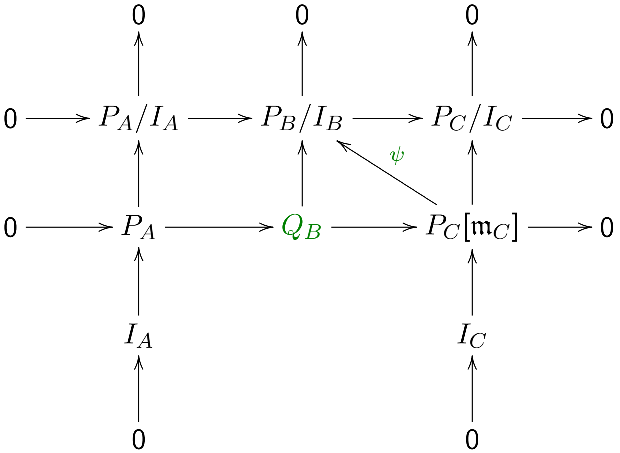

$Q_B = P_A \oplus P_C$

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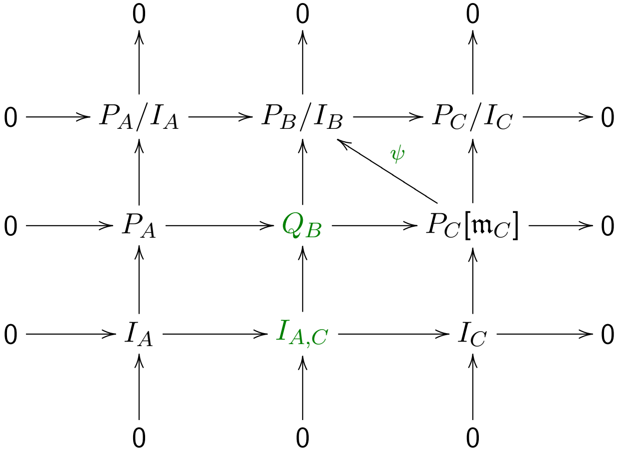

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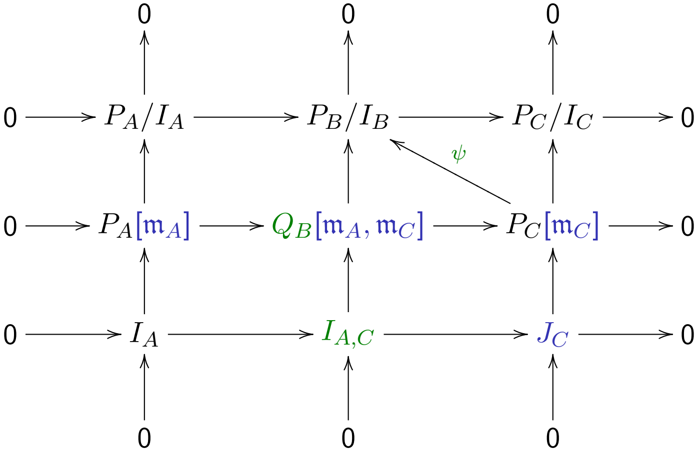

$Q_B = P_A \oplus P_C$ $\qquad$ J_c V_d-strict Gröbner basis

Two V_d-strict short exact sequences

Now change two sequences simultaneously:

$$0 \longrightarrow P_A/I_A \longrightarrow P_B/I_B \longrightarrow P_C[\mathfrak{m}_C]/I_C \longrightarrow 0$$

$$0 \longrightarrow P_D/I_D \longrightarrow P_A/I_A \longrightarrow P_F/I_F \longrightarrow 0$$

Two V_d-strict short exact sequences

Now change two sequences simultaneously:

$$0 \longrightarrow P_A/I_A \longrightarrow P_B/I_B \longrightarrow P_C[\mathfrak{m}_C]/I_C \longrightarrow 0$$

$$0 \longrightarrow P_D/I_D \longrightarrow Q_A/I_{D,F} \longrightarrow P_F/I_F \longrightarrow 0$$

