Localization

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Introduction

This talk deals with localisation of holonomic Weyl algebra modules and their localisation. Consider left modules an d left ideals for this talk. Instead of the talks before we will localise a general holonomic module and not the polynomial ring. It will follow algorithms given by [6] and [5, D-modules and Cohomology of Varieties]. Note that there is a more general algorithm in [4]. It is planned to implement this algorithm in Singular ([2],[1])

1 Notation. For this talk let $R_n := K[\underline{x}] := K[x_1, .., x_n]$ the polynomial ring in n indeterminates and let $D_n := R\langle\underline{\partial}\rangle := R_n\langle\partial_1, .., \partial_n\rangle$ be the n-th Weyl algebra over a computable field K of characteristic 0 contained in $\mathbb{C}$.

2 Motivation. Let $f \in R_n$ and M a holonomic (left) D_n-module. Since M is cyclic, $M \cong D_n/I$ for a left ideal $I = \langle P_1, .., P_r\rangle$ in D_n. Our goal is to compute $M[f^{-1}] := R_n[f^{-1}] \otimes_{R_n} M$, i.e. to find generators and relations. As mentioned above, the algorithm can be generalised to the case, where M only needs to be holonomic on $K^n \setminus \mathcal{V}(f)$. This would allows us, to get rid of the non-holonomic locus by "localising it away" and go on computing with a holonomic module.

Since this module is holonomic again, we need to find a generator $f^a \otimes 1$ and its annihilator. First call this generator $f^s \otimes 1 \otimes 1 \in f^s \otimes_K R_n[f^{-1}, s] \otimes_{R_n} M$. Here f^s is used as abstract generator, that behaves as the factor f^s under all operations of the Weyl algebra D_n. Now the algorithm will work in two steps:

1. Compute $J^I(f^s) := \mathrm{Ann}_{D_n[s]}(f^s \otimes 1 \otimes 1)$.

2. Compute a suitable number $a \in K$ for substituting s by a.

On the above modules let $\underline{x}$ operate by left multiplication on the right factor (this is equivalent to left or right multiplication on the middle factor), let s operate on the middle factor and let $\underline{\partial}$ operate by product rule. Since f^s is just a symbol, the action three factors of ∂_i on the tensor product works the following way:

$$\partial_i \bullet (f^s \otimes \frac{g(\underline{x}, s)}{f^k} \otimes Q) = f^s \otimes \frac{sg(\underline{x}, s)f_i}{f^{k+1}} \otimes Q + f^s \otimes \partial_i(\frac{g(\underline{x}, s)}{f^k}) \otimes Q + f^s \otimes \frac{g(\underline{x}, s)}{f^k} \otimes \partial_i Q$$

Here $f_i := \frac{\mathrm{d}f}{\mathrm{d}x_i}$.

1 The Annihilator

3 Remark. The way of proceding here is quite similar to the case of $M = R_n$ seen two weeks ago, but one has to consider that holonomic M in the general setting complicates matters. Before the annihilator of $M = R_n$ was given by $\langle \underline{\partial} \rangle$ and now it is given by I. But the same changes to the generators of the new annihilator are done by the ringautomorphism ϕ introduced later.

4 Remark. Since it is quite hard to compute $J^I(f^s)$, we will extend $D_n[s]$ to a new algebra $D_{n+1} := D_n\langle t, \partial_t \rangle$ to compute $J^I_{n+1}(f^s) := \mathrm{Ann}_{D_{n+1}}(f^s \otimes 1 \otimes 1)$ and then "intersect" this algebra with $D_n[s]$. But first we need to define the action of D_{n+1} on $f^s \otimes_K R_n[f^{-1}, s] \otimes_{R_n} M$. In short t acts by shifting s up by one and ∂_t acts by shifting s down by one and a differential factor. More formally we have:

$$\begin{aligned} t \bullet (f^s \otimes \frac{g(\underline{x}, s)}{f^k} \otimes Q) &:= f^s \otimes \frac{g(\underline{x}, s+1)f}{f^k} \otimes Q \\ \partial_t \bullet (f^s \otimes \frac{g(\underline{x}, s)}{f^k} \otimes Q) &:= f^s \otimes \frac{-sg(\underline{x}, s-1)}{f^{k+1}} \otimes Q \end{aligned}$$

Note that this actually defines a module structure and that $-\partial_t t$ acts by s (so $D_n[s] \hookrightarrow D_{n+1}$).

