Hom(M, N) for holonomic D-modules M and N

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Notations

- $K \subseteq \mathbb{C}$ subfield
- $K[\underline{x}] := K[x_1, \ldots, x_n]$
- $K[\underline{\partial}] := K[\partial_1, \ldots, \partial_n]$
- $D := K[\underline{x}]\langle\underline{\partial}\rangle$
- M and N holonomic D-modules.
- $\tau\colon D \to D$ the standard transposition.

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- $\mathrm{Hom}(M, N)$

The algorithm is based on the proof of the following theorem.

Theorem (BJÖRK, 1979)

Let M and N be holonomic left D-modules. Then

$$\mathrm{Ext}^i_D(M, N) \cong \mathrm{Tor}^D_{n-i}(\mathrm{Ext}^n_D(M, D), N).$$

Proof: Let the following be a free resolution of M

$$X^\bullet : \quad 0 \to D^{r_{-a}} \xrightarrow{\cdot M_{-a+1}} \cdots \longrightarrow D^{r_{-1}} \xrightarrow{\cdot M_0} D^{r_0} \to M \to 0$$

Proof: Let the following be a free resolution of M

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Applying the Hom functor to the sequence induces a complex of right D-modules $\mathrm{Hom}_D(X^\bullet, D)$:

$$0 \leftarrow \underbrace{(D^{r_{-a}})^T}_{degree\ a} \xleftarrow{M_{-a+1}\cdot} \cdots \longleftarrow (D^{r_{-1}})^T \xleftarrow{M_0\cdot} (D^{r_0})^T \leftarrow 0$$

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Since $\mathrm{Hom}_D(D^r, D) \otimes_D N \cong \mathrm{Hom}_D(D^r, N)$, we obtain $\mathrm{Hom}_D(X^\bullet, D) \otimes_D N \cong \mathrm{Hom}_D(X^\bullet, N)$. But the cohomology groups of this complex are exactly $\mathrm{Ext}^i_D(M, N)$.

We replace N by a free resolution $Y^\bullet$ to obtain the following double complex:

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$$
\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & 0 & \\
 & & \uparrow & & & & \uparrow & & \uparrow & \\
0 & \longleftarrow & (D^{r_{-a}})^T \otimes_D D^{s_0} & \xleftarrow{(M_{-a+1}\cdot)\otimes \mathrm{Id}_{s_0}} & \cdots & \longleftarrow & (D^{r_{-1}})^T \otimes_D D^{s_0} & \xleftarrow{(M_0\cdot)\otimes \mathrm{Id}_{s_0}} & (D^{r_0})^T \otimes_D D^{s_0} & \longleftarrow & 0 \\
 & & \uparrow{\scriptstyle -\mathrm{Id}_{r_{-a}} \otimes (\cdot N_0)} & & & & \uparrow{\scriptstyle -\mathrm{Id}_{r_{-1}} \otimes (\cdot N_0)} & & \uparrow{\scriptstyle -\mathrm{Id}_{r_0} \otimes (\cdot N_0)} & \\
0 & \longleftarrow & (D^{r_{-a}})^T \otimes_D D^{s_{-1}} & \xleftarrow{(M_{-a+1}\cdot)\otimes \mathrm{Id}_{s_{-1}}} & \cdots & \longleftarrow & (D^{r_{-1}})^T \otimes_D D^{s_{-1}} & \xleftarrow{(M_0\cdot)\otimes \mathrm{Id}_{s_{-1}}} & (D^{r_0})^T \otimes_D D^{s_{-1}} & \longleftarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow & \\
 & & \vdots & & & & \vdots & & \vdots & \\
 & & \uparrow{\scriptstyle -\mathrm{Id}_{r_{-a}} \otimes (\cdot N_{-b+1})} & & & & \uparrow{\scriptstyle -\mathrm{Id}_{r_{-1}} \otimes (\cdot N_{-b+1})} & & \uparrow{\scriptstyle -\mathrm{Id}_{r_0} \otimes (\cdot N_{-b+1})} & \\
0 & \longleftarrow & (D^{r_{-a}})^T \otimes_D D^{s_{-b}} & \xleftarrow{(M_{-a+1}\cdot)\otimes \mathrm{Id}_{s_{-b}}} & \cdots & \longleftarrow & (D^{r_{-1}})^T \otimes_D D^{s_{-b}} & \xleftarrow{(M_0\cdot)\otimes \mathrm{Id}_{s_{-b}}} & (D^{r_0})^T \otimes_D D^{s_{-b}} & \longleftarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow & \\
 & & 0 & & & & 0 & & 0 &
\end{array}
$$

