Enhancing the classical algorithm by Oaku for the computation of Bernstein-Sato polynomials

Jorge Martín-Morales

Department of Mathematics
University of Zaragoza

Seminar on D-modules
Aachen, January 8, 2008

Introduction and notations

Basic notations

- $\mathbb{C}$ the field of the complex numbers.
- $\mathbb{C}[s]$ the ring of polynomials in one variable over $\mathbb{C}$.
- $R_n = \mathbb{C}[x_1, \ldots, x_n]$ the ring of polynomials in n variables.
- $D_n = \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$ the ring of $\mathbb{C}$-linear differential operators on R_n, the n-th Weyl algebra:

$$\partial_i x_i = x_i \partial_i + 1$$

- $D_n[s]$ the ring of polynomials in one variable over D_n.

THE $D_n[s]$-MODULE $R_n[s, \frac{1}{f}] \cdot f^s$

- Let $f \in R_n$ be a non-zero polynomial.
- By $R_n[s, \frac{1}{f}]$ we denote the ring of rational functions of the form
$$\frac{g(\mathbf{x}, s)}{f^r}$$
where $g(\mathbf{x}, s) \in R_n[s] = \mathbb{C}[x_1, \ldots, x_n, s]$.
- We denote by $M = R_n[s, \frac{1}{f}] \cdot f^s$ the free $R_n[s, \frac{1}{f}]$-module of rank one generated by the symbol f^s.
- $R_n[s, \frac{1}{f}] \cdot f^s$ has a natural structure of left $D_n[s]$-module.

$$\partial_i \cdot f^s = s \frac{\partial f}{\partial x_i} \frac{1}{f} \cdot f^s \quad \in \quad R_n[s, \frac{1}{f}] \cdot f^s$$

The global b-function

Theorem (Bernstein)

For every polynomial $f \in R_n$ there exists a non-zero polynomial $b(s) \in \mathbb{C}[s]$ and a differential operator $P(s) \in D_n[s]$ such that

$$P(s) f^{s+1} = b(s) f^s \quad \in \quad R_n[s, \frac{1}{f}] \cdot f^s.$$

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Definition (Bernstein & Sato)

The set of all possible polynomials $b(s)$ satisfying the above equation is an ideal of $\mathbb{C}[s]$. The monic generator of this ideal is denoted by $b_f(s)$ and called the Bernstein-Sato polynomial of f.

THE LOCAL b-FUNCTION

Now assume that

- $f \in \mathcal{O} = \mathbb{C}\{x_1, \ldots, x_n\}$ is a convergent power series.
- $\mathcal{D}_n$ is the ring of differential operators with coefficients in $\mathcal{O}$.

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Theorem (Björk & Kashiwara)

For every $f \in \mathcal{O}$ there exists a non-zero polynomial $b(s) \in \mathbb{C}[s]$ and a differential operator $P(s) \in \mathcal{D}_n[s]$ such that

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DEFINITION

The monic polynomial in $\mathbb{C}[s]$ of lowest degree which satisfies the above equation is denoted by $b_{f,0}(s)$ and called the local b-function of f.

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3. The set $\{e^{2\pi i\alpha} \mid b_{f,0}(\alpha)=0\}$ is a topological invariant of the singularity $f=0$. (Malgrange).

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6. $b_f(s)=\text{lcm}_{p\in\mathbb{C}^n}(b_{f,p}(s))$ (Briançon-Maisonobe, see also Mebkhout-Narváez).

ALGORITHMS FOR COMPUTING THE b-FUNCTION

Algorithms for computing the b-function

1. Global b-function.
   - Isolated case: use the algorithm implemented by Mathias Schulze in SINGULAR for computing the local b-functions and then apply the formula $b_f(s) = \text{lcm}_{p\in\mathbb{C}^n}(b_{f,p}(s))$.

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1. Obtain an upper bound for $b_f(s)$: find $B(s) \in \mathbb{C}[s]$ such that $b_f(s)$ divides $B(s)$.

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Remark

- There are some well-known methods to obtain such $B(s)$: Resolution of Singularities.
- We need two algorithms.