5 Remark. Let $\phi : D_{n+1} \xrightarrow{\sim} D_{n+1} : x_i \mapsto x_i, t \mapsto t - f, \partial_i \mapsto \partial_i + f_i\partial_t, \partial_t \mapsto \partial_t$ be a ring automorphism. It is clear, that this map is invertible.

Short computation on the relations: $[x_i, \partial_i + f_i\partial_t] = [x_i, \partial_i]$, $[t - f, \partial_t] = [t, \partial_t]$, $[\partial_i + f_i\partial_t, \partial_t] = 0$ etc.

Now a lemma that gives concrete generators for $J^I_{n+1}(f^s)$:

6 Lemma. Let I be f-saturated. Then $J^I_{n+1}(f^s) =_{D_{n+1}} \langle \phi(I), t - f \rangle$ holds.
Proof:

"$\supseteq$" For $P \in I$ write P as polynom in the ∂_i's with coefficents in R_n. Then it holds, that $\phi(P(\underline{\partial})) = P(\underline{\partial} + f_{\underline{i}}\partial_t)$. With above definition it is easy to check, that:

$$(\partial_i + f_i\partial_t) \bullet (f^s \otimes 1 \otimes 1) = (f^s \otimes 1 \otimes \partial_i)$$

and so also

$$\phi(P(\underline{\partial})) \bullet (f^s \otimes 1 \otimes 1) = f^s \otimes 1 \otimes P(\underline{\partial})$$

holds. Further we get:

$$t \bullet (f^s \otimes 1 \otimes 1) = (f^s \otimes f \otimes 1) = f \bullet (f^s \otimes 1 \otimes 1)$$

So $t - f$ is included in the annihilator.

"⊆" Now let $Q \bullet (f^s \otimes 1 \otimes 1) = 0$. $f - t$ is contained in the annihilator, hence we can asume that all powers of t in Q are substituted by f. So write Q in the form

$$Q = \sum_{\alpha,\beta} \partial_t^\alpha \underline{x}^\beta Q_{\alpha,\beta}(\partial_1 + f_1\partial_t, .., \partial_n + f_n\partial_t)$$

for $Q_{\alpha,\beta} \in K[\underline{y}]$. Then, as seen in "⊇", it holds that:

$$Q \bullet (f^s \otimes 1 \otimes 1) = \sum_{\alpha,\beta} \partial_t^\alpha \bullet (f^s \otimes 1 \otimes \underline{x}^\beta Q_{\alpha,\beta}(\underline{\partial}))$$

Let $\overline{\alpha}$ be the largest α with nonzero $Q_{\alpha,\beta}$ in above representaion of Q. We claim, that:

$$\sum_\beta \underline{x}^\beta Q_{\overline{\alpha},\beta}(\partial_1 + f_1\partial_t, .., \partial_n + f_n\partial_t) \in \phi(I)$$

Then by induction over α also $Q \in \phi(I)$ and we are finished. To prove this, observe that in the term

$$\sum_\beta \underline{x}^\beta Q_{\overline{\alpha},\beta}(\partial_1 + f_1\partial_t, .., \partial_n + f_n\partial_t) \bullet (f^s \otimes 1 \otimes 1) = f^s \otimes \frac{s^{\overline{\alpha}} + LOT(s)}{f^{\overline{\alpha}}} \otimes (\sum_\beta \underline{x}^\beta Q_{\overline{\alpha},\beta}(\underline{\partial}))$$