Using a result from the theory of spectral sequences we get that the cohomology of the total complex equals the cohomology of the complex

$$0 \longleftarrow \mathrm{Hom}_D(D^{r_{-a}}, N) \xleftarrow{\mathrm{Hom}_D(M_{-a+1}\cdot, N)} \cdots \xleftarrow{\mathrm{Hom}_D(M_0\cdot, N)} \mathrm{Hom}_D(D^{r_0}, N) \longleftarrow 0 \; .$$

The cohomologies of this sequence are the $\mathrm{Ext}^i_D(M, N)$.

With a similar argumentation as above the cohomology of the total complex also equals the cohomology of the complex

$$0 \longrightarrow \mathrm{Ext}^n_D(M, D) \otimes_D D^{s-b} \longrightarrow \cdots \xrightarrow{\mathrm{Id}_{\mathrm{Ext}^n_D(M,D)} \otimes (\cdot N_0)} \mathrm{Ext}^n_D(M, D) \otimes_D D^{s_0} \longrightarrow 0$$

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By definition the cohomology groups of this complex are the $\mathrm{Tor}_D^j(\mathrm{Ext}_D^n(M, D), N)$, which concludes the proof.

q.e.d.

Polynomial solutions

Suppose the left D-module M given by $M := D^{r_0}/D \cdot \{L_1, \ldots, L_{r_1}\}$ for $L_1, \ldots, L_{r_1} \in D^{r_0}$. Our aim is to construct the vector space $\mathrm{Hom}_D(M, K[x])$. These homomorphisms can be identified with the subset of $K[x]^{r_0}$ of all elements that are annihilated by $L_1, \ldots, L_{r_1} \in D^{r_0}$.

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(3) Let $\pi_n \colon D^{s_{-n}} \to \ker(\cdot\tau(M_{-n}))$ be a surjection and find the pre-image $\tau(P) := \pi_n^{-1}(\mathrm{im}(\cdot\tau(M_{-n+1})))$.

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$$\tilde{\pi} \colon D^{s_{-n}} \to \ker(\cdot\tau(M_{-n}))/\,\mathrm{im}(\cdot\tau(M_{-n+1}))\colon$$
$$x \mapsto \pi_n(x) + \mathrm{im}(\cdot\tau(M_{-n+1})),$$

whose kernel is exactly P. So we get a presentation $D^{s_{-n}}/\tau(P) \cong \tau(\mathrm{Ext}^n_D(M, D))$.

(4) Compute the derived integration module

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using the integration algorithm. We get

(i) A $\tilde{V}$-adapted free resolution $E^\bullet$ of $D^{s-n}/\tau(P)$ of length $n+1$.

(ii) Elements $g_1, \ldots, g_k \in D^{s_0}$ whose images modulo $\mathrm{im}(\tau(K[\underline{x}]) \otimes E^\bullet)$ form a K-basis for

$$\begin{aligned} &H^0((D/\{\partial_1, \ldots, \partial_n\} \cdot D \otimes^L_D (D^{s-n}/\tau(P))[n]) \\ &\cong H^0(D/\{\partial_1, \ldots, \partial_n\} \cdot D \otimes_D E^\bullet) \end{aligned}$$

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$$H^0((D/\{\partial_1, \ldots, \partial_n\} \cdot D \otimes^L_D (D^{s-n}/\tau(P))[n]) \cong H^0(D/\{\partial_1, \ldots, \partial_n\} \cdot D \otimes_D E^\bullet)$$

(5) Lift the map π_n to a chain map $\pi_\bullet \colon E^\bullet \to \tau(\mathrm{Hom}_D(X^\bullet, D))$ and denote the maps by $\pi_i \colon D^{s-i} \to D^{r-i}$.

(6) Evaluate $\{\tau(\pi_0(g_1)), \dots, \tau(\pi_0(g_k))\} \subseteq (D/D \cdot \{\partial_1, \dots, \partial_n\})^{r_0}$ and write $\{R_1(\underline{x}), \dots, R_n(\underline{x})\}$ for these elements understood in $K[\underline{x}]^{r_0}$.