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- By definition, $(\text{ann}_{D_n[s]}(f^s) + \langle f \rangle) \cap \mathbb{C}[s] = \langle b_f(s) \rangle$.

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$$(\mathrm{ann}_{D_n[s]}(f^s) + \langle f, s + \alpha \rangle) \cap \mathbb{C}[s] = \langle b_f(s), s + \alpha \rangle$$

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Corollary

The following conditions are equivalent:

1. $\alpha \in \mathbb{Q}$ is a root of $b_f(-s)$.
2. $\mathrm{ann}_{D_n[s]}(f^s) + \langle f, s + \alpha \rangle \neq D_n[s]$.
3. $\mathrm{ann}_{D_n[s]}(f^s)_{|s=-\alpha} + \langle f \rangle \neq D$.

Algorithm 1

Algorithm 1 (check whether $\alpha \in \mathbb{Q}$ is a root of the b-function)

Input: $I = \mathrm{ann}_{D_n[s]}(f^s)$, f a polynomial in R_n, $\alpha \in \mathbb{Q}$;
Output: **true** if α is a root of $b_f(-s)$, **false** otherwise;

1. $J := I_{|s=-\alpha} + \langle f \rangle$; $\triangleright$ $J \subseteq D_n$
2. G a reduced Gröbner basis of J w.r.t. any term ordering;
3. **if** $G \neq \{1\}$ **then**
   return true
   else
   return false
   end if

What about the multiplicity ?

- By definition, $(\mathrm{ann}_{D_n[s]}(f^s) + \langle f \rangle) \cap \mathbb{C}[s] = \langle b_f(s) \rangle$.
- $(\mathrm{ann}_{D_n[s]}(f^s) + \langle f \rangle) \cap \mathbb{C}[s] + \langle q(s) \rangle = \langle b_f(s), q(s) \rangle, \quad q(s) \in \mathbb{C}[s]$

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Corollary

- m_α the multiplicity of α as a root of $b_f(-s)$.
- $J_i = \operatorname{ann}_{D_n[s]}(f^s) + \langle f, (s+\alpha)^{i+1} \rangle \subseteq D_n[s]$.

The following conditions are equivalent:

1. $m_\alpha > i$.
2. $(s+\alpha)^i \notin J_i$.

Algorithm 2

Algorithm 2 (compute the multiplicity of α as a root of $b_f(-s)$)

Input: $I = \mathrm{ann}_{D_n[s]}(f^s)$, f a polynomial in R_n, α in $\mathbb{Q}$;
Output: m_α, the multiplicity of α as a root of $b_f(-s)$;

for $i = 0$ to n **do**

1. $J := I + \langle f, (s+\alpha)^{i+1}\rangle$; $\triangleright$ $J_i \subseteq D_n[s]$
2. G a reduced Gröbner basis of J w.r.t. any term ordering;
3. r normal form of $(s+\alpha)^i$ with respect to G;
4. **if** $r = 0$ **then** $\triangleright$ $r = 0 \iff (s+\alpha)^i \in J_i$
   $m_\alpha = i$; **break** $\triangleright$ leave the **for** block
   end if

end for
return m_α

Remember the idea for computing $b_f(s)$

1. Obtain an upper bound for $b_f(s)$: find $B(s) \in \mathbb{C}[s]$ such that $b_f(s)$ divides $B(s)$.

$$B(s) = \prod_{i=1}^{d} (s - \alpha_i)^{m_i}.$$

2. Check whether α_i is a root of the b-function.
3. Compute its multiplicity m_i.

What about the first step ?

Applications

Let us see the following applications:

1. Computations of the b-functions via embedded resolutions.
2. Computations of the b-function of deformation of singularities.
3. An algorithm for computing the minimal integral root of $b_f(s)$ without computing the whole Bernstein-Sato polynomial.

Resolution of Singularities

- Let $f \in \mathcal{O}$ be a convergent power series, $f : \Delta \subseteq \mathbb{C}^n \to \mathbb{C}$.
- Assume that $f(0) = 0$, otherwise $b_{f,0}(s) = 1$.
- Let $\varphi : Y \to \Delta$ be an embedded resolution of $\{f = 0\}$.
- If $F = f \circ \varphi$, then $F^{-1}(0)$ is a normal crossing divisor.