$Q_A = P_D \oplus P_F$

Two V_d-strict short exact sequences

Now change two sequences simultaneously:

$$0 \longrightarrow Q_A/I_{D,F} \longrightarrow Q_B/I_{D,F,C} \longrightarrow P_C[\mathfrak{m}_C]/J_C \longrightarrow 0$$

$$0 \longrightarrow P_D/I_D \longrightarrow Q_A/I_{D,F} \longrightarrow P_F/I_F \longrightarrow 0$$

$$Q_A = P_D \oplus P_F \qquad Q_B = Q_A \oplus P_C = P_D \oplus P_F \oplus P_C$$

Two V_d-strict short exact sequences

Now change two sequences simultaneously:

$$0 \longrightarrow Q_A/I_{D,F}[\cdot, \mathfrak{m}_F] \longrightarrow Q_B/I_{D,F,C}[\cdot, \mathfrak{m}_F, \mathfrak{m}_C] \longrightarrow P_C[\mathfrak{m}_C]/J_C \longrightarrow 0$$

$$0 \longrightarrow P_D/I_D \longrightarrow Q_A/I_{D,F} \longrightarrow P_F/I_F \longrightarrow 0$$

$$Q_A = P_D \oplus P_F \qquad Q_B = Q_A \oplus P_C = P_D \oplus P_F \oplus P_C$$

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Now change two sequences simultaneously:

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$$0 \longrightarrow Q_A/I_{D,F}[\mathfrak{m}_D, \mathfrak{m}_F] \longrightarrow Q_B/I_{D,F,C}[\mathfrak{m}_D, \mathfrak{m}_F, \mathfrak{m}_C] \longrightarrow P_C[\mathfrak{m}_C]/J_C \longrightarrow$$

$$0 \longrightarrow P_D[\mathfrak{m}_D]/I_D \longrightarrow Q_A[\mathfrak{m}_D, \mathfrak{m}_F]/I_{D,F} \longrightarrow P_F[\mathfrak{m}_F]/J_F \longrightarrow 0$$

$$Q_A = P_D \oplus P_F \qquad Q_B = Q_A \oplus P_C = P_D \oplus P_F \oplus P_C$$

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2. *Make both* V_d*-strict with agreeing shift vectors for* B^i, Z^i.
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V_d-strict CE resolutions (1)

Lemma (Walther [5], La 4.3)

A V_d-strict exact sequence ($I_i \leq F_i[\mathfrak{m}_i]$) can be completed:

$$0 \longrightarrow I_A[\mathfrak{m}_A] \xrightarrow{\phi_A} I_B[\mathfrak{m}_B] \xrightarrow{\phi_B} I_C[\mathfrak{m}_C] \longrightarrow 0$$

V_d-strict CE resolutions (1)

Lemma (Walther [5], La 4.3)

A V_d-strict exact sequence ($I_i \leq F_i[\mathfrak{m}_i]$) can be completed:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I_A[\mathfrak{m}_A] & \xrightarrow{\phi_A} & I_B[\mathfrak{m}_B] & \xrightarrow{\phi_B} & I_C[\mathfrak{m}_C] & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P_A[\mathfrak{n}_A] & \longrightarrow & P_B[\mathfrak{n}_B] & \longrightarrow & P_C[\mathfrak{n}_C] & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & J_A & \longrightarrow & J_B & \longrightarrow & J_C & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

with P_i free, $\mathfrak{n}_i$ such that the rows and columns are V_d-strict.

V_d-strict CE resolutions (2)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B^i & \longrightarrow & Z^i & \longrightarrow & H^i & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P^0_{B,i} & \longrightarrow & P^0_{Z,i} & \longrightarrow & P^0_{H,i} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^0_{B,i} & \longrightarrow & I^0_{Z,i} & \longrightarrow & I^0_{H,i} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

V_d-strict CE resolutions (2)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B^i & \longrightarrow & Z^i & \longrightarrow & H^i & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P^0_{B,i} & \longrightarrow & P^0_{Z,i} & \longrightarrow & P^0_{H,i} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P^1_{B,i} & \longrightarrow & P^1_{Z,i} & \longrightarrow & P^1_{H,i} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^1_{B,i} & \longrightarrow & I^1_{Z,i} & \longrightarrow & I^1_{H,i} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