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(7) Return $\{R_1(\underline{x}), \ldots, R_n(\underline{x})\}$.

Example:

Let $n = 2$ and $M := D/_D\langle u, v\rangle$ where

$$u = x_1\partial_1 + 2x_2\partial_2 - 5 \text{ and } v = \partial_1^2 - \partial_2.$$

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We resolve the module M by

$$X^\bullet : \qquad 0 \longrightarrow D^1 \xrightarrow{\cdot\begin{pmatrix} -v & u+2 \end{pmatrix}} D^2 \xrightarrow{\cdot\begin{pmatrix} u \\ v \end{pmatrix}} D^1 \longrightarrow M \longrightarrow 0.$$

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On the other hand $K[x_1, x_2]$ can be resolved by the KOSZUL-complex

$$\mathcal{K}^\bullet : \qquad 0 \longrightarrow D \xrightarrow{\cdot\begin{pmatrix} \partial_1 & \partial_2 \end{pmatrix}} D^2 \xrightarrow{\cdot\begin{pmatrix} \partial_2 \\ -\partial_1 \end{pmatrix}} D^1 \longrightarrow K[x_1, x_2] \longrightarrow 0.$$

By applying $\mathsf{Hom}(-, D)$ to $X^\bullet$ we get the complex

$$0 \longleftarrow D^1 \xleftarrow{\begin{pmatrix} -v & u+2 \end{pmatrix}\cdot} D^2 \xleftarrow{\begin{pmatrix} u \\ v \end{pmatrix}\cdot} D^1 \longleftarrow M \longleftarrow 0 \ .$$

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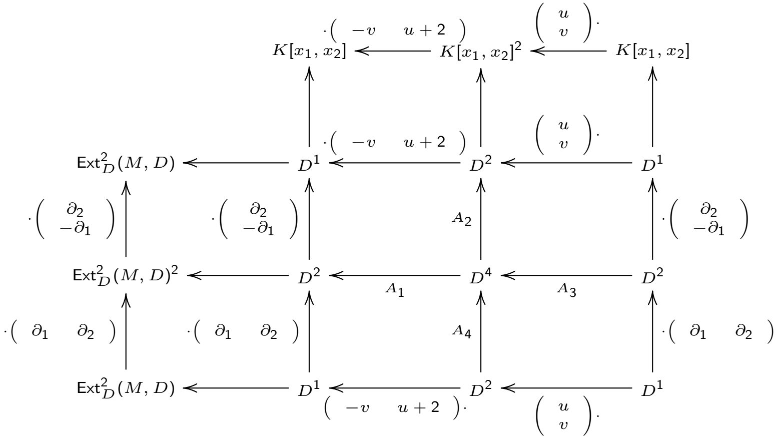
The complex

$$\mathrm{Ext}^2_D(M, D) \longleftarrow D^1 \xleftarrow{\begin{pmatrix} -v & u+2 \end{pmatrix}\cdot} D^2 \xleftarrow{\begin{pmatrix} u \\ v \end{pmatrix}\cdot} D^1 \longleftarrow M \longleftarrow 0$$

is exact.

$$
\begin{array}{ccccccc}
 & & K[x_1,x_2] & \xleftarrow{\cdot\begin{pmatrix} -v & u+2 \end{pmatrix}} & K[x_1,x_2]^2 & \xleftarrow{\begin{pmatrix} u \\ v \end{pmatrix}\cdot} & K[x_1,x_2] \\
 & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow \\
\mathrm{Ext}^2_D(M,D) & \longleftarrow & D^1 & \xleftarrow{\cdot\begin{pmatrix} -v & u+2 \end{pmatrix}} & D^2 & \xleftarrow{\begin{pmatrix} u \\ v \end{pmatrix}\cdot} & D^1 \\
\cdot\begin{pmatrix} \partial_2 \\ -\partial_1 \end{pmatrix}\Big\uparrow & & \cdot\begin{pmatrix} \partial_2 \\ -\partial_1 \end{pmatrix}\Big\uparrow & & A_2\Big\uparrow & & \Big\uparrow\cdot\begin{pmatrix} \partial_2 \\ -\partial_1 \end{pmatrix} \\
\mathrm{Ext}^2_D(M,D)^2 & \longleftarrow & D^2 & \xleftarrow[A_1]{} & D^4 & \xleftarrow[A_3]{} & D^2 \\
\cdot\begin{pmatrix} \partial_1 & \partial_2 \end{pmatrix}\Big\uparrow & & \cdot\begin{pmatrix} \partial_1 & \partial_2 \end{pmatrix}\Big\uparrow & & A_4\Big\uparrow & & \Big\uparrow\cdot\begin{pmatrix} \partial_1 & \partial_2 \end{pmatrix} \\
\mathrm{Ext}^2_D(M,D) & \longleftarrow & D^1 & \xleftarrow[\begin{pmatrix} -v & u+2 \end{pmatrix}\cdot]{} & D^2 & \xleftarrow[\begin{pmatrix} u \\ v \end{pmatrix}\cdot]{} & D^1
\end{array}
$$