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Theorem (Kashiwara).

There exists an integer $k \geq 0$ such that $b_f(s)$ is a divisor of the product $b_F(s) b_F(s+1) \cdots b_F(s+k)$.

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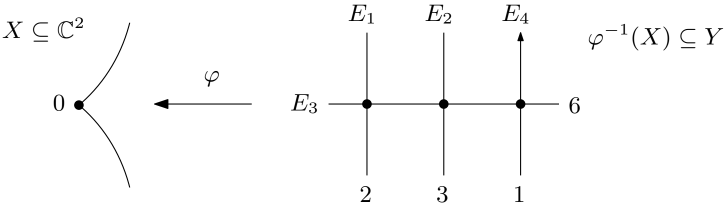

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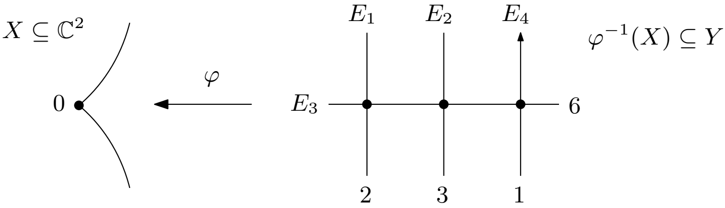

- From Kashiwara, the possible roots of $b_f(-s)$ are:

$$\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1, \frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{11}{6}.$$

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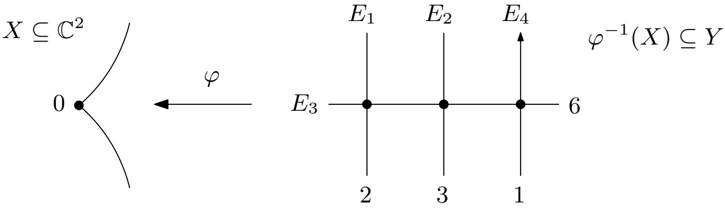

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- Using algorithms 1 and 2, we have proved that the numbers in red are the roots of $b_f(s)$, all of them with multiplicity one.

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Using this method we have computed the b-function of $f = (xz + y)(x^4 + y^5 + xy^4)$ which is a non-isolated singularity.

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- Then we use algorithms 1 and 2 for computing $b_g(s)$.

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Using this method we have computed the Bernstein polynomial for $g = z^4 + x^6y^5 + x^5y^4z$. We chose $f = z^4 + x^6y^5$ which is topologically equivalent to g.

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- Δ_i determinant of the minor resulting from deleting the i-th column of A, $i = 1, 2, 3, 4$.
- $f = \Delta_1\Delta_2\Delta_3\Delta_4 \in \mathbb{C}[x_1, \ldots, x_{12}]$.

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Example

Let us consider the following example:

- $A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} \end{pmatrix}$
- Δ_i determinant of the minor resulting from deleting the i-th column of A, $i = 1, 2, 3, 4$.
- $f = \Delta_1 \Delta_2 \Delta_3 \Delta_4 \in \mathbb{C}[x_1, \ldots, x_{12}]$.

From Kashiwara, the possible integral roots of $b_f(-s)$ are

$$11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1.$$

Using the algorithm 1, we have proved that the minimal integral root of $b_f(s)$ is -1.

dmod.lib Joint work with V. Levandovskyy

At the moment the Singular library dmod.lib for algebraic D-modules contains the following main procedures:

- Sannfs: computes a system of generators of $\mathrm{ann}_{D[s]}(f^s)$.
- Sannfslog: computes a system of generators of $\mathrm{ann}^{(1)}_{D[s]}(f^s)$.
- SannfsParam: computes a system of generators of $\mathrm{ann}_{D[s]}(f^s)$ when f has parameters.
- **checkRoot**
- annfs
- operator: computes $P(s)$ such that $P(s)f^{s+1} = b_f(s)f^s$.
- isHolonomic: checks whether a module given by a presentation is holonomic.

Thank you very much!

J. Martín-Morales (jorge@unizar.es)

Department of Mathematics
University of Zaragoza

Seminar on *D*-modules
Aachen, January 8, 2008