V_d-strict CE resolutions (2)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B^i & \longrightarrow & Z^i & \longrightarrow & H^i & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P^0_{B,i} & \longrightarrow & P^0_{Z,i} & \longrightarrow & P^0_{H,i} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P^1_{B,i} & \longrightarrow & P^1_{Z,i} & \longrightarrow & P^1_{H,i} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P^2_{B,i} & \longrightarrow & P^2_{Z,i} & \longrightarrow & P^2_{H,i} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Special feature: $P^i_{Z,i} = P^i_{B,i} \oplus P^i_{H,i}$

V_d-strict CE resolutions (3)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & B^i & \to & Z^i & \to & H^i & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^0_{B,i} & \to & P^0_{Z,i} & \to & P^0_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & I^0_{B,i} & \to & I^0_{Z,i} & \to & I^0_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Z^i & \to & C^i & \to & B^{i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^0_{Z,i} & \to & P^0_{C,i} & \to & P^0_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & I^0_{Z,i} & \to & I^0_{C,i} & \to & I^0_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

V_d-strict CE resolutions (3)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B^i & \longrightarrow & Z^i & \longrightarrow & H^i & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P^0_{B,i} & \longrightarrow & P^0_{Z,i} & \longrightarrow & P^0_{H,i} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & P^1_{B,i} & & I^0_{Z,i} & \longrightarrow & I^0_{H,i} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & I^1_{B,i} & & 0 & & 0 & & \\
 & & \uparrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Z^i & \longrightarrow & C^i & \longrightarrow & B^{i+1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P^0_{Z,i} & \longrightarrow & P^0_{C,i} & \longrightarrow & P^0_{B,i+1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^0_{Z,i} & \longrightarrow & I^0_{C,i} & \longrightarrow & I^0_{B,i+1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

V_d-strict CE resolutions (3)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & B^i & \to & Z^i & \to & H^i & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^0_{B,i} & \to & P^0_{Z,i} & \to & P^0_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^1_{B,i} & \to & P^1_{Z,i} & \to & P^1_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & I^1_{B,i} & \to & I^1_{Z,i} & \to & I^1_{H,i} & \to & 0 \\
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\end{array}
\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Z^i & \to & C^i & \to & B^{i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^0_{Z,i} & \to & P^0_{C,i} & \to & P^0_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & P^1_{Z,i} & & I^0_{C,i} & \to & I^0_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & I^1_{Z,i} & & 0 & & 0 & & \\
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\end{array}
$$

V_d-strict CE resolutions (3)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & B^i & \to & Z^i & \to & H^i & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^0_{B,i} & \to & P^0_{Z,i} & \to & P^0_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^1_{B,i} & \to & P^1_{Z,i} & \to & P^1_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & I^1_{B,i} & \to & I^1_{Z,i} & \to & I^1_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Z^i & \to & C^i & \to & B^{i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^0_{Z,i} & \to & P^0_{C,i} & \to & P^0_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^1_{Z,i} & \to & P^1_{C,i} & \to & P^1_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & I^1_{Z,i} & \to & I^1_{C,i} & \to & I^1_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

V_d-strict CE resolutions (3)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & B^i & \to & Z^i & \to & H^i & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^0_{B,i} & \to & P^0_{Z,i} & \to & P^0_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^1_{B,i} & \to & P^1_{Z,i} & \to & P^1_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^2_{B,i} & \to & P^2_{Z,i} & \to & P^2_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Z^i & \to & C^i & \to & B^{i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^0_{Z,i} & \to & P^0_{C,i} & \to & P^0_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^1_{Z,i} & \to & P^1_{C,i} & \to & P^1_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^2_{Z,i} & \to & P^2_{C,i} & \to & P^2_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

V_d-strict CE resolutions (3)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & B^i & \to & Z^i & \to & H^i & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^0_{B,i} & \to & P^0_{Z,i} & \to & P^0_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^1_{B,i} & \to & P^1_{Z,i} & \to & P^1_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^2_{B,i} & \to & P^2_{Z,i} & \to & P^2_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^3_{B,i} & \to & P^3_{Z,i} & \to & P^3_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Z^i & \to & C^i & \to & B^{i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^0_{Z,i} & \to & P^0_{C,i} & \to & P^0_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^1_{Z,i} & \to & P^1_{C,i} & \to & P^1_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^2_{Z,i} & \to & P^2_{C,i} & \to & P^2_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^3_{Z,i} & \to & P^3_{C,i} & \to & P^3_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Special features: $P^i_{Z,i} = P^i_{B,i} \oplus P^i_{H,i}$