where

$$A_1 = \begin{pmatrix} -v & u+2 & 0 & 0 \\ 0 & 0 & -v & u+2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} -v & u+2 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} \partial_2 & 0 \\ 0 & \partial_2 \\ -\partial_1 & 0 \\ 0 & -\partial_1 \end{pmatrix} = \begin{pmatrix} \partial_2 \\ -\partial_1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} u & 0 \\ v & 0 \\ 0 & u \\ 0 & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} u \\ v \end{pmatrix}$$

$$A_4 = \begin{pmatrix} \partial_1 & 0 & \partial_2 & 0 \\ 0 & \partial_1 & 0 & \partial_2 \end{pmatrix} = \begin{pmatrix} \partial_1 & \partial_2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

With the integration algorithm we obtain the cohomology at the left D^1 in the last row of the augmented complex, which in this case is one-dimensional and spanned by

$$L_{1,0} = -(2x_1^5x_2 - 40x_1^3x_2^2 + 120x_1x_2^3)\partial_1 - (x_1^6 - 30x_1^4x_2 + 180x_1^2x_2^2 - 120x_2^3).$$

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By first mapping $L_{1,0}$ with $\cdot \begin{pmatrix} \partial_1 & \partial_2 \end{pmatrix}$, then taking the preimage we get $L_{1,1} \in D^4$ and repeating this we obtain

$$(x_1^5 - 20x_1^3x_2 + 60x_1x_2^2) = L_{1,2} \in D^1.$$

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The residue class of this element spans the space of polynomial solutions.

Rational solutions

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$$\begin{aligned}\mathrm{Ext}^i_D(M, K[\underline{x}][f^{-1}]) &\cong \mathrm{Tor}^D_{n-i}(\mathrm{Ext}^n_D(M, D), K[\underline{x}][f^{-1}]) \\ &\cong \mathrm{Tor}^D_{n-i}(\mathrm{Ext}^n_D(M, D)[f^{-1}], K[\underline{x}]) \\ &\cong \mathrm{Tor}^D_{n-i}(D/\{\partial_1, \ldots, \partial_n\} \cdot D, \tau(\mathrm{Ext}^n_D(M, D))[f^{-1}])\end{aligned}$$

Algorithm for computing dimensions of rational solution spaces

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We use the above isomorphism.

1. Compute a polynomial f vanishing on the codimension 1 component of the singular locus of M.

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4. Use the integration algorithm to compute the derived integration $\mathrm{Tor}^D_{n-i}(D/\{\partial_1, \ldots, \partial_n\} \cdot D, \tau(\mathrm{Ext}^n_D(M, D))[f^{-1}])$

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5. Return the dimensions.

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With some slight modifications the algorithm for computing $\mathrm{Hom}(M, K[\underline{x}])$ in the polynomial case can also be applied in this situation.

Holonomic solutions

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Assumptions:

M and N are holonomic D-modules. Let D_x be the n-th WEYL algebra on the variables $x_1, \ldots, x_n$ and derivations $\partial_1, \ldots, \partial_n$, D_y the n-th WEYL algebra on $y_1, \ldots, y_n$ with derivations $\delta_1, \ldots, \delta_n$.

Holonomic solutions

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M and N are holonomic D-modules. Let D_x be the n-th WEYL algebra on the variables $x_1, \ldots, x_n$ and derivations $\partial_1, \ldots, \partial_n$, D_y the n-th WEYL algebra on $y_1, \ldots, y_n$ with derivations $\delta_1, \ldots, \delta_n$.

If X and Y are D_x- respectively D_Y-modules, let $X \boxtimes Y$ denote the exterior product. As an additive group this equals tensor product of X and Y regarded as a $D_{2n} \cong D_x \otimes D_y$ module.