$s^{\overline{\alpha}}$ cannot be reduced by lower summmands of Q in α. So already

$$f^s \otimes \frac{s^{\overline{\alpha}}}{f^{\overline{\alpha}}} \otimes (\sum_\beta \underline{x}^\beta Q_{\overline{\alpha},\beta}(\underline{\partial}))$$

has to vanish. Since I is f-saturated and no other factor than f could "pass" the right tensor product, this implies, that

$$(\sum_\beta \underline{x}^\beta Q_{\overline{\alpha},\beta}(\underline{\partial})) \in I \Leftrightarrow \sum_\beta \underline{x}^\beta Q_{\overline{\alpha},\beta}(\partial_1 + f_1\partial_t, .., \partial_n + f_n\partial_t) \in \phi(I)$$

□

Now identify $D_n[-\partial_t t] \subset D_{n+1}$ with $D_n[s]$ and - to finish the task of this section - compute:

$$J^I(f^s) = J^I_{n+1}(f^s) \cap D_n[s] = J^I_{n+1}(f^s) \cap D_n[-\partial_t t]$$

For this give an algorithm due to Oaku [3] that computes the intersection of any left ideal J of D_{n+1} with $D_n[-\partial_t t]$:

7 Algorithm. Input: Left ideal I of D_{n+1}

Output: $J = I \cap D_n[s] = I \cap D_n[-\partial_t t]$

On D_{n+1} define the weight vector w by $w(t) = 1, w(\partial_t) = -1, w(x_i) = w(\partial_i) = 0$. Then extend this weights to $D_{n+1}[y_1, y_2]$ by $w(y_1) = 1, w(y_2) = -1$.

Now homogenize the generators of I by y_1 according to the weight vector w.

Compute a Gröbner basis $\tilde{J}$ of this homogenized ideal and $1 - y_1 y_2$ eliminating y_1 and y_2. Note that the resulting generators are also homogenous w.r.t. w, since the input and all relations of the ring are homogenous. So, even though there are negative weights, the Buchberger algorithm works.

Take all elements of $\tilde{J}$ not having y_1 or y_2 and multiply them (from the left) with appropriate powers of t and ∂_t to give them a w-degree of 0. Return these elements. □

Combining the above lemma and algorithm yields a method to compute $J^I(f^s)$ for f-saturated I.

2 The Generator

This sections aims at finding an appropriate number a for substituting s. As main tool for this use the Berstein(-Sato) polynomial. This polynomial needs to be redefined for this situation:

8 Definition. Let the Bernstein polynomial $b_f^I(s) \in K[s]$ be the monic generator for the ideal of all elements $b \in K[s]$, such that there exists a $Q(s) \in D_n[s]$ with:

$$b(s) \bullet (f^s \otimes 1 \otimes 1) = Q(s) \bullet (f^s \otimes f \otimes 1) = Q(s)f \bullet (f^s \otimes 1 \otimes 1)$$

For $b_f^I(s)$ fix $Q_f^I(s)$ as the operator with above properties.

9 Remark. As always with the Bernstein polynomial the idea ist, that $\frac{Q_f^I(s)}{b_f^I(s)}$ is some kind of inverse for f under certain circumstances.

In the case of $M = R_n$ it is known, that $b_f(s)$ factors over the rationals with negative roots. In the general case there are counterexamples to this, compare [5, D-modules and Cohomology of Varieties, 3.10].

10 Algorithm. Input: $f \in R_n$ and f-saturated holonomic ideal $I \trianglelefteq D_n$

Output: $b_f^I(s)$

Compute the (unique) monic generator of ${}_{D_n[s]}\langle f, J^I(f^s)\rangle \cap K[s]$ by computing a Gröbner basis of ${}_{D_n[s]}\langle f, J^I(f^s)\rangle$ with an order eliminating the x_i and ∂_i. Here $J^I(f^s)$ can be computed by means of section 1.