$P^i_{C,i} = P^i_{B,i} \oplus P^i_{H,i} \oplus P^i_{B,i+1}$

Replace $C^\bullet$ by a quasi-isomorphic V_d-strict complex

Algorithm

Given $C^\bullet : \; 0 \longrightarrow C^0 \longrightarrow \cdots \longrightarrow C^r \longrightarrow 0,$ *a shift vector* $\mathfrak{m}_r$ *for* C^r*, compute quasi-isomorphic* V_d*-strict complex* $A^\bullet[\mathfrak{m}_\bullet]$*:*

1. *Break* $C^\bullet$ *into short exact sequences:*

$$0 \to B^i \to Z^i \to H^i \to 0, \quad 0 \to Z^i \to C^i \to B^{i+1} \to 0.$$

2. *Make both* V_d*-strict with agreeing shift vectors for* B^i, Z^i.
3. *Compute a Cartan-Eilenberg resolution for both, such that the resolutions for* B^i, Z^i *coincide.*
4. *From (3), assemble a* V_d*-strict CE resolution of* $C^\bullet$*.*
5. *Take the total complex.*

Cartan-Eilenberg resolution for $C^\bullet$

Use the V_d-strict resolutions of C^i:

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & C^0 & \longrightarrow & \cdots & \longrightarrow & C^i & \longrightarrow & C^{i+1} & \longrightarrow & \cdots & \longrightarrow & C^r & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & P^0_{C,0} & & \cdots & & P^0_{C,i} & & P^0_{C,i+1} & & \cdots & & P^0_{C,r} & & \\
 & & \uparrow & & & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & P^0_{C,1} & & \cdots & & P^1_{C,i} & \xrightarrow{\delta^1_i} & P^1_{C,i+1} & & \cdots & & P^0_{C,r} & & \\
 & & \uparrow & & & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & P^0_{C,2} & & \cdots & & P^2_{C,i} & & P^2_{C,i+1} & & \cdots & & P^0_{C,r} & & \\
 & & \uparrow & & & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & \vdots & & & & \vdots & & \vdots & & & & \vdots & &
\end{array}
$$

The horizontal maps are defined as:

$$\pm\delta^l_i : \; P^l_{C,i} \twoheadrightarrow P^l_{B,i+1} \hookrightarrow P^l_{Z,i+1} \hookrightarrow P^l_{C,i+1}$$

V_d-strict CE resolutions (3)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & B^i & \to & Z^i & \to & H^i & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^0_{B,i} & \to & P^0_{Z,i} & \to & P^0_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^1_{B,i} & \to & P^1_{Z,i} & \to & P^1_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^2_{B,i} & \to & P^2_{Z,i} & \to & P^2_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^3_{B,i} & \to & P^3_{Z,i} & \to & P^3_{H,i} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Z^i & \to & C^i & \to & B^{i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^0_{Z,i} & \to & P^0_{C,i} & \to & P^0_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^1_{Z,i} & \to & P^1_{C,i} & \to & P^1_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^2_{Z,i} & \to & P^2_{C,i} & \to & P^2_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & P^3_{Z,i} & \to & P^3_{C,i} & \to & P^3_{B,i+1} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Special features: $P^i_{Z,i} = P^i_{B,i} \oplus P^i_{H,i}$

$P^i_{C,i} = P^i_{B,i} \oplus P^i_{H,i} \oplus P^i_{B,i+1}$

Cartan-Eilenberg resolution for $C^\bullet$

Use the V_d-strict resolutions of C^i:

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & C^0 & \longrightarrow & \cdots & \longrightarrow & C^i & \longrightarrow & C^{i+1} & \longrightarrow & \cdots & \longrightarrow & C^r & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & P^0_{C,0} & \longrightarrow & \cdots & \longrightarrow & P^0_{C,i} & \longrightarrow & P^0_{C,i+1} & \longrightarrow & \cdots & \longrightarrow & P^0_{C,r} & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & P^0_{C,1} & \longrightarrow & \cdots & \longrightarrow & P^1_{C,i} & \xrightarrow{\delta^1_i} & P^1_{C,i+1} & \longrightarrow & \cdots & \longrightarrow & P^0_{C,r} & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & P^0_{C,2} & \longrightarrow & \cdots & \longrightarrow & P^2_{C,i} & \longrightarrow & P^2_{C,i+1} & \longrightarrow & \cdots & \longrightarrow & P^0_{C,r} & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & \vdots & & & & \vdots & & \vdots & & & & \vdots & &
\end{array}
$$

The horizontal maps are defined as:

$$
\pm\delta^l_i : \; P^l_{C,i} \twoheadrightarrow P^l_{B,i+1} \hookrightarrow P^l_{Z,i+1} \hookrightarrow P^l_{C,i+1}
$$

V_d-strict complex $A^\bullet[\mathfrak{m}_\bullet]$

Proposition (Walther [5], Prop 4.5)

$P^\bullet_{C,\bullet}[\mathfrak{m}_{C,\bullet,\bullet}]$ is a V_d-strict double complex. Its total complex $\mathrm{Tot}^\bullet(P^\bullet_{C,\bullet})$ is also V_d-strict.

In general, the total complex of a V_d-strict double complex is not V_d-strict.

Define $A^\bullet[\mathfrak{m}_\bullet] := \mathrm{Tot}^\bullet(P^\bullet_{C,\bullet})$.

Replace $C^\bullet$ by a quasi-isomorphic V_d-strict complex

Algorithm

Given $C^\bullet : 0 \longrightarrow C^0 \longrightarrow \cdots \longrightarrow C^r \longrightarrow 0,$ *a shift vector* $\mathfrak{m}_r$ *for* C^r*, compute quasi-isomorphic* V_d*-strict complex* $A^\bullet[\mathfrak{m}_\bullet]$*:*

1. *Break* $C^\bullet$ *into short exact sequences:*

$$0 \to B^i \to Z^i \to H^i \to 0, \quad 0 \to Z^i \to C^i \to B^{i+1} \to 0.$$

2. *Make both* V_d*-strict with agreeing shift vectors for* B^i, Z^i.
3. *Compute a Cartan-Eilenberg resolution for both, such that the resolutions for* B^i, Z^i *coincide.*
4. *From (3), assemble a* V_d*-strict CE resolution of* $C^\bullet$.
5. *Take the total complex.*

Main Theorem – computations

The cohomology of $H^{i-n}(\Omega \otimes^L_{D_n} MV^\bullet)$ is calculated as follows:

1. $MV^\bullet: \; 0 \longrightarrow \check{C}^1 \longrightarrow \cdots \longrightarrow \check{C}^{r+1} \longrightarrow 0,$
2. V_d-strict D_n-free complex quasi-isomorphic to $\tilde{MV}^\bullet$:

$$\cdots \longrightarrow A^{r-1}[\mathfrak{m}_{r-1}] \longrightarrow A^r[\mathfrak{m}_r] \longrightarrow 0,$$

3. b-function $b(s) = b_{A^\bullet[\mathfrak{m}_\bullet]}(s)$ for $A^\bullet[\mathfrak{m}_\bullet]$ restricted to the origin.
4. Find $k_0, k_1 \in \mathbb{Z}$: $b(k) = 0$ for $k \in \mathbb{Z} \Longrightarrow k_0 \leq k \leq k_1$.
5. $\Omega \otimes^L_{D_n} MV^\bullet$ is quasi-isomorphic to a complex of finite-dimensional $\mathbb{C}$-vector spaces.

b-function

The b-function for the restriction of a V_d-strict $(A^\bullet[\mathfrak{m}_\bullet], \phi^\bullet)$ to $H = \mathrm{Var}(x_1, \ldots, x_d)$ is defined as follows:

For $\kappa \in \ker(\phi^i)$ set $b_\kappa(s) \in K[s]$:

$$b_\kappa(\theta_d + \ldots) F_H^{k}(D_n) \cdot \kappa \subseteq F_H^{k-1}(D_n) \cdot \kappa + \mathrm{im}(\phi^{i-1}).$$

b-function

The b-function for the restriction of a V_d-strict $(A^\bullet[\mathfrak{m}_\bullet], \phi^\bullet)$ to $H = \mathrm{Var}(x_1, \ldots, x_d)$ is defined as follows:

For $\kappa \in \ker(\phi^i)$ set $b_\kappa(s) \in K[s]$:

$$b_\kappa(\theta_d + k + V_d \deg(\kappa)) F_H^k(D_n) \cdot \kappa \subseteq F_H^{k-1}(D_n) \cdot \kappa + \mathrm{im}(\phi^{i-1}).$$

b-function

The b-function for the restriction of a V_d-strict $(A^\bullet[\mathfrak{m}_\bullet], \phi^\bullet)$ to $H = \mathrm{Var}(x_1, \ldots, x_d)$ is defined as follows:

For $\kappa \in \ker(\phi^i)$ set $b_\kappa(s) \in K[s]$:

$$b_\kappa(\theta_d + k + V_d \deg(\kappa)) F_H^k(D_n) \cdot \kappa \subseteq F_H^{k-1}(D_n) \cdot \kappa + \mathrm{im}(\phi^{i-1}).$$

For $A^\bullet[\mathfrak{m}_\bullet]$ set:

$$b_{A^\bullet[\mathfrak{m}_\bullet]} := \mathrm{lcm}\{b_\kappa(s)\},$$

where lcm runs over all $\kappa \in ker(\phi^i)$ and all $i \in \mathbb{Z}$.

Computing cohomology

Theorem (Walther [5], Thm 5.7)

For a V_d-strict complex $A^\bullet[\mathfrak{m}_\bullet]$ of free D_n-modules the restriction to the origin $H = \mathrm{Var}(x_1, \ldots, x_n)$ is computed as:

- *Compute $b(s) = b_{A^\bullet[\mathfrak{m}_\bullet]}(s)$ for $A^\bullet[\mathfrak{m}_\bullet]$ restricted to the origin.*
- *Find $k_0, k_1 \in \mathbb{Z}$: $b(k) = 0$ for $k \in \mathbb{Z} \Longrightarrow k_0 \leq k \leq k_1$.*
- *$\tilde{\Omega} \otimes^L_{D_n} A^\bullet$ is quasi-isomorphic to a complex of finite-dimensional $\mathbb{C}$-vector spaces:*

$$\cdots \longrightarrow \frac{F_H^{k_1}(\tilde{\Omega} \otimes_{D_n} A^i[\mathfrak{m}_i])}{F_H^{k_0-1}(\tilde{\Omega} \otimes_{D_n} A^i[\mathfrak{m}_i])} \longrightarrow \frac{F_H^{k_1}(\tilde{\Omega} \otimes_{D_n} A^i[\mathfrak{m}_{i+1}])}{F_H^{k_0-1}(\tilde{\Omega} \otimes_{D_n} A^{i+1}[\mathfrak{m}_{i+1}])} \longrightarrow \cdots$$

Here, $\tilde{\Omega} = \mathbb{C}[\partial_1, \ldots, \partial_n]$.

Main Theorem – computations

The cohomology of $H^{i-n}(\Omega \otimes^L_{D_n} MV^\bullet)$ is calculated as follows:

1. $MV^\bullet$: $0 \longrightarrow \check{C}^1 \longrightarrow \cdots \longrightarrow \check{C}^{r+1} \longrightarrow 0$,
2. V_d-strict D_n-free complex quasi-isomorphic to $M\tilde{V}^\bullet$:
$$\cdots \longrightarrow A^{r-1}[\mathfrak{m}_{r-1}] \longrightarrow A^r[\mathfrak{m}_r] \longrightarrow 0,$$
3. b-function $b(s) = b_{A^\bullet[\mathfrak{m}_\bullet]}(s)$ for $A^\bullet[\mathfrak{m}_\bullet]$ restricted to the origin.
4. Find $k_0, k_1 \in \mathbb{Z}$: $b(k) = 0$ for $k \in \mathbb{Z} \Longrightarrow k_0 \leq k \leq k_1$.
5. $\Omega \otimes^L_{D_n} MV^\bullet$ is quasi-isomorphic to a complex of finite-dimensional $\mathbb{C}$-vector spaces.

The end.

Thanks!

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