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By μ we define the isomorphism

$$\mu \colon D_{2n} \to D_{2n} \colon \left\{ \begin{array}{ll} x_i \mapsto \frac{1}{2}x_i - \delta_i & \partial_i \mapsto \frac{1}{2}y_i + \partial_i \\ y_i \mapsto \frac{1}{2}x_i - \delta_i & \delta_i \mapsto \frac{1}{2}y_i - \partial_i \end{array} \right\}_{i=1}^{n},$$

and let Δ and Λ be the following right D_{2n}-modules

$$\Delta := D_{2n}/(\{x_i - y_i, \partial_i + \delta_i \,|\, 1 \le i \le n\} \cdot D_{2n})$$
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Combining those two and setting $M' = \mathrm{Ext}^n_D(M, D)$ one obtains

$$Ext^i_D(M, N) \cong$$
$$\mathrm{Tor}^{D_{2n}}_{n-i}(D_{2n}/\{x_i - y_i, \partial_i + \delta_i \mid 1 \le i \le n\} \cdot D_{2n}, \tau(\mathrm{Ext}^n_D(M, D)) \boxtimes N).$$

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(1) Compute finite free resolutions $X^\bullet$ and $Y^\bullet$ of M and N, say

$$X^\bullet : \quad 0 \to D^{r_{-a}} \xrightarrow{\cdot M_{-a+1}} \cdots \longrightarrow D^{r_{-1}} \xrightarrow{\cdot M_0} D^{r_0} \to M \to 0$$

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Dualize $X^\bullet$ and apply the standard transposition to obtain

$$\tau(\mathrm{Hom}_D(X^\bullet, D)) \colon 0 \leftarrow D^{r_{-a}} \xleftarrow{\cdot \tau(M_{-a+1})} \cdots \leftarrow D^{r_{-1}} \xleftarrow{\cdot \tau(M_0)} D^{r_0} \leftarrow 0$$

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(2) Build the double complex $\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet$ (of left D_{2n}-modules) and let

$$Z^\bullet : \quad 0 \leftarrow D_{2n}^{t_a} \xleftarrow{\cdot T_{a-1}} \cdots \leftarrow D_{2n}^{t_0} \leftarrow \cdots \leftarrow D_{2n}^{t_{-b}} \leftarrow 0$$

be its total complex.

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(3) Let $\pi_n \colon D_{2n}^{u_n} \to \ker(\cdot\mu(T_n))$ be a surjection and find the pre-image $P := \pi_n^{-1}(\mathrm{im}(\cdot\mu(T_{n-1})))$.

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 (i) A V-adapted free resolution $E^\bullet$ of $D_{2n}^{u_n}/P$ of length $n+1$.

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$$\begin{aligned} &H^0((D/\{\underline{x}, \underline{y}\} \cdot D \otimes_{D_{2n}}^{L} (D_{2n}^{u_n}/P))[n]) \\ &\cong H^0(D/\{\underline{x}, \underline{y}\} \cdot D \otimes_{D_{2n}} E^\bullet) \end{aligned}$$

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(5) Lift π_n to a chain map $\pi_\bullet \colon E^\bullet \to \mu(Z^\bullet)$ with $\pi_i \colon D_{2n}^{u_i} \to D_{2n}^{t_i}$.

(6) Evaluate $\{L_1 = \mu^{-1}(\pi_0(g_1)), \ldots, L_k = \mu^{-1}(\pi_0(g_k))\}$ and write each L_i as

$$\sum_j L_{i,j} \in \bigoplus_j D^{r-j} \boxtimes D^{s-j} = D_{2n}^{t_0}$$

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$$L_{i,0} = l_{i,1} e_1 + \cdots + l_{i,r_0} e_{r_0} \in (D_y)^{s_0} e_1 \oplus \cdots \oplus (D_y)^{s_0} e_{r_0}.$$

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Let $\{\overline{l_{i,1}}, \ldots, \overline{l_{i,r_0}}\}$ be the images in $(D^{s_0}/N_0) \cong N$. Finally set $\varphi_i \in \mathrm{Hom}_D(M, N)$ to be the map induced by

$$e_j \mapsto \overline{l_{i,j}}.$$

(7) Return $\{\varphi_1, \ldots, \varphi_k\}$.