Proof:

By definition of $b_f^I(s)$ it holds that $b_f^I(s) \bullet (f^s \otimes 1 \otimes 1) = (Q_f^I(s) f) \bullet (f^s \otimes 1 \otimes 1)$ for a $Q_f^I(s) \in D_n[s]$. So $b_f^I(s)$ is also an element of

$$_{D_n[s]}\langle f\rangle + \mathrm{Ann}_{D_n[s]}(f^s \otimes 1 \otimes 1) =_{D_n[s]} \langle f\rangle +_{D_n[s]} \langle J^I(f^s)\rangle =_{D_n[s]} \langle f, J^I(f^s)\rangle$$

□

This implies a theorem which will only partly be proven here. It allows us to connect the Berstein polynomial $b_f^I(s)$ and the exponents by means of finding integer roots of $b_f^I(s)$.

11 Theorem. If $M = D_n/I$ is holonomic and $a \in K^*$, such that no element of $\{a-1, a-2, ..\}$ is a root of $b_f^I(s)$, then we have

$$f^a \otimes_K R_n[f^{-1}] \otimes_{R_n} M \cong D_n \bullet (f^a \otimes 1 \otimes 1) \cong (D_n[s]/J^I(f^s))_{|s=a}$$

as D_n-modules.

Proof: (partly/1. isomorphism)

For the first isomorphism we get

$$\begin{array}{rrcll} & b_f^I(s) \bullet (f^s \otimes 1 \otimes 1) & = & Q_f^I(s) f \bullet (f^s \otimes 1 \otimes 1) & \mid (-\partial_t) \cdot \\ \Rightarrow & s b_f^I(s-1) \bullet (f^s \otimes f^{-1} \otimes 1) & = & s Q_f^I(s-1) \bullet (f^s \otimes 1 \otimes 1) & \end{array}$$

from definition of $b_f^I(s)$ and Q_f^I. Since $b_f^I(a-1) \neq 0$, we can substitute s by a and invert $b_f^I(a-1)$ to get:

$$f^a \otimes f^{-1} \otimes 1 = (b_f^I(a-1))^{-1} Q_f^I(a-1) \bullet (f^a \otimes 1 \otimes 1)$$

So at least all powers of f are generated. Now let Q be a monomial in $K[\partial_1, .., \partial_n]$ of degree m. Assume that all such monomials of degree smaller than m can be generated and let Q' be such that $\partial_i Q' = Q$ and with a P' such that $P' \bullet (f^a \otimes 1 \otimes 1) = f^a \otimes f^k \otimes Q'$ for any fixed $k \in \mathbb{Z}$. But then it holds, that:

$$f^a \otimes f^k \otimes Q = \partial_i P' \bullet (f^a \otimes 1 \otimes 1) - f^a \otimes f_i(a+k) f^{k-1} \otimes Q'$$

By induction over m we can get any monomial Q.

The first isomorphism follows, because one can add up these monomials and multiply them with elements of R_n by definition of the action of D_n.

12 Remark. Now one takes a as the smallest negative integer root of $b_f^I(s)$. If no such number exists, then set $a := -1$. With these choices the prerequisite of the last theorem are fulfilled.

Conclusion

Now the part to compute the localisation are assembled:

13 Algorithm. Input: $f \in R_n$, $M = D_n/I$ holonomic and f-saturated

Output: $J \trianglelefteq D_n$ (by generators) and $a \in \mathbb{Z}$ with $R_n[f^{-1}] \otimes_{R_n} M \cong D_n/J$, where $R_n[f^{-1}] \otimes_{R_n} M$ is generated by $f^a \otimes 1$.

Determine $J^I(f^s)$ as in section 1.

Determine $b_f^I(s)$ as in section 2.

Find the smallest integer root a of $b_f^I(s)$ using rational factorisation or by testing (there are bounds depending on coeffcients for roots of a polynomial). If no such root exisits, set $a := -1$.

Replace s by a in each generator of $J^I(f^s)$ and return these elements (calling them J) together with a.

